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# AN ESTIMATE OF THE EXPECTATION OF THE EXCESS OF A RENEWAL SEQUENCE GENERATED BY A TIME-INHOMOGENEOUS MARKOV CHAIN IF A SQUARE-INTEGRABLE MAJORIZING SEQUENCE EXISTS

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Abstract. We consider sufficient conditions providing the existence of the expectation of the excess of a renewal sequence for a time-inhomogeneous Markov chain with an arbitrary space of states. For such a chain, we study the behavior of the corresponding renewal process, the sequence of moments when the chain returns to a certain set $C$. The main aim of the paper is to derive a numerical bound for the expectation of the excess of the renewal sequence defined as the time between a moment $t$ and the first renewal after $t$.

## 1. Introduction

The existence of the coupling moment is studied in the paper [31] for time-inhomogeneous Markov chains. Some sufficient conditions are found in [31] for the finiteness of the coupling moment. However the results obtained in [31] are not easy to apply in practice for two reasons. First, the checking of assumptions of the main theorem in [31] is a rather complicated task and, second, a numerical bound for the expectation of the coupling moment is not obtained in [31].

Therefore it is natural to find other assumptions, even more restrictive than in [31], allowing one to evaluate the expectation of the coupling moment effectively.

It is worth mentioning that the key ingredient of the proof in [31] is an estimation of the expectation of the excess for a certain renewal sequence constructed from a time-inhomogeneous Markov chain, namely for the random time between an arbitrary moment $t$ and the first renewal after $t$.

Under quite general assumptions, the results in [31] imply a linear bound for the expectation of the excess, namely $\mathsf{E}[R_t] \le \rho t + C$. A similar bound can be found in the monograph by Lindvall [13] for the coupling of two copies of the same chain in the time-homogeneous case. However, the result in [13] is not completed with a bound for the expectation of the coupling moment.

At the same time, there exist papers (see, for example, [28]), where some numerical bounds are obtained for coupling moments of two different time-homogeneous Markov chains. The key assumption in [28] is that the second moment exists for the renewal sequence and the proof in [28] is based on a Daley inequality [23].

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A similar result is obtained in [31] for coupling moments of two different time-homogeneous Markov chains under the same assumption on the existence of the second moment and again by using the Daley inequality.

It is natural to extend the results of [31] to the time-inhomogeneous case. Unfortunately the Daley inequality (meaning, in fact, that the expectation of the excess is finite) has no counterpart in the time-inhomogeneous case. Instead, we present another result of the same type in the current paper. Namely, we prove the finiteness of the expectation of the excess of a renewal sequence generated by a time-inhomogeneous Markov chain under the assumption that the conditional distributions of renewals are bounded from above by some square integrable distribution. As a byproduct, we obtain a numerical bound for the expectation of the excess.

The result obtained in this paper is much easier to apply in practice than the main result of [31] in the time-inhomogeneous case.

Note that answers to the questions on the finiteness of the expectation of the coupling moment and estimation of this expectation are very important, since they play a crucial role when constructing estimates for stability of transient probabilities over $n$ steps for two time-inhomogeneous Markov chains.

The stability of time-inhomogeneous Markov chains is studied in [25, 26, 34]. The stability of transient probabilities over $n$ steps are considered in these papers for both time-homogeneous and time-inhomogeneous cases. The so-called $C$-coupling is used in those papers which means that two chains may couple if they simultaneously happen to be in the same set $C$. Despite the fact that $C$-coupling is a widely used tool ([19, 22]), it finds an application only in the analysis of the stability of a single time-homogeneous Markov chain for different initial distributions. This method is adjusted correspondingly in this paper in order to apply it to the analysis of stability of two different (possibly time-inhomogeneous) chains.

The papers [29, 30, 33] deal with the so-called maximal coupling of chains. It means that (discrete) chains couple when they meet at a certain point. In the papers mentioned above, some results are obtained concerning the stability of transient probabilities over $n$ steps and, which is important, concerning finite dimensional distributions. These results are proved under the uniform mixing condition for matrices of transient probabilities which is an analogue of the uniform ergodicity.

The main tool of the proof below is the coupling method. General results concerning the coupling method can be found in the classical monographs [13, 17]. The coupling method is a useful tool for various studies of Markov chains (see [4, 5, 8–10, 19, 20, 22, 24]). The coupling method or similar constructions appear in the pioneering papers [1, 7, 11].

## 2. The renewal sequence generated by a time-inhomogeneous Markov chain

We follow the notation introduced in [31] throughout this paper. The renewal sequence generated by a time-inhomogeneous Markov chain is introduced and some of its properties are studied in [31]. We briefly recall the notation of [31].

Let $X_t$, $t \geq 0$, be some time-inhomogeneous Markov chain (with discrete time) assuming values in a phase space $(E, \mathbb{E})$. The transient probabilities at step $t$ are denoted by $P_t(x, A)$.

The symbols $\mathsf{P}$ and $\mathsf{E}$ stand for the probability and expectation, respectively, generated by the Markov chain.

Let $C$ be a set, $C \in \mathbb{E}$. The main object studied in this paper is the sequence of returns to the set $C$. Assume that $X_0 \in C$. In contrast to the paper [31], we do not assume here that $C$ consists of a single element. Although this assumption (that $C$ consists of a single element) is not crucial, it simplifies the general construction.

Let

$$\begin{aligned} \theta_1 &= \inf\{t > 0\colon X_t \in C\}, \\ \theta_2 &= \inf\{t > \theta_1\colon X_t \in C\}, \\ &\dots \\ \theta_m &= \inf\{t > \theta_{m-1}\colon X_t \in C\}, \qquad m > 1. \end{aligned} \tag{1}$$

Therefore $\theta_k$ is the time between the $(k-1)^{\text{th}}$ and $k^{\text{th}}$ returns to $C$. We also need the notation for the return moments. Let

$$\tau_k = \sum_{j=1}^{k} \theta_j, \qquad k > 0, \tag{2}$$

be the moment of the $k^{\text{th}}$ return, $\tau_0 = 0$.

We consider a process without delay in this paper, that is, $\theta_0 = \tau_0 = 0$ or, in other words, $X_0 \in C$. However this assumption is not crucial.

Consider a flow of $\sigma$-algebras generated by the sequences $\theta_n$ and $\tau_n$ and values of the process at renewal moments,

$$\mathscr{F}_n = \sigma[\theta_k, X_{\tau_k}, 1 \le k \le n] = \sigma[\tau_k, X_{\tau_k}, 1 \le k \le n]. \tag{3}$$

The structure of dependence between the random variables $\theta_k$ is worth mentioning. These random variables are independent in the homogeneous case. This property, however, is not valid for the time-inhomogeneous case. This is explained by the fact that the distribution of the next return moment to the set $C$ depends on the initial moment. Nevertheless the distribution of the random variable $\theta_k$ is completely determined by the preceding return moment (and by the value of the chain at the return moment if $C$ consists of more than one element). More precisely,

$$\mathsf{E}[\theta_{n+1} \mid \mathscr{F}_n] = \mathsf{E}[\theta_{n+1} \mid \tau_n, X_{\tau_n}]. \tag{4}$$

It is interesting that the distribution of random variables $\theta_{n+1}$ does not depend on the index $n$ but it depends only on the moment of the preceding return $\tau_n$ as well as on the values of the process at the return moment. The latter property is crucial only if the cardinality of $C$ is larger than 1.

In other words, the total number of return moments does not matter; the important characteristics are the preceding return moment and the value of the process at this moment.

Therefore we consider the two parameter family of distributions

$$g_k^{(s,x)} = \mathsf{P}\{\theta_{n+1} = k \mid \tau_n = s, X_s = x\}. \tag{5}$$

Note that the analogous distributions in the paper [31] depend on a single parameter, since the set $C$ consists of a single element in [31].

Properties of the renewal sequence generated by a time-inhomogeneous Markov chain are described in Sections 2, 3 and 4 of the paper [31].

In what follows we make use of the following condition.

(A) *Existence of a stochastic majorizing sequence.* Assume that, for all $s$ and $x$, the distribution of $g_n^{(s,x)}$, $n \ge 1$ (therefore the conditional distribution of $\theta_n$, as well) is bounded from above by a sequence of nonnegative numbers $\hat{g} = (\hat{g}_k, k \ge 1)$, that is, for all $k > 0$,

$$\sum_{i \ge k} g_i^{(s,x)} \le \sum_{i \ge k} \hat{g}_i$$

and, in addition, $\sum_{k \ge 1} g_k < \infty$.

Put

$$\hat{G}_n = \sum_{k \ge n} \hat{g}_k. \tag{6}$$

The sequence $(g_k, k \ge 1)$ is called a *stochastic majorizing sequence* and usually is a probability distribution. This property however is not important for the purposes of the current paper.

The proof of the main result requires a more restrictive condition as compared to the existence of a stochastic majorizing sequence, namely it requires the finiteness of the second moment of the majorizing distribution.

(A2) Let the sequence $\theta_n$ admit the condition of the existence of a majorizing sequence (A). Assume that the second moment of the sequence $(\hat{g}_n, n \ge 1)$ is finite,

$$\hat{\mu}_2 = \sum_{k \ge 1} k^2 \hat{g}_k < \infty.$$

It is obvious that the first moment is finite under condition (A2). Denote the first moment by

$$\hat{\mu} = \sum_{k \ge 1} k g_k.$$

Let $R_t = \inf\{\tau_k > t\} - t$ be the time between a given instance $t$ and the first return to the set $C$ after $t$. Also, for every $t \ge 0$, we introduce the sequential number of the first return after $t$,

$$N(t) = \inf\{k \ge 1 \colon \tau_k > t\}.$$

We agree that $N(-1) = 0$.

*Remark* 2.1. Conditions (A) and (A2) hold for a wide class of Markov chains. In fact, condition (A2) means that the sequence of returns is square integrable. This holds, for example, in the case of an aperiodic Markov chain.

Therefore

$$R_t + t = \tau_{N(t)}.$$

We introduce the random events $A_k$, $k > 1$, meaning that a return occurs at the moment $k$,

$$A_k = \{X_k \in C\} = \{\exists m \colon \tau_m = k\}. \tag{7}$$

We also introduce the random events $B_k$ and $B_{k,n}$ meaning that the return does not happen at the moment $k$ and between the moments $k$ and $n$, respectively, that is,

$$B_k = \{X_k \notin C\} = \overline{A}_k, \tag{8}$$

$$B_{k,n} = \{X_k \notin C, \dots, X_n \notin C\} = \overline{A}_k \cap \overline{A}_{k+1} \cap \dots \overline{A}_n. \tag{9}$$

We agree that $\mathbb{1}_{B_{k,n}} = 1$ if $k > n$.

We also use the following condition.

(B) There are numbers $\gamma > 0$ and $n_0 \geq 0$ such that

$$\mathsf{P}\{B_{k,n}\} \leq (1-\gamma)^{(n-k-n_0)^+}$$

for all $n \geq k$ with $n - k \geq n_0$.

Here $x^+ = \max\{x, 0\}$.

*Remark* 2.2. Condition (B) means, in fact, that the renewal sequence is separated from zero. If $u_n$ is the probability that a renewal happens at a moment $n$, then condition (B) follows if there exist $\gamma > 0$ and $n_0 \geq 0$ such that

$$\inf_{n \geq n_0} u_n > \gamma.$$

More detail can be found in [31]. See, in particular, condition 2 of Theorem 5.1 and Lemma 8.5 in [31].

*Remark* 2.3. The assumption that the renewal sequence is separated from zero is an analogue of the renewal theorem in a certain sense for the homogeneous case. Recall that the latter means $u_n \to 1/m > 0$, whence we conclude that $u_n$ is separated from zero starting with some index $n$. It is also true that the separation condition holds for a rather wide class of time-inhomogeneous Markov chains. As mentioned above, it holds for time-homogeneous Markov chains with an integrable renewal sequence.

## 3. Main results

**Theorem 3.1.** *Let conditions (A), (A1), and (B) hold. Then the expectation of $R_t$ is such that*

$$\mathsf{E}[R_t] \leq \hat{\mu}_2 + \hat{\mu}(1/\gamma + n_0). \tag{10}$$

**Corollary 3.1.** *Put*

$$\mu_- = \inf_{n,s,x} \mathsf{E}[\theta_n \mid \tau_{n-1} = s, X_{\tau_{n-1}} = x] \geq 1$$

*and let all assumptions of Theorem* 3.1 *hold. Then the mean number of renewals during the time $t$ is such that*

$$\mathsf{E}[N(t)] \leq \hat{\mu}_2/\mu_- + \hat{\mu}(1/\gamma + n_0)/\mu_- + t + 1 \leq \hat{\mu}_2 + \hat{\mu}(1/\gamma + n_0) + t + 1.$$

## 4. Auxiliary results

**Lemma 4.1.** *The random variable $R_t$ is such that*

$$R_{t+1} = \mathbb{1}_{A_{t+1}} \theta_{N(t)+1} + \mathbb{1}_{B_{t+1}} (R_t - 1). \tag{11}$$

*Proof.* It is obvious that if a renewal happens at an instance $t+1$, then its number is $N(t)$ and the waiting time until the next renewal is equal to $\theta_{N(t)+1}$. Otherwise, if a renewal does not happen at an instance $t+1$, then the waiting time equals the renewal moment after the time $t$ minus 1. More precisely,

$$R_{t+1} = \mathbb{1}_{A_{t+1}} R_{t+1} + \mathbb{1}_{B_{t+1}} R_{t+1} = \mathbb{1}_{A_{t+1}} \theta_{N(t)+1} + \mathbb{1}_{B_{t+1}} (R_t - 1). \qquad \square$$

**Lemma 4.2.** *We have*

$$R_{t+1} \leq \mathbb{1}_{B_{t+1}} R_t + \theta_{N(t)+1}. \tag{12}$$

*Proof.* According to Lemma 4.1,

$$R_{t+1} = \mathbb{1}_{A_{t+1}} \theta_{N(t)+1} + \mathbb{1}_{B_{t+1}} (R_t - 1) = \mathbb{1}_{A_{t+1}} \theta_{N(t)+1} + \mathbb{1}_{B_{t+1}} R_t - \mathbb{1}_{B_{t+1}}.$$

Then we neglect the negative term $-\mathbb{1}_{B_{t+1}}$ and note that $\mathbb{1}_{A_{t+1}} \leq 1$:

$$R_{t+1} = \mathbb{1}_{A_{t+1}} \theta_{N(t)+1} + \mathbb{1}_{B_{t+1}} R_t - \mathbb{1}_{B_{t+1}} \leq \theta_{N(t)+1} + \mathbb{1}_{B_{t+1}} R_t. \qquad \square$$

**Lemma 4.3.** *For all $t \geq 0$,*

$$R_{t+1} \leq \sum_{k=0}^{t+1} \theta_{N(k-1)+1} \mathbb{1}_{B_{k+1,t+1}}, \tag{13}$$

*where $N(-1) = 0$ and $\mathbb{1}_{B_{t+2,t+1}} = 1$ as defined above.*

*Proof.* We follow the method of mathematical induction. First we consider the instance $t = 0$.

Note that $N(0) = 1$ and $\tau_{N(0)} = \tau_1 = \theta_1$. Moreover, $R_0 = \theta_1$. Two cases are possible, namely either a renewal occurs at the instance $t = 1$ or it does not.

If a renewal happens at the instance $t = 1$, then $\theta_1 = 1$ and $R_1 = \theta_2$.

Otherwise, if a renewal does not happen at the instance $\theta_1 = 1$, then $\theta_1 > 1$ and $R_1 = \theta_1 - 1$. Moreover, $N(0) = N(1) = 1$ in this case.

Combining the above results,

$$\begin{aligned} R_1 &= \mathbb{1}_{A_1} R_1 + \mathbb{1}_{B_1} R_1 = \mathbb{1}_{A_1} \theta_2 + \mathbb{1}_{B_1}(\theta_1 - 1) = \mathbb{1}_{A_1} \theta_{N(0)+1} + \mathbb{1}_{B_1}(\theta_1 - 1) \\ &\leq \theta_{N(0)+1} + \mathbb{1}_{B_1} \theta_1 = \theta_{N(0)+1} + \mathbb{1}_{B_1} \theta_{N(-1)+1}, \end{aligned}$$

where $N(-1) = 0$.

Now assume that inequality (13) holds for $t+1$ and prove it for $t+2$. According to inequality (12)

$$R_{t+2} \leq \mathbb{1}_{B_{t+2}} R_{t+1} + \theta_{N(t+1)+1}.$$

Then we use the assumption of the induction and obtain

$$\begin{aligned} R_{t+2} &\leq \mathbb{1}_{B_{t+2}} \sum_{k=0}^{t+1} \theta_{N(k-1)+1} \mathbb{1}_{B_{k+1,t+1}} + \theta_{N(t+1)+1} \\ &\leq \sum_{k=0}^{t+1} \theta_{N(k-1)+1} \mathbb{1}_{B_{k+1,t+1} \cap B_{t+2}} + \theta_{N(t+1)+1} = \sum_{k=0}^{t+2} \theta_{N(k-1)+1} \mathbb{1}_{B_{k+1,t+2}}. \end{aligned}$$ □

**Lemma 4.4.** *Assume that conditions (A), (A1), and (B) hold. Then*

$$\mathsf{E}[\theta_{N(k)+1} \mathbb{1}_{B_{k+1,t}}] \leq \hat{\mu}(1-\gamma)^{(t-k-1-n_0)^+}. \tag{14}$$

First we make some comments concerning inequality (14). Consider the random variable $\theta_{N(k)+1} \mathbb{1}_{B_{k+1,t}}$ for some $t \geq k+1$. The random event $B_{k+1,t}$ means that a renewal does not happen during the period between $k+1$ and $t$. This, in turn, means that a renewal next to $k$ happens after the instance $t$. In other words, $\tau_{N(k)} > t$. This means that the random event $B_{k+1,t}$ belongs to the $\sigma$-algebra $\mathscr{F}_t$, while the event $\tau_{N(k)} > t$ belongs to the $\sigma$-algebra $\mathscr{F}_{N(k)}$. On the other hand, $\theta_{N(k)+1}$ depends on the $\sigma$-algebra $\mathscr{F}_{N(k)}$ via the values $\tau_{N(k)}$ and the conditional expectation $\theta_{N(k)+1}$ is bounded from above by $\hat{\mu}$ whatever $\tau_{N(k)}$ is.

At the same time, the probability that a renewal does not happen during the period from $k+1$ to $t$ does not exceed $(1-\gamma)^{(t-k-1-n_0)^+}$ as follows from condition (B).

*Proof.* Now we can write the above reasoning more precisely:

$$\begin{aligned} \mathsf{E}[\theta_{N(k)+1} \mathbb{1}_{B_{k+1,t}}] &= \sum_{s>t} \mathsf{E}[\theta_{N(k)+1} \mathbb{1}_{B_{k+1,t}} \mathbb{1}_{\tau_{N(k)}=s}] \\ &= \sum_{j\geq 0} \sum_{s>t} \mathsf{E}[\theta_{j+1} \mathbb{1}_{B_{k+1,t}} \mathbb{1}_{\tau_j=s} \mathbb{1}_{N(k)=j}]. \end{aligned}$$

Note that $N(k)$ is a stopping time and hence $\{N(k)=j\} \in \mathscr{F}_j$, whence

$$\begin{aligned}
\mathsf{E}[\theta_{N(k)+1}\mathbb{1}_{B_{k+1,t}}] &= \sum_{j\ge 0}\sum_{s>t} \mathsf{E}[\mathsf{E}[\theta_{j+1}\mathbb{1}_{B_{k+1,t}}\mathbb{1}_{\tau_j=s}\mathbb{1}_{N(k)=j} \mid \mathscr{F}_j]] \\
&= \sum_{j\ge 0}\sum_{s>t} \mathsf{E}[\mathsf{E}[\theta_{j+1} \mid \mathscr{F}_j]\mathbb{1}_{B_{k+1,t}}\mathbb{1}_{\tau_j=s}\mathbb{1}_{N(k)=j}] \\
&= \sum_{j\ge 0}\sum_{s>t} \mathsf{E}[\mathsf{E}[\theta_{j+1} \mid \tau_j, X_{\tau_j}]\mathbb{1}_{B_{k+1,t}}\mathbb{1}_{\tau_j=s}\mathbb{1}_{N(k)=j}] \\
&= \sum_{j\ge 0}\sum_{s>t} \mathsf{E}[\mathsf{E}[\theta_{j+1} \mid \tau_j = s, X_s]\mathbb{1}_{B_{k+1,t}}\mathbb{1}_{N(k)=j}\mathbb{1}_{\tau_j=s}] \\
&\le \hat{\mu}\sum_{j\ge 0}\sum_{s>t} \mathsf{E}[\mathbb{1}_{B_{k+1,t}}\mathbb{1}_{N(k)=j}\mathbb{1}_{\tau_j=s}] = \hat{\mu}\,\mathsf{P}\{B_{k+1,t}\} \\
&\le \hat{\mu}(1-\gamma)^{(t-k-1-n_0)^+}.
\end{aligned}$$

Here we used equality (4) and the condition on the existence of a stochastic majorizing sequence. □

**Lemma 4.5.** *We have*

$$\sum_{k=0}^{t+1} \mathsf{E}[\theta_{N(k)+1}\mathbb{1}_{\theta_{N(k)+1}>t-k-2}] \le \hat{\mu}_2. \tag{15}$$

*Proof.* First

$$\begin{aligned}
\mathsf{E}[\theta_{N(k)+1}\mathbb{1}_{\theta_{N(k)+1}>t-k-2}] &= \sum_{j,s} \mathsf{E}[\theta_{j+1}\mathbb{1}_{\theta_{j+1}>t-k-2}\mathbb{1}_{\tau_j=s}\mathbb{1}_{N(k)=j}] \\
&= \sum_{j,s} \mathsf{E}[\mathsf{E}[\theta_{j+1}\mathbb{1}_{\theta_{j+1}>t-k-2} \mid \tau_j = s, X_s]\mathbb{1}_{\tau_j=s}\mathbb{1}_{N(k)=j}].
\end{aligned}$$

Then

$$\mathsf{E}[\theta_{j+1}\mathbb{1}_{\theta_{s+1}>t-k-2} \mid \tau_j = s, X_s] = \sum_{i>t-k-2} i g_i^{(s,X_s)} = \sum_{i>t-k-2} G_i^{(s,X_s)} \le \sum_{i>t-k-2} \hat{G}_i.$$

Substituting this result into the preceding equality, we obtain

$$\begin{aligned}
\mathsf{E}[\theta_{N(k)+1}\mathbb{1}_{\theta_{N(k)+1}>t-k-2}] &= \sum_{j,s} \mathsf{E}[\mathsf{E}[\theta_{j+1}\mathbb{1}_{\theta_{j+1}>t-k-2} \mid \tau_j = s, X_s]\mathbb{1}_{\tau_j=s}\mathbb{1}_{N(k)=j}] \\
&\le \sum_{i>t-k-2} \hat{G}_i.
\end{aligned}$$

Consider the sum

$$\begin{aligned}
\sum_{k=0}^{t+1} \mathsf{E}[\theta_{N(k)+1}\mathbb{1}_{\theta_{N(k)+1}>t-k-2}] &\le \sum_{k=0}^{t+1}\sum_{i>t-k-2} \hat{G}_i = \sum_{k=0}^{t-2}\sum_{j=k}^{\infty} \hat{G}_j \\
&\le \sum_{k=0}^{\infty}\sum_{j=k}^{\infty} \hat{G}_j = \sum_{k=0}^{\infty}(k+1)\hat{G}_k = \hat{\mu}_2.
\end{aligned}$$

□

## 5. Proof of the main theorem

Using Lemma 4.3 we rewrite equality (13) as follows:

$$R_{t+1} \le \sum_{k=0}^{t+1} \theta_{N(k-1)+1}\mathbb{1}_{B_{k+1,t+1}}.$$

Consider $\theta_{N(k)+1}\mathbb{1}_{B_{k+2,t}}$ for some $t \geq k+2$. It is clear that

$$\begin{aligned}\theta_{N(k)+1}\mathbb{1}_{B_{k+2,t}} &= \theta_{N(k)+1}(\mathbb{1}_{A_{k+1}} + \mathbb{1}_{B_{k+1}})\mathbb{1}_{B_{k+2,t}} \\ &= \theta_{N(k)+1}\mathbb{1}_{A_{k+1}}\mathbb{1}_{B_{k+2,t}} + \theta_{N(k)+1}\mathbb{1}_{B_{k+1,t}}.\end{aligned}$$

Now consider the random event

$$A_{k+1} \cap B_{k+2,t}.$$

This random event means that a renewal occurs at an instance $k+1$ and the next renewal does not happen until $t$. On the other hand the number of the next renewal is equal to $N(k)+1$, since $\tau_{N(k)} = k+1$. Thus

$$\theta_{N(k)+1}\mathbb{1}_{A_{k+1}\cap B_{k+2,t}} \leq \theta_{N(k)+1}\mathbb{1}_{\theta_{N(k)+1}>t-k-2}. \tag{16}$$

Now we pass to the expectation on both sides of equality (13) and substitute there expressions from both (15) and (16):

$$\begin{aligned}\mathsf{E}[R_{t+1}] &\leq \sum_{k=0}^{t+1} \mathsf{E}[\theta_{N(k-1)+1}\mathbb{1}_{B_{k+1,t+1}}] \\ &\leq \sum_{k=0}^{t+1} \mathsf{E}[\theta_{N(k-1)+1}\mathbb{1}_{A_k}\mathbb{1}_{B_{k+1,t+1}}] + \sum_{k=0}^{t+1} \mathsf{E}[\theta_{N(k-1)+1}\mathbb{1}_{B_{k,t+1}}] \\ &\leq \sum_{k=0}^{t+1} \mathsf{E}[\theta_{N(k-1)+1}\mathbb{1}_{\theta_{N(k-1)+1>t-k-2}}] + \sum_{k=0}^{t+1} \mathsf{E}[\theta_{N(k-1)+1}\mathbb{1}_{B_{k,t+1}}] \\ &\leq \hat{\mu}_2 + \sum_{k=0}^{t+1} \hat{\mu}(1-\gamma)^{(t-k+1-n_0)^+} \leq \hat{\mu}_2 + \hat{\mu} \sum_{k=-n_0}^{\infty} (1-\gamma)^{k^+} \\ &= \hat{\mu}_2 + \hat{\mu}\left(n_0 + \sum_{k=0}^{\infty}(1-\gamma)^k\right) = \hat{\mu}_2 + \hat{\mu}(1/\gamma + n_0).\end{aligned}$$

## 6. Proof of the corollary

Note that the Wald identity holds in the homogeneous case,

$$\mathsf{E}[S_{N(t)}] = \mathsf{E}[\theta_1]\,\mathsf{E}[N(t)].$$

This is not the case if the chain is time-inhomogeneous. However

$$\begin{aligned}\mathsf{E}[R_t] + t = \mathsf{E}[\tau_{N(t)}] &= \sum_{k\geq 1} E[\tau_k \mathbb{1}_{N(t)=k}] = \sum_{k\geq 1} \mathsf{E}[\tau_{k-1}\mathbb{1}_{N(t)=k}\,\mathsf{E}[\theta_k \mid \tau_{k-1}, X_{\tau_{k-1}}]] \\ &\geq \sum_{k\geq 1} \mu_-\,\mathsf{E}[\tau_{k-1}\mathbb{1}_{N(t)=k}]] \geq \sum_{k\geq 1} \mu_-\,\mathsf{E}[(k-1)\mathbb{1}_{N(t)=k}] \\ &= \mu_-(\mathsf{E}[N(t)] - 1).\end{aligned}$$

This yields

$$\begin{aligned}\mathsf{E}[N(t)] &\leq (\mathsf{E}[R_t] + t)/\mu_- + 1 \leq (\hat{\mu}_2 + \hat{\mu}(1/\gamma + n_0) + t)/\mu_- + 1 \\ &\leq \hat{\mu}_2 + \hat{\mu}(1/\gamma + n_0) + t + 1.\end{aligned}$$

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