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MINIMAX INTERPOLATION OF STOCHASTIC PROCESSES WITH STATIONARY INCREMENTS FROM OBSERVATIONS WITH NOISE

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Abstract. The problem of optimal estimation of the linear functional

$$A_T\xi = \int_0^T a(t)\xi(t)\,dt$$

depending on unknown values of the stochastic process $\xi(t)$ with stationary increments from observations of the process $\xi(t)+\eta(t)$ at points $t\in\mathbb{R}\setminus[0;T]$ is considered, where $\eta(t)$ is a stationary stochastic process uncorrelated with $\xi(t)$. Formulas for calculating the mean square error and spectral characteristic of the optimal linear estimate of the functional are proposed in the case where spectral densities are known. Otherwise relations that determine the least favorable spectral densities and minimax spectral characteristics are proposed for given sets of admissible spectral densities.

1. Introduction

The problem of estimating missing data is studied in the current paper for stochastic processes with stationary increments. A similar problem for stationary processes is considered by Kolmogorov [10], Wiener [23], and Yaglom [25, 26]. Yaglom [24] and Pinsker [19] proposed a more general model of processes with stationary increments of order n. In particular, they constructed canonical representations for processes with stationary increments and for their spectral densities, proved Wold decomposition, and studied problems of extrapolation and filtration. These and other properties of processes with stationary increments as well as of other generalizations of the notion of stationarity are described in detail by Yaglom [25, 26]. The problem of extrapolation of a nonstationary signal from observations with noise are studied by Bell [1].

The classical methods for extrapolation, interpolation, and filtration of stochastic processes use spectral densities for constructing corresponding estimates. However, the closed form of spectral densities is not available in practice but, on the other hand, a certain set of admissible densities is known. The minimax (robust) method is reasonable for estimating in this case. The minimax method is used to minimize the mean square error of the estimate for all densities of a given set. Grenander [4] was the first to apply this method for solving the problem of extrapolation of a functional of unknown values of a stationary stochastic process. Franke [6] studied the problem of the minimax extrapolation and interpolation by using the methods of convex optimization. The basic results concerning robust methods of estimation obtained prior to 1985 are described in a survey paper by Kassam and Poor [9]. Moklyachuk [16, 17] and Moklyachuk and Masyutka [18]

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describe modern results concerning the problem of minimax estimation of linear functionals of stationary processes. Similar problems for periodically correlated processes are studied by Moklyachuk and Dubovets'ka [5] (also see the book by Golichenko and Moklyachuk [3]).

Some classical and minimax problems of extrapolation, interpolation, and filtration for stochastic processes with stationary increments are considered by Luz and Moklyachuk [11]–[14].

The problem of interpolation is studied in the current paper for the functional

$$A_T\xi = \int_0^T a(t)\xi(t)\,dt$$

depending on unknown values of a stochastic process $\xi(t)$, $t \in \mathbb{R}$, with stationary increments from observations of the stochastic process $\xi(t) + \eta(t)$ for $t \in \mathbb{R} \setminus [0;T]$, where $\eta(t)$, $t \in \mathbb{R}$, is a stationary stochastic process uncorrelated with $\xi(t)$. A similar problem is considered in the paper by Luz and Moklyachuk [15] for the discrete time.

2. Stationary increments. Spectral decomposition

This section contains a brief survey of the spectral theory of stochastic processes with stationary increments of order n developed by Yaglom and Pinsker [19, 24] for the case of stochastic processes with continuous time.

Definition 2.1. The function

$$\xi^{(n)}(t,\tau) = (1 - B_\tau)^n \xi(t) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} \xi(t - l\tau) \tag{1}$$

is called a stochastic increment of order n with step $\tau \in \mathbb{R}$ constructed from a stochastic process $\xi(t)$, $t \in \mathbb{R}$, where B_τ is the shift operator with step τ, that is $B_\tau\xi(t) = \xi(t - \tau)$, $\tau \in \mathbb{R}$.

Definition 2.2. A stochastic increment $\xi^{(n)}(t,\tau)$ of order n constructed from a stochastic process $\xi(t)$, $t \in \mathbb{R}$, is called stationary (in the wide sense) if the expectations

$$\mathsf{E}\xi^{(n)}(t_0,\tau) = c^{(n)}(\tau),$$
$$\mathsf{E}\xi^{(n)}(t_0 + t,\tau_1)\xi^{(n)}(t_0,\tau_2) = D^{(n)}(t,\tau_1,\tau_2)$$

exist for all real t_0, τ, t, τ_1, and τ_2 and they do not depend on t_0. The function $c^{(n)}(\tau)$ is called the mean value of a stationary increment of order n, while $D^{(n)}(t,\tau_1,\tau_2)$ is called the structural function of a stationary increment of order n (or, the structural function of order n for the stochastic process $\xi(t)$, $t \in \mathbb{R}$).

The stochastic process $\xi(t)$, $t \in \mathbb{R}$, generating the stationary increment $\xi^{(n)}(t,\tau)$ of order n by equality (1) is called a process with stationary increments of order n.

Theorem 2.1. *The mean value $c^{(n)}(\tau)$ of a stationary stochastic increment $\xi^{(n)}(t,\tau)$ of order n and its structural function $D^{(n)}(t,\tau_1,\tau_2)$ admit the following representations:*

$$c^{(n)}(\tau) = c\tau^n, \tag{2}$$

$$D^{(n)}(t,\tau_1,\tau_2) = \int_{-\infty}^{\infty} e^{i\lambda t} \left(1 - e^{-i\tau_1\lambda}\right)^n \left(1 - e^{i\tau_2\lambda}\right)^n \frac{(1+\lambda^2)^n}{\lambda^{2n}}\, dF(\lambda), \tag{3}$$

where c is a certain constant and $F(\lambda)$ is a left continuous nondecreasing bounded function, $F(-\infty) = 0$. Moreover, the constant c and function $F(\lambda)$ are uniquely determined by the increment $\xi^{(n)}(t,\tau)$.

On the other hand, a function $c^{(n)}(\tau)$ of the form (2) with some constant c and function $D^{(n)}(t,\tau_1,\tau_2)$ of the form (3), where $F(\lambda)$ satisfies the conditions above, are the mean value and structural function of some stationary increment $\xi^{(n)}(t,\tau)$ of order n.

Using representation (3) of the structural function of a stationary increment $\xi^{(n)}(t,\tau)$ of order n together with Karhunen's theorem [8] (also see [2]) we derive the following representation of a stationary increment $\xi^{(n)}(t,\tau)$ of order n:

$$\xi^{(n)}(t,\tau)=\int_{-\infty}^{\infty} e^{it\lambda}\left(1-e^{-i\lambda\tau}\right)^n \frac{(1+i\lambda)^n}{(i\lambda)^n}\,dZ_{\xi^{(n)}}(\lambda), \tag{4}$$

where $Z_{\xi^{(n)}}(\lambda)$ is a stochastic process with orthogonal increments on $\mathbb{R}$ subordinated to the structural function $F(\lambda)$:

$$\mathsf{E}\left|Z_{\xi^{(n)}}(t_2)-Z_{\xi^{(n)}}(t_1)\right|^2=F(t_2)-F(t_1)<\infty, \qquad -\infty<t_1<t_2<\infty. \tag{5}$$

We use spectral decomposition (4) for finding an optimal linear estimate of unknown values of a stochastic process with stationary increments.

3. The classical method of interpolation

Consider a stochastic process $\xi(t)$, $t\in\mathbb{R}$, that generates a stationary stochastic increment $\xi^{(n)}(t,\tau)$ with an absolutely continuous spectral function $F(\lambda)$ whose spectral density is $f(\lambda)$. We also consider a stationary stochastic process $\eta(t)$, $t\in\mathbb{R}$, being uncorrelated with the process $\xi(t)$ and that has an absolutely continuous spectral function $G(\lambda)$ and corresponding spectral density $g(\lambda)$. Without loss of generality assume that the stochastic increment $\xi^{(n)}(t,\tau)$ and stochastic process $\eta(t)$ are centered, $\mathsf{E}\xi^{(n)}(t,\tau)=0$ and $\mathsf{E}\eta(t)=0$, and that the step is positive, $\tau>0$.

The classical problem of interpolation is to find the optimal mean square linear estimate of the functional

$$A_T\xi=\int_0^T a(t)\xi(t)\,dt$$

of unknown values of the process $\xi(t)$ by using the observations of the process $\xi(t)+\eta(t)$ for $t\in\mathbb{R}\setminus[0;T]$. When solving this problem, we suppose that the spectral densities $f(\lambda)$ and $g(\lambda)$ satisfy the following condition of minimality:

$$\int_{-\infty}^{\infty}\frac{|\gamma(\lambda)|^2\lambda^{2n}}{\left|1-e^{i\lambda\tau}\right|^{2n}(1+\lambda^2)^n\left(f(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda)\right)}\,d\lambda<\infty \tag{6}$$

for some nonzero function $\gamma(\lambda)$ of the exponential type. The latter means that the function is written as follows:

$$\gamma(\lambda)=\int_0^{T+\tau n}\alpha(t)e^{i\lambda t}\,dt.$$

The condition of minimality is necessary and sufficient to the condition that there is no error free estimate of the functional $A_T\xi$ in the problem of interpolation; see [22].

We solve the problem of interpolation by using the orthogonal projection method in a Hilbert space proposed by Kolmogorov [10]. Consider the Hilbert space $H=L_2(\Omega,\mathcal{F},\mathsf{P})$ of random variables with zero expectation and finite variance. The inner product of two elements $\gamma_1,\gamma_2\in H$ is defined as $(\gamma_1;\gamma_2)=\mathsf{E}\gamma_1\overline{\gamma_2}$.

The orthogonal projection method consists of the following three steps: Step 1 is to determine a subspace $H^0\subset H$ generated by known observations of the process; Step 2 is to describe an element $\gamma\in H$ for which an estimate is to be found; Step 3 is to find the orthogonal projection of the element $\gamma\in H$ onto the subspace H^0.

Step 1. Denote by $H^{0-}\big(\xi_\tau^{(n)}+\eta_\tau^{(n)}\big)$ the closed linear subspace of the space $H=L_2(\Omega,\mathcal{F},\mathsf{P})$ generated by increments

$$\left\{\xi^{(n)}(t,\tau)+\eta^{(n)}(t,\tau)\colon t<0\right\},$$

and by $H^{T+}\big(\xi_{-\tau}^{(n)}+\eta_{-\tau}^{(n)}\big)$ the closed linear subspace in the space H generated by the increments

$$\left\{\xi^{(n)}(t,-\tau)+\eta^{(n)}(t,-\tau)\colon t>T\right\}.$$

Since

$$\xi^{(n)}(t,-\tau)+\eta^{(n)}(t,-\tau)=(-1)^n\left(\xi^{(n)}(t+\tau n,\tau)+\eta^{(n)}(t+\tau n,\tau)\right),$$

we have

$$H^{T+}\left(\xi_{-\tau}^{(n)}+\eta_{-\tau}^{(n)}\right)=H^{(T+\tau n)+}\left(\xi_{\tau}^{(n)}+\eta_{\tau}^{(n)}\right).$$

Then

$$H^0=H^{0-}\left(\xi_{\tau}^{(n)}+\eta_{\tau}^{(n)}\right)\oplus H^{(T+\tau n)+}\left(\xi_{\tau}^{(n)}+\eta_{\tau}^{(n)}\right).$$

We further introduce two subspaces $L_2^{0-}(p)$ and $L_2^{T+}(p)$ in the Hilbert space $L_2(p)$ of square integrable functions defined on the real axis $\mathbb{R}$ and generated by the families of functions

$$\left\{e^{i\lambda t}(1-e^{-i\lambda\tau})^n\frac{(1+i\lambda)^n}{(i\lambda)^n}\colon t<0\right\}\quad\text{and}\quad\left\{e^{i\lambda t}(1-e^{-i\lambda\tau})^n\frac{(1+i\lambda)^n}{(i\lambda)^n}\colon t>T\right\},$$

respectively, where

$$p(\lambda)=f(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda)$$

is the spectral density of the stochastic process $\zeta(t)=\xi(t)+\eta(t)$; see [13].

Step 2. Let $q_\tau(t)=\big[-\frac{t}{\tau}\big]$, $q'_\tau(t)=\big[-\frac{t}{\tau}\big]'$, $r_\tau(t,T)=\big[\frac{T-t}{\tau}\big]$, and $r'_\tau(t,T)=\big[\frac{T-t}{\tau}\big]'$, where $[x]'$ denotes the least integer number that is greater than or equal to x. The functional $A_T\xi$ admits the following representation:

$$A_T\xi=A_T\zeta-A_T\eta=B_T\zeta-A_T\eta-V_T\zeta=H_T\xi-V_T\zeta,\qquad H_T\xi:=B_T\zeta-A_T\eta,$$

$$A_T\zeta=\int_0^T a(t)\zeta(t)dt,\quad A_T\eta=\int_0^T a(t)\eta(t)\,dt,$$

$$B_T\zeta=\int_0^T b_{\tau,T}(t)\zeta^{(n)}(t,\tau)\,dt,\qquad V_T\zeta=\int_{-\tau n}^0 v_{\tau,T}(t)\zeta(t)\,dt,$$

where the functions $v_{\tau,T}(t)$, $t\in[-\tau n;0)$, and $b_{\tau,T}(t)$, $t\in[0;T]$, are calculated according to

$$v_{\tau,T}(t)=\sum_{l=q'_\tau(t)}^{\min\{r_\tau(t,T),n\}}(-1)^l\binom{n}{l}b_{\tau,T}(t+l\tau),\qquad t\in[-\tau n;0)\tag{7}$$

and

$$b_{\tau,T}(t)=\sum_{k=0}^{r_\tau(t,T)}a(t+\tau k)d(k)=D_T^\tau\mathbf{a}(t),\qquad t\in[0;T],\tag{8}$$

respectively; see [11, 12]. The coefficients $\{d(k)\colon k\geq 0\}$ in (8) are defined from

$$\sum_{k=0}^{\infty}d(k)x^k=\left(\sum_{j=0}^{\infty}x^j\right)^n,$$

where D^{τ}_T is a linear transform acting at an arbitrary function $x(t)$, $t\in[0;T]$, as follows:

$$D^{\tau}_T\mathbf{x}(t)=\sum_{k=0}^{r_\tau(t,T)}x(t+\tau k)d(k).$$

The functional $H_T\xi$ has a finite second moment. This allows one to apply the methods of convex optimization in Hilbert spaces. Denote by $\widehat{A}_T\xi$ the optimal in the mean square sense linear estimate of the functional $A_T\xi$ constructed from known observations of the stochastic process $\xi(t)+\eta(t)$ for $t\in\mathbb{R}\setminus[0;T]$. Also let $\widehat{H}_T\xi$ be the optimal in the mean square sense linear estimate of the functional $H_T\xi$ constructed from observations of the stochastic increment $\xi^{(n)}(t,\tau)+\eta^{(n)}(t,\tau)$ of order n for $t\in\mathbb{R}\setminus[0;T+\tau n]$. Denote by $\Delta(f,g;\widehat{A}_T\xi)=\mathsf{E}|A_T\xi-\widehat{A}_T\xi|^2$ and $\Delta(f,g;\widehat{H}_T\xi)=\mathsf{E}|H_T\xi-\widehat{H}_T\xi|^2$ the mean square errors of the estimates $\widehat{A}_T\xi$ and $\widehat{H}_T\xi$, respectively. Since the functional $V_T\zeta$ depends on known observations $\zeta(t)$, $-\tau n\leq t<0$, we have

$$\widehat{A}_T\xi=\widehat{H}_T\xi-V_T\zeta, \tag{9}$$

$$\begin{aligned}\Delta\left(f,g;\widehat{A}_T\xi\right)&=\mathsf{E}\left|A_T\xi-\widehat{A}_T\xi\right|^2=\mathsf{E}\left|H_T\xi-V_T\zeta-\widehat{H}_T\xi+V_T\zeta\right|^2\\&=\mathsf{E}\left|H_T\xi-\widehat{H}_T\xi\right|^2=\Delta\left(f,g;\widehat{H}_T\xi\right).\end{aligned}$$

Hence it is sufficient to construct an estimate of the functional $H_T\xi$ in order to construct the optimal estimate of the functional $A_T\xi$. Then the linear estimate $\widehat{A}_T\xi$ of the functional $A_T\xi$ is such that

$$\widehat{A}_T\xi=\int_{-\infty}^{\infty}h_\tau(\lambda)dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda)-\int_{-\tau n}^{0}v_{\tau,T}(t)(\xi(t)+\eta(t))\,dt, \tag{10}$$

where $h_\tau(\lambda)$ is the spectral characteristic of the estimate $\widehat{H}_T\xi$.

Note that the functional $H_T\xi$ admits the spectral representation

$$H_T\xi=\int_{-\infty}^{\infty}B^{\tau}_T(\lambda)\left(1-e^{-i\lambda\tau}\right)^n\frac{(1+i\lambda)^n}{(i\lambda)^n}\,dZ_{\xi^{(n)}+\eta^{(n)}}(\lambda)-\int_{-\infty}^{\infty}A_T(\lambda)\,dZ_\eta(\lambda),$$

where

$$B^{\tau}_T(\lambda)=\int_0^T b_{\tau,T}(t)e^{i\lambda t}\,dt=\int_0^T D^{\tau}_T\mathbf{a}(t)e^{i\lambda t}\,dt,\qquad A_T(\lambda)\,dt=\int_0^T a(t)e^{i\lambda t}\,dt.$$

In addition,

$$dZ_{\eta^{(n)}}(\lambda)=(i\lambda)^n(1+i\lambda)^{-n}\,dZ_\eta(\lambda)$$

for $\lambda\in\mathbb{R}$; see [13].

Step 3. The optimal estimate $\widehat{H}_T\xi$ is a projection of the element $H_T\xi$ of the space H onto the subspace $H^{0-}\big(\xi^{(n)}_\tau+\eta^{(n)}_\tau\big)\oplus H^{(T+\tau n)+}\big(\xi^{(n)}_\tau+\eta^{(n)}_\tau\big)$. This projection is characterized by the following two conditions:

1) $\widehat{H}_T\xi\in H^{0-}\big(\xi^{(n)}_\tau+\eta^{(n)}_\tau\big)\oplus H^{(T+\tau n)+}\big(\xi^{(n)}_\tau+\eta^{(n)}_\tau\big)$;
2) $(H_T\xi-\widehat{H}_T\xi)\perp H^{0-}\big(\xi^{(n)}_\tau+\eta^{(n)}_\tau\big)\oplus H^{(T+\tau n)+}\big(\xi^{(n)}_\tau+\eta^{(n)}_\tau\big)$.

Using condition 2) above we get

$$\begin{aligned}\int_{-\infty}^{\infty}\left[\left(B_\tau(\lambda)\,(1-e^{-i\lambda\tau})^n-\frac{(i\lambda)^n h_\tau(\lambda)}{(1+i\lambda)^n}\right)\frac{(1+\lambda^2)^n}{\lambda^{2n}}p(\lambda)\;-A(e^{i\lambda})g(\lambda)\right]&\\ \times e^{-i\lambda t}\left(1-e^{i\lambda\tau}\right)^n\,d\lambda=0&\end{aligned}$$

for all $t \in \mathbb{R} \setminus [0; T+\tau n]$, whence we derive the general form of the spectral characteristic $h_\tau(\lambda)$:

$$\begin{aligned} h_\tau(\lambda) &= B_T^\tau(\lambda)\frac{(1-e^{-i\lambda\tau})^n(1+i\lambda)^n}{(i\lambda)^n} - A_T(\lambda)\frac{(-i\lambda)^n g(\lambda)}{(1-i\lambda)^n p(\lambda)} \\ &\quad - \frac{(-i\lambda)^n C_T^\tau(\lambda)}{(1-e^{i\lambda\tau})^n(1-i\lambda)^n p(\lambda)}, \\ C_T^\tau(\lambda) &= \int_0^{T+\tau n} \mathbf{c}_\tau(t) e^{i\lambda t}\, dt, \end{aligned}$$

where $\mathbf{c}_\tau(t)$, $t \in [0; T+\tau n]$ is an unknown function to be found. Condition 1) above implies

$$\begin{aligned} \int_{-\infty}^{\infty} \Bigg[B_T^\tau(\lambda) - \frac{A_T(\lambda)\left(1-e^{-i\lambda\tau}\right)^{-n}\lambda^{2n} g(\lambda)}{(1+\lambda^2)^n p(\lambda)} \\ - \frac{\left|1-e^{i\lambda\tau}\right|^{-2n}\lambda^{2n} C_T^\tau(\lambda)}{(1+\lambda^2)^n p(\lambda)} \Bigg] e^{-i\lambda s}\, d\lambda = 0 \end{aligned} \tag{11}$$

for all $s \in [0; T+\tau n]$.

The following equation is derived from relation (11):

$$[D_T^\tau \mathbf{a}]_{+\tau n}(s) - (\mathbf{T}_T^\tau \mathbf{a}_{\tau,T})(s) = (\mathbf{P}_T^\tau \mathbf{c}_\tau)(s), \qquad s \in [0; T+\tau n], \tag{12}$$

where $[D_T^\tau \mathbf{a}]_{+\tau n}(t) = D_T^\tau \mathbf{a}(t)$ for $t \in [0; T]$ and $[D_T^\tau \mathbf{a}]_{+\tau n}(t) = 0$ for $t \in (T; T+\tau n]$, $\mathbf{a}(t) = 0$ for $t > T$, the function $\mathbf{a}_{\tau,T}(t)$, $t \in [0; T+\tau n]$ is evaluated from

$$\mathbf{a}_{\tau,T}(t) = \sum_{l=\max\{r'_\tau(t,T),0\}}^{\min\{q_\tau(t),n\}} (-1)^l \binom{n}{l} a(t-\tau l), \qquad 0 \le t \le T+\tau n; \tag{13}$$

see [14]. The linear operators $\mathbf{T}_T^\tau$ and $\mathbf{P}_T^\tau$ acting in the space $L_2[0; T+\tau n]$ are such that

$$(\mathbf{T}_T^\tau \mathbf{x})(s) = \frac{1}{2\pi}\int_0^{T+\tau n} \mathbf{x}(t) \int_{-\infty}^{\infty} \frac{e^{i\lambda(t-s)}\lambda^{2n} g(\lambda)}{|1-e^{i\lambda\tau}|^{2n}(1+\lambda^2)^n p(\lambda)}\, d\lambda\, dt, \qquad s \in [0; T+\tau n],$$

$$(\mathbf{P}_T^\tau \mathbf{y})(s) = \frac{1}{2\pi}\int_0^{T+\tau n} \mathbf{y}(t) \int_{-\infty}^{\infty} \frac{e^{i\lambda(t-s)}\lambda^{2n}}{|1-e^{i\lambda\tau}|^{2n}(1+\lambda^2)^n p(\lambda)}\, d\lambda\, dt, \qquad s \in [0; T+\tau n].$$

Now the unknown function $\mathbf{c}_\tau(t)$, $t \in [0; T+\tau n]$, is obtained from

$$\mathbf{c}_\tau(t) = \left((\mathbf{P}_T^\tau)^{-1}[D_T^\tau \mathbf{a}]_{+\tau n} - (\mathbf{P}_T^\tau)^{-1}\mathbf{T}_T^\tau \mathbf{a}_{\tau,T} \right)(t).$$

Thus the spectral characteristic $h_\tau(\lambda)$ of the optimal estimate $\widehat{H}_T\xi$ of the functional $H_T\xi$ can be evaluated from

$$\begin{aligned} h_\tau(\lambda) &= B_T^\tau(\lambda)\frac{(1-e^{-i\lambda\tau})^n(1+i\lambda)^n}{(i\lambda)^n} - \frac{A_T(\lambda)(1+i\lambda)^n(-i\lambda)^n g(\lambda)}{(1+\lambda^2)^n f(\lambda) + \lambda^{2n} g(\lambda)} \\ &\quad - \frac{(1+i\lambda)^n(-i\lambda)^n C_T^\tau(\lambda)}{(1-e^{i\lambda\tau})^n\left((1+\lambda^2)^n f(\lambda) + \lambda^{2n} g(\lambda)\right)}, \\ C_T^\tau(\lambda) &= \int_0^{T+\tau n} \left((\mathbf{P}_T^\tau)^{-1}[D_T^\tau \mathbf{a}]_{+\tau n} - (\mathbf{P}_T^\tau)^{-1}\mathbf{T}_T^\tau \mathbf{a}_{\tau,T} \right)(t) e^{i\lambda t}\, dt. \end{aligned} \tag{14}$$

The mean square error of the estimate $\widehat{A}_T\xi$ is such that

$$\begin{aligned}
\Delta\left(f,g;\widehat{A}_T\xi\right) &= \Delta\left(f,g;\widehat{H}_T\xi\right) = \mathsf{E}\left|H_T\xi-\widehat{H}_T\xi\right|^2 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|A_T(\lambda)(1-e^{i\lambda\tau})^n(1+\lambda^2)^nf(\lambda)-\lambda^{2n}C_T^{\tau}(\lambda)\right|^2}{\left|1-e^{i\lambda\tau}\right|^{2n}(1+\lambda^2)^{2n}(f(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda))^2}g(\lambda)d\lambda \\
&+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|A_T(\lambda)(1-e^{i\lambda\tau})^n(-i\lambda)^ng(\lambda)+(-i\lambda)^nC_T^{\tau}(\lambda)\right|^2}{\left|1-e^{i\lambda\tau}\right|^{2n}(1+\lambda^2)^n\left(f(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g(\lambda)\right)^2}f(\lambda)d\lambda \\
&= \left\langle[D_T^{\tau}\mathbf{a}]_{+\tau n}-\mathbf{T}_T^{\tau}\mathbf{a}_{\tau,T},(\mathbf{P}_T^{\tau})^{-1}[D_T^{\tau}\mathbf{a}]_{+\tau n}-(\mathbf{P}_T^{\tau})^{-1}\mathbf{T}_T^{\tau}\mathbf{a}_{\tau,T}\right\rangle+\langle\mathbf{Q}_T\mathbf{a},\mathbf{a}\rangle,
\end{aligned}$$

where the linear operator $\mathbf{Q}_T$ acts in the space $L_2[0;T]$ as follows:

$$(\mathbf{Q}_T\mathbf{z})(s)=\frac{1}{2\pi}\int_0^T\mathbf{z}(t)\int_{-\infty}^{\infty}e^{i\lambda(t-s)}\frac{f(\lambda)g(\lambda)}{p(\lambda)}\,d\lambda\,dt,\qquad s\in[0;T].$$

Now we justify the existence of the inverse operator $(\mathbf{P}_T^{\tau})^{-1}$. Instead of equation (12) we consider the equation

$$\widetilde{\mathbf{b}}_{\tau}(t)=(\mathbf{P}_T^{\tau}\mathbf{c}_{\tau})(t),\qquad t\in[0;T+\tau n]. \tag{15}$$

This leads to the problem of constructing the projection of the element

$$B_{T+\tau n}\eta=\int_0^{T+\tau n}\widetilde{\mathbf{b}}_{\tau}(t)\eta^{(n)}(t,\tau)\,dt$$

of the space H onto the subspace

$$H^{0-}\left(\xi_{\tau}^{(n)}+\eta_{\tau}^{(n)}\right)\oplus H^{(T+\tau n)+}\left(\xi_{\tau}^{(n)}+\eta_{\tau}^{(n)}\right).$$

Since the subspace $H^{0-}\big(\xi_{\tau}^{(n)}+\eta_{\tau}^{(n)}\big)\oplus H^{(T+\tau n)+}\big(\xi_{\tau}^{(n)}+\eta_{\tau}^{(n)}\big)$ is closed and convex, the projection is uniquely determined for an arbitrary function $\widetilde{\mathbf{b}}_{\tau}(t)$ which differs from zero on the interval $[0;T+\tau n]$. This means that system (15) possesses a unique solution for an arbitrary function $\widetilde{\mathbf{b}}_{\tau}(t)$, whence we deduce that the inverse operator $(\mathbf{P}_T^{\tau})^{-1}$ for $\mathbf{P}_T^{\tau}$ exists.

The reasoning above implies the following result.

Theorem 3.1. *Let $\xi(t)$, $t\in\mathbb{R}$, be a stochastic process that generates the stationary increment $\xi^{(n)}(t,\tau)$ of order n and let $\eta(t)$, $t\in\mathbb{R}$, be a stationary stochastic process uncorrelated with $\xi(t)$. Assume that the spectral densities $f(\lambda)$ and $g(\lambda)$ of stochastic processes $\xi(t)$ and $\eta(t)$ satisfy condition of minimality* (6)*. Then the optimal linear estimate $\widehat{A}_T\xi$ of the functional $A_T\xi$ constructed from observations of the process $\xi(t)+\eta(t)$ for $t\in\mathbb{R}\backslash[0;T]$ is given by* (10)*. The spectral characteristic $h_{\tau}(\lambda)$ and mean square error $\Delta(f,g;\widehat{A}_T\xi)$ of the optimal estimate $\widehat{A}_T\xi$ are evaluated by* (14) *and* (3)*, respectively.*

Remark 3.1. The spectral characteristic $h_{\tau}(\lambda)$ of the optimal estimate $\widehat{A}_T\xi$ can be written as follows:

$$h_{\tau}(\lambda)=h_{\tau}^1(\lambda)-h_{\tau}^2(\lambda),$$

where

$$h^1_\tau(\lambda)=B^\tau_T(\lambda)\frac{\left(1-e^{-i\lambda\tau}\right)^n(1+i\lambda)^n}{(i\lambda)^n} \tag{16}$$
$$-\frac{(1+i\lambda)^n(-i\lambda)^n\int_0^{T+\tau n}\left((\mathbf{P}^\tau_T)^{-1}[D^\tau_T\mathbf{a}]_{+\tau n}\right)(t)e^{i\lambda t}\,dt}{\left(1-e^{i\lambda\tau}\right)^n\left((1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)\right)},$$

$$h^2_\tau(\lambda)=-\frac{A_T(\lambda)(1+i\lambda)^n(-i\lambda)^n g(\lambda)}{(1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)} \tag{17}$$
$$-\frac{(1+i\lambda)^n(-i\lambda)^n\int_0^{T+\tau n}((\mathbf{P}^\tau_T)^{-1}\mathbf{T}^\tau_T\mathbf{a}_{\tau,T})(t)e^{i\lambda t}dt}{(1-e^{i\lambda\tau})^n\left((1+\lambda^2)^n f(\lambda)+\lambda^{2n}g(\lambda)\right)}.$$

The functions $h^1_\tau(\lambda)$ and $h^2_\tau(\lambda)$ are spectral characteristics of the optimal estimates $\widehat{B}_T\zeta$ and $\widehat{A}_T\eta$ of the functionals $B_T\zeta$ and $A_T\eta$, respectively, constructed from observations of the process $\xi(t)+\eta(t)$ for $t\in\mathbb{R}\setminus[0;T]$.

Consider the problem of interpolation of the linear functional $A_T\xi$ from observations of the stochastic process $\xi(t)$ without noise for $t\in\mathbb{R}\setminus[0;T]$. The linear estimate $\widehat{A}_T\xi$ of the functional $A_T\xi$ is represented as follows:

$$\widehat{A}_T\xi=\int_{-\infty}^{\infty}h^\xi_\tau(\lambda)\,dZ_{\xi^{(n)}}(\lambda)-\int_{-\tau n}^{0}v_{\tau,T}(t)\xi(t)\,dt, \tag{18}$$

where the spectral characteristic of the estimate is given by

$$h^\xi_\tau(\lambda)=B^\tau_T(\lambda)\frac{(1-e^{-i\lambda\tau})^n(1+i\lambda)^n}{(i\lambda)^n} \tag{19}$$
$$-\frac{(-i\lambda)^n\int_0^{T+\tau n}((\mathbf{F}^\tau_T)^{-1}[D^\tau_T\mathbf{a}]_{+\tau n})(t)e^{i\lambda t}\,dt}{(1-i\lambda)^n(1-e^{i\lambda\tau})^n f(\lambda)}.$$

For an arbitrary function $\mathbf{x}(t)\in L_2[0;T+\tau n]$, the linear operator $\mathbf{F}^\tau_T$ is defined by the following relation:

$$(\mathbf{F}^\tau_T\mathbf{x})(s)=\frac{1}{2\pi}\int_0^{T+\tau n}\mathbf{x}(t)\int_{-\infty}^{\infty}\frac{e^{i\lambda(t-s)}\lambda^{2n}}{\left|1-e^{i\lambda\tau}\right|^{2n}\left(1+\lambda^2\right)^n f(\lambda)}\,d\lambda\,dt,\qquad s\in[0;T+\tau n].$$

The mean square error of the estimate $\widehat{A}_T\xi$ is given by

$$\Delta\left(f;\widehat{A}_T\xi\right)=\Delta\left(f;\widehat{B}_T\xi\right)=\mathsf{E}\left|B_T\xi-\widehat{B}_T\xi\right|^2 \tag{20}$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\lambda^{2n}\left|\int_0^{T+\tau n}((\mathbf{F}^\tau_T)^{-1}[D^\tau_T\mathbf{a}]_{+\tau n})(t)e^{i\lambda t}\,dt\right|^2}{|1-e^{i\lambda\tau}|^{2n}(1+\lambda^2)^n f(\lambda)}\,d\lambda$$
$$=\left\langle[D^\tau_T\mathbf{a}]_{+\tau n},(\mathbf{F}^\tau_T)^{-1}[D^\tau_T\mathbf{a}]_{+\tau n}\right\rangle.$$

The reasoning above leads to the following result.

Theorem 3.2. *Let $\xi(t)$, $t\in\mathbb{R}$, be a stochastic process that generates the stationary increment $\xi^{(n)}(t,\tau)$ of order n. Assume that the spectral density $f(\lambda)$ of the stochastic process $\xi(t)$ satisfies the condition of minimality:*

$$\int_{-\infty}^{+\infty}\frac{|\gamma(\lambda)|^2\lambda^{2n}}{|1-e^{i\lambda\tau}|^{2n}(1+\lambda^2)^n f(\lambda)}\,d\lambda<\infty. \tag{21}$$

Then the optimal linear estimate $\widehat{A}_T\xi$ of the functional $A_T\xi$ constructed from observations of the process $\xi(t)$ for $t \in \mathbb{R} \setminus [0;T]$ is calculated by (18). *The spectral characteristic $h^{\xi}_{\tau}(\lambda)$ and mean square error $\Delta(f;\widehat{A}_T\xi)$ of the optimal estimate $\widehat{A}_T\xi$ are given by* (19) *and* (20), *respectively.*

4. The minimax (robust) method of interpolation

The mean square error $\Delta(h_\tau(f,g);f,g) := \Delta(f,g;\widehat{A}_T\xi)$ and spectral characteristic $h_\tau(f,g) := h_\tau(\lambda)$ of the optimal linear estimate $\widehat{A}_T\xi$ of the functional $A_T\xi$ depending on unknown values of a stochastic process $\xi(t)$ with stationary increments of order n are constructed from observations of the process $\xi(t)+\eta(t)$ (or process $\xi(t)$) by using the relations obtained in Section 3 if the spectral densities $f(\lambda)$ and $g(\lambda)$ of the stochastic processes $\xi(t)$ and $\eta(t)$ are known. If the densities are unknown, but a set $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$ of admissible spectral densities is specified instead, then one can apply the minimax approach to solving problems of estimation of the functional. According to this approach, one chooses an estimate that minimizes the mean square error for all pairs of spectral densities of the class $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$.

Definition 4.1. For a given a class $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$, a pair of densities $f^0(\lambda)\in\mathcal{D}_f$ and $g^0(\lambda)\in\mathcal{D}_g$ is called the least favorable in the class $\mathcal{D}$ for the optimal linear interpolation of the functional $A_T\xi$ if

$$\Delta\left(f^0,g^0\right)=\Delta\left(h\left(f^0,g^0\right);f^0,g^0\right)=\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta(h(f,g);f,g).$$

Definition 4.2. For a given a class of spectral densities $\mathcal{D}=\mathcal{D}_f\times\mathcal{D}_g$, a spectral characteristic $h^0(\lambda)$ of the optimal estimate of the functional $A_T\xi$ is called minimax (robust) if

$$h^0(\lambda)\in H_{\mathcal{D}}=\bigcap_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}L_2^{0-}(p)\oplus L_2^{(T+\tau n)+}(p),$$

$$\min_{h\in H_{\mathcal{D}}}\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta(h;f,g)=\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}\Delta(h^0;f,g).$$

According to the above definitions of least favorable densities and minimax spectral characteristic, Theorems 3.1 and 3.2 imply the following results.

Lemma 4.1. *Spectral densities $f^0\in\mathcal{D}_f$ and $g^0\in\mathcal{D}_g$ that satisfy the condition of minimality* (6) *are the least favorable in the class $\mathcal{D}$ for the optimal linear interpolation of the functional $A_T\xi$ from observations of the process $\xi(t)+\eta(t)$ for $t\in\mathbb{R}\setminus[0;T]$ if the operators $(\mathbf{P}_T^\tau)^0$, $(\mathbf{T}_T^\tau)^0$, and $\mathbf{Q}_T^0$, constructed with the help of the Fourier coefficients of the functions*

$$\frac{\lambda^{2n}\left|1-e^{i\lambda\tau}\right|^{-2n}}{(1+\lambda^2)^n p^0(\lambda)},\qquad \frac{\lambda^{2n}g^0(\lambda)\left|1-e^{i\lambda\tau}\right|^{-2n}}{(1+\lambda^2)^n p^0(\lambda)},\qquad \frac{f^0(\lambda)g^0(\lambda)}{p^0(\lambda)},\tag{22}$$
$$p^0(\lambda)=f^0(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g^0(\lambda),$$

determine a solution of the following constrained optimization problem:

$$\begin{aligned}\max_{(f,g)\in\mathcal{D}_f\times\mathcal{D}_g}&\Big(\big\langle[D_T^\tau\mathbf{a}]_{+\tau n}-\mathbf{T}_T^\tau\mathbf{a}_\tau,(\mathbf{P}_T^\tau)^{-1}[D_T^\tau\mathbf{a}]_{+\tau n}-(\mathbf{P}_T^\tau)^{-1}\mathbf{T}_T^\tau\mathbf{a}_{\tau,T}\big\rangle\\&\qquad\qquad+\langle\mathbf{Q}_T\mathbf{a},\mathbf{a}\rangle\Big)\\&=\big\langle[D_T^\tau\mathbf{a}]_{+\tau n}-(\mathbf{T}_T^\tau)^0\mathbf{a}_\tau,((\mathbf{P}_T^\tau)^0)^{-1}[D_T^\tau\mathbf{a}]_{+\tau n}-((\mathbf{P}_T^\tau)^0)^{-1}(\mathbf{T}_T^\tau)^0\mathbf{a}_{\tau,T}\big\rangle\\&\qquad\qquad+\langle\mathbf{Q}_T^0\mathbf{a},\mathbf{a}\rangle.\end{aligned}\tag{23}$$

The minimax spectral characteristic $h^0 = h_\tau(f^0, g^0)$ *is evaluated by equality* (14) *if* $h_\tau(f^0, g^0) \in H_{\mathcal{D}}$.

Corollary 4.1. *A spectral density* $f^0 \in \mathcal{D}_f$ *that satisfies the condition of minimality* (21) *is the least favorable among densities of the class* $\mathcal{D}_f$ *for the optimal linear interpolation of the functional* $A_T\xi$ *from observations of the process* $\xi(t)$ *for* $t \in \mathbb{R}\setminus[0;T]$ *if the operator* $(\mathbf{F}_T^\tau)^0$ *constructed with the help of the Fourier transform of the function*

$$\lambda^{2n}|1-e^{i\lambda\tau}|^{-2n}(1+\lambda^2)^{-n}(f^0(\lambda))^{-1}$$

determines a solution of the following constrained optimization problem

$$\max_{f\in\mathcal{D}_f}\left\langle (\mathbf{F}_T^\tau)^{-1}\,[D_T^\tau\mathbf{a}]_{+\tau n}, [D_T^\tau\mathbf{a}]_{+\tau n}\right\rangle = \left\langle \left((\mathbf{F}_T^\tau)^0\right)^{-1}[D_T^\tau\mathbf{a}]_{+\tau n}, [D_T^\tau\mathbf{a}]_{+\tau n}\right\rangle. \tag{24}$$

The minimax spectral characteristic $h^0 = h_\tau^\xi(f^0)$ *is evaluated by equality* (19) *if*

$$h_\tau^\xi(f^0) \in H_{\mathcal{D}}.$$

The minimax (robust) spectral characteristic h^0 and a pair of the least favorable spectral densities (f^0, g^0) form a saddle point of the function $\Delta(h; f, g)$ in the set $H_{\mathcal{D}}\times\mathcal{D}$. The saddle point inequalities

$$\Delta\left(h; f^0, g^0\right) \geq \Delta\left(h^0; f^0, g^0\right) \geq \Delta\left(h^0; f, g\right) \qquad \forall f\in\mathcal{D}_f, \forall g\in\mathcal{D}_g, \forall h\in H_{\mathcal{D}}$$

hold if $h^0 = h_\tau(f^0, g^0)$, $h_\tau(f^0, g^0)\in H_{\mathcal{D}}$, and a pair (f^0, g^0) determines a solution of the following constrained optimization problem:

$$\widetilde{\Delta}(f,g) = -\Delta\left(h_\tau\left(f^0,g^0\right); f, g\right) \to \inf, \qquad (f,g)\in\mathcal{D},$$

where

$$\begin{aligned}
&\Delta\left(h_\tau(f^0,g^0); f, g\right)\\
&\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|A_T(\lambda)\left(1-e^{i\lambda\tau}\right)^n\left(1+\lambda^2\right)^n f^0(\lambda) - \lambda^{2n}C^0_{\tau,T}(\lambda)\right|^2}{|1-e^{i\lambda\tau}|^{2n}\,(1+\lambda^2)^{2n}\left(f^0(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g^0(\lambda)\right)^2}g(\lambda)\,d\lambda\\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|A_T(\lambda)(1-e^{i\lambda\tau})^n(-i\lambda)^n g^0(\lambda) + (-i\lambda)^n C^0_{\tau,T}(\lambda)\right|^2}{|1-e^{i\lambda\tau}|^{2n}\,(1+\lambda^2)^n\left(f^0(\lambda)+\frac{\lambda^{2n}}{(1+\lambda^2)^n}g^0(\lambda)\right)^2}f(\lambda)\,d\lambda,
\end{aligned}$$

$$C^0_{\tau,T}\left(e^{i\lambda}\right) = \int_0^{T+\tau n}\left(\left((\mathbf{P}_T^\tau)^0\right)^{-1}[D_T^\tau\mathbf{a}]_{+\tau n} - \left((\mathbf{P}_T^\tau)^0\right)^{-1}(\mathbf{T}_T^\tau)^0\mathbf{a}_{\tau,T}\right)(t)e^{i\lambda t}\,dt.$$

The above constrained optimization problem is equivalent to a usual unconstrained optimization problem

$$\Delta_{\mathcal{D}}(f,g) = \widetilde{\Delta}(f,g) + \delta(f,g|\mathcal{D}_f\times\mathcal{D}_g)\to\inf,$$

where $\delta(f,g|\mathcal{D}_f\times\mathcal{D}_g)$ is the indicator function for the set $\mathcal{D}_f\times\mathcal{D}_g$. A solution (f^0,g^0) of the latter problem is characterized by the condition $0\in\partial\Delta_{\mathcal{D}}(f^0,g^0)$ which is a necessary and sufficient condition for the pair (f^0,g^0) to belong to the set of minimum of the convex functional $\Delta_{\mathcal{D}}(f,g)$; see [6, 16, 20, 21].

5. The least favorable spectral densities in the class $\mathcal{D}^0_{1/f} \times \mathcal{D}^0_{1/g}$

Consider the problem of minimax estimation of the functional $A_T\xi$ depending on unknown values of a process with stationary increments $\xi(t)$ from observations of the process $\xi(t) + \eta(t)$ for $t \in \mathbb{R} \setminus [0;T]$ in the set of admissible spectral densities $\mathcal{D} = \mathcal{D}^0_{1/f} \times \mathcal{D}^0_{1/g}$, where

$$\begin{aligned}\mathcal{D}^0_{1/f} &= \left\{ f(\lambda) \,\middle|\, \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{f(\lambda)}\, d\lambda \geq P_1 \right\},\\ \mathcal{D}^0_{1/g} &= \left\{ g(\lambda) \,\middle|\, \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{g(\lambda)}\, d\lambda \geq P_2 \right\}.\end{aligned} \tag{25}$$

Assume that the spectral densities $f^0 \in \mathcal{D}^0_{1/f}$ and $g^0 \in \mathcal{D}^0_{1/g}$ and functions

$$h_{\tau,f}\left(f^0, g^0\right) = \frac{\left|A_T(\lambda)(1-e^{i\lambda\tau})^n(-i\lambda)^n g^0(\lambda) + (-i\lambda)^n C^0_{\tau,T}(\lambda)\right|}{|1-e^{i\lambda\tau}|^n(1+\lambda^2)^{-n/2}((1+\lambda^2)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda))}, \tag{26}$$

$$h_{\tau,g}\left(f^0, g^0\right) = \frac{\left|A_T(\lambda)(1-e^{i\lambda\tau})^n(1+\lambda^2)^n f^0(\lambda) - \lambda^{2n} C^0_{\tau,T}(\lambda)\right|}{|1-e^{i\lambda\tau}|^n((1+\lambda^2)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda))} \tag{27}$$

are bounded. Under these conditions, the functional $\Delta(h_\tau(f^0, g^0); f, g)$ is continuous and bounded in the space $L_1 \times L_1$. The inclusion $0 \in \partial\Delta_{\mathcal{D}}(f^0, g^0)$ yields the following relations that determine the spectral densities $f^0 \in \mathcal{D}^0_{1/f}$ and $g^0 \in \mathcal{D}^0_{1/g}$:

$$\begin{aligned} g^0(\lambda)&\left|A_T(\lambda)\left(1-e^{i\lambda\tau}\right)^n\left(1+\lambda^2\right)^n f^0(\lambda) - \lambda^{2n} C^0_{\tau,T}(\lambda)\right| \\ &= \alpha_2\left|1-e^{i\lambda\tau}\right|^n\left(\left(1+\lambda^2\right)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda)\right),\end{aligned} \tag{28}$$

$$\begin{aligned} f^0(\lambda)&\left|A_T(\lambda)(1-e^{i\lambda\tau})^n(-i\lambda)^n g^0(\lambda) + (-i\lambda)^n C^0_{\tau,T}(\lambda)\right| \\ &= \alpha_1|1-e^{i\lambda\tau}|^n(1+\lambda^2)^{-n/2}\left((1+\lambda^2)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda)\right),\end{aligned} \tag{29}$$

where α_1 and α_2 are some constants such that $\alpha_1 \geq 0$, $\alpha_2 \geq 0$; $\alpha_1 \neq 0$ if

$$\int_{-\infty}^{\infty} f^0(\lambda)\, d\lambda = 2\pi P_1,$$

and $\alpha_2 \neq 0$ if $\int_{-\infty}^{\infty} g^0(\lambda)\, d\lambda = 2\pi P_2$.

The reasoning above allows us to formulate the following results.

Theorem 5.1. *Assume that two spectral densities $f^0(\lambda) \in \mathcal{D}^0_{1/f}$ and $g^0(\lambda) \in \mathcal{D}^0_{1/g}$ satisfy condition* (6). *Let the functions $h_{\tau,f}(f^0, g^0)$ and $h_{\tau,g}(f^0, g^0)$ obtained from equalities* (26) *and* (27) *be bounded. The spectral densities $f^0(\lambda)$ and $g^0(\lambda)$ calculated from equalities* (28) *and* (29) *are the least favorable among densities of the class $\mathcal{D} = \mathcal{D}^0_{1/f} \times \mathcal{D}^0_{1/g}$ for the linear interpolation of the functional $A_T\xi$ if they determine a solution of extremum problem* (23). *The function $h_\tau(f^0, g^0)$ defined by equality* (14) *is the minimax spectral characteristic of the optimal estimate of the functional $A_T\xi$.*

If one of the densities is known, we obtain the following results.

Theorem 5.2. *Assume that the spectral density $f(\lambda)$ is known and that $g^0(\lambda) \in \mathcal{D}^0_{1/g}$. Assume that the pair (f, g) satisfies condition* (6). *Also assume that the function*

$$h_{\tau,g}(f, g^0)$$

defined by equality (27) *is bounded. Then the spectral density*

$$g^0(\lambda) = f(\lambda)\left[f_1(\lambda) - \frac{\lambda^{2n}}{(1+\lambda^2)^n}\right]_+^{-1},$$

$$f_1(\lambda) = \frac{\left|A_T(\lambda)(1-e^{i\lambda\tau})^n(1+\lambda^2)^n f(\lambda) - \lambda^{2n}C^0_{\tau,T}(\lambda)\right|}{\alpha_2|1-e^{i\lambda\tau}|^n(1+\lambda^2)^n}, \tag{30}$$

is the least favorable in the class $\mathcal{D}^0_{1/g}$ *for the linear interpolation of the functional* $A_T\xi$ *if the functions* $f(\lambda) + (1+\lambda^2)^{-n}\lambda^{2n}g^0(\lambda)$ *and* $g^0(\lambda)$ *determine a solution of extremum problem* (23). *The function* $h_\tau(f, g^0)$ *defined by equality* (14) *is the minimax spectral characteristic of the optimal estimate of the functional* $A_T\xi$.

Theorem 5.3. *Assume that the spectral density* $g(\lambda)$ *is known,* $f^0(\lambda) \in \mathcal{D}^0_{1/f}$, *and that a pair* (f, g) *satisfies condition* (6). *We also assume that the function* $h_{\tau,f}(f^0, g)$ *defined by* (26) *is bounded. Then the spectral density*

$$f^0(\lambda) = \frac{\lambda^{2n}g(\lambda)}{(1+\lambda^2)^n[g_2(\lambda)-1]_+},$$

$$g_2(\lambda) = \frac{\left|A_T(\lambda)(1-e^{i\lambda\tau})^n(-i\lambda)^n g(\lambda) + (-i\lambda)^n C^0_{\tau,T}(\lambda)\right|}{\alpha_1|1-e^{i\lambda\tau}|^n(1+\lambda^2)^{n/2}}, \tag{31}$$

is the least favorable in the class $\mathcal{D}^0_{1/f}$ *for the linear interpolation of the functional* $A_T\xi$ *if the function* $f^0(\lambda) + (1+\lambda^2)^{-n}\lambda^{2n}g(\lambda)$ *determines a solution of extremum problem* (23). *The function* $h_\tau(f^0, g)$ *defined by equality* (14) *is the minimax spectral characteristic of the optimal estimate of the functional* $A_T\xi$.

As a corollary of the above results, we obtain a relation that determines the least favorable density in the class $\mathcal{D}^0_{1/f}$ for the optimal interpolation of the functional $A_T\xi$ from observations of the process $\xi(t)$ without noise at points $t \in \mathbb{R}\setminus[0;T]$. Assume that $f^0(\lambda) \in \mathcal{D}^0_{1/f}$ and that the function

$$h_f(f^0) = \frac{\lambda^n\left|C^0_{\tau,T}(\lambda)\right|}{|1-e^{i\lambda\tau}|^n|1+i\lambda|^n f^0(\lambda)} \tag{32}$$

is bounded. Then the least favorable spectral density satisfies the following relation:

$$\left|\int_0^{T+\tau n}\left(((\mathbf{F}^\tau_T)^0)^{-1}[D^\tau_T\mathbf{a}]_{+\tau n}\right)(t)e^{i\lambda t}\,dt\right| = \beta|\lambda|^{-n}|1-e^{i\lambda\tau}|^n|1+i\lambda|^n, \tag{33}$$

where β is a constant such that $\beta \geq 0$ and $\beta \neq 0$ if $\int_{-\infty}^{\infty} f^0(\lambda)\,d\lambda = 2\pi P$.

The argument above leads to the following result.

Corollary 5.1. *Let a spectral density* $f^0(\lambda) \in \mathcal{D}_{1/f}$ *satisfy the condition of minimality* (21) *and let the function* $h_f(f^0)$ *defined by* (32) *be bounded. Then a spectral density* $f^0(\lambda)$ *is the least favorable in the class* $\mathcal{D}_{1/f}$ *for the optimal interpolation of the functional* $A_T\xi$ *from observations of the stochastic process* $\xi(t)$ *for* $t \in \mathbb{R}\setminus[0;T]$ *if the linear operator* $(\mathbf{F}^\tau_T)^0$ *constructed with the help of the Fourier transform of the function*

$$\lambda^{2n}\left|1-e^{i\lambda\tau}\right|^{-2n}\left(1+\lambda^2\right)^{-n}\left(f^0(\lambda)\right)^{-1}$$

satisfies condition (33) *and determines a solution of extremum problem* (24). *The minimax spectral characteristic* $h^0 = h^\xi_\tau(f^0)$ *of the optimal estimate* $\widehat{A}_T\xi$ *is given by equality* (19).

6. The least favorable densities in the class $\mathcal{D} = \mathcal{D}_v^u \times \mathcal{D}_\varepsilon$

Consider the problem of minimax interpolation of the linear functional $A_T\xi$ from observations of a stochastic process $\xi(t) + \eta(t)$ at moments $t \in \mathbb{R} \setminus [0;T]$ in the set of admissible spectral densities $\mathcal{D} = \mathcal{D}_v^u \times \mathcal{D}_\varepsilon$, where

$$\mathcal{D}_v^u = \left\{ f(\lambda) \,\middle|\, v(\lambda) \le f(\lambda) \le u(\lambda),\, \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\lambda)\,d\lambda = P_1 \right\},$$

$$\mathcal{D}_\varepsilon = \left\{ g(\lambda) \,\middle|\, g(\lambda) = (1-\varepsilon)g_1(\lambda) + \varepsilon w(\lambda),\, \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\lambda)\,d\lambda = P_2 \right\}.$$

The spectral densities $u(\lambda)$, $v(\lambda)$, and $g_1(\lambda)$ are assumed to be known and, moreover, the spectral densities $u(\lambda)$ and $v(\lambda)$ are bounded.

Using the condition $0 \in \partial\Delta_{\mathcal{D}}(f^0, g^0)$ we obtain the following relations determining the least favorable spectral densities $f^0 \in \mathcal{D}_v^u$ and $g^0 \in \mathcal{D}_\varepsilon$:

$$\begin{aligned}(34)\quad &\left| A_T(\lambda)\left(1-e^{i\lambda\tau}\right)^n \left(1+\lambda^2\right)^n f^0(\lambda) - \lambda^{2n} C^0_{\tau,T}(\lambda) \right| \\ &\quad = \left|1-e^{i\lambda\tau}\right|^n \left(\left(1+\lambda^2\right)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda)\right)(\beta(\lambda)+\alpha_2),\end{aligned}$$

$$\begin{aligned}(35)\quad &\left| A_T(\lambda)\left(1-e^{i\lambda\tau}\right)^n (-i\lambda)^n g^0(\lambda) + (-i\lambda)^n C^0_{\tau,T}(\lambda) \right| \\ &\quad = \left|1-e^{i\lambda\tau}\right|^n \left(1+\lambda^2\right)^{-n/2} \left(\left(1+\lambda^2\right)^n f^0(\lambda) + \lambda^{2n} g^0(\lambda)\right) \\ &\quad\quad \times (\gamma_1(\lambda)+\gamma_2(\lambda)+\alpha_1),\end{aligned}$$

where $\gamma_1(\lambda) \le 0$ and $\gamma_1(\lambda) = 0$ if $f^0(\lambda) \ge v(\lambda)$; $\gamma_2(\lambda) \ge 0$ and $\gamma_2(\lambda) = 0$ if $f^0(\lambda) \le u(\lambda)$; $\beta(\lambda) \le 0$ and $\beta(\lambda) = 0$ if $g^0(\lambda) \ge (1-\varepsilon)g_1(\lambda)$.

The following results hold.

Theorem 6.1. *Let two spectral densities $f^0(\lambda) \in \mathcal{D}_v^u$ and $g^0(\lambda) \in \mathcal{D}_\varepsilon$ satisfy condition* (6) *and let the functions $h_{\tau,f}(f^0, g^0)$ and $h_{\tau,g}(f^0, g^0)$ defined by equalities* (26) *and* (27)*, respectively, be bounded.*

Then the functions $f^0(\lambda)$ and $g^0(\lambda)$ defined by relations (34) *and* (35)*, respectively, are the least favorable spectral densities in the class $\mathcal{D} = \mathcal{D}_v^u \times \mathcal{D}_\varepsilon$ for the optimal linear interpolation of the functional $A_T\xi$ if they determine a solution of optimization problem* (23)*. The function $h_\tau(f^0, g^0)$ obtained from* (14) *is the minimax (robust) spectral characteristic of the optimal estimate of the functional $A_T\xi$.*

Theorem 6.2. *Let a spectral density $f(\lambda)$ be known and let the spectral density*

$$g^0(\lambda) \in \mathcal{D}_\varepsilon$$

satisfy the condition of minimality (6)*. Assume that the function $h_{\tau,g}(f, g^0)$ defined by relation* (27) *is bounded.*

Then the spectral density

$$g^0(\lambda) = \max\left\{(1-\varepsilon)g_2(\lambda), f_1(\lambda) - \left(1+\lambda^2\right)^n \lambda^{-2n} f(\lambda)\right\},$$

is the least favorable in the class $\mathcal{D}_\varepsilon$ for the optimal linear interpolation of the functional $A_T\xi$ if the functions $f(\lambda) + (1+\lambda^2)^{-n}\lambda^{2n} g^0(\lambda)$ and $g^0(\lambda)$ determine a solution of optimization problem (23)*, where the function $f_1(\lambda)$ is evaluated by* (30)*. The function $h_\tau(f, g^0)$ defined by* (14) *is the minimax (robust) spectral characteristic of the optimal estimate of the functional $A_T\xi$.*

Theorem 6.3. *Assume that a spectral density $g(\lambda)$ is known, while a spectral density $f^0(\lambda) \in \mathcal{D}_v^u$ satisfies minimality condition* (6)*. Finally, let the function $h_{\tau,f}(f^0, g)$ defined by equality* (26) *be bounded.*

Then the function

$$f^0(\lambda) = \min\left\{u(\lambda), \max\left\{v(\lambda), g_2(\lambda) - \left(1+\lambda^2\right)^{-n} \lambda^{2n} g(\lambda)\right\}\right\},$$

where $g_2(\lambda)$ *is established from relation* (31), *is the least favorable among spectral densities of the class* $\mathcal{D}_v^u$ *for the optimal linear interpolation of the functional* $A_T\xi$ *if the function* $f^0(\lambda) + (1+\lambda^2)^{-n}\lambda^{2n} g(\lambda)$ *determines a solution of optimization problem* (23). *The function* $h_\tau(f^0, g)$ *defined by equality* (14) *is the minimax (robust) spectral characteristic of the optimal estimate of the functional* $A_T\xi$.

Consider the problem of the minimax interpolation of the functional $A_T\xi$ from observations of the process $\xi(t)$ without noise at points $t \in \mathbb{R} \setminus [0;T]$ in the class $\mathcal{D}_v^u$ of admissible spectral densities. Then the least favorable spectral density $f^0(\lambda) \in \mathcal{D}_v^u$ satisfies the following relation:

$$\left| \int_0^{T+\tau n} \big(((\mathbf{F}_T^\tau)^0)^{-1} [D_T^\tau \mathbf{a}]_{+\tau n}\big)(t) e^{i\lambda t}\, dt \right| = \frac{f^0(\lambda)(\gamma_1(\lambda) + \gamma_2(\lambda) + \beta)}{|\lambda|^n |1 - e^{i\lambda\tau}|^{-n} |1 + i\lambda|^{-n}}, \tag{36}$$

where γ_1 is such that $\gamma_1(\lambda) \le 0$ and $\gamma_1(\lambda) = 0$ if $f^0(\lambda) \ge v(\lambda)$; γ_2 is such that $\gamma_2(\lambda) \ge 0$ and $\gamma_2(\lambda) = 0$ if $f^0(\lambda) \le u(\lambda)$.

These results lead to the following corollary.

Corollary 6.1. *Let a spectral density* $f^0(\lambda) \in \mathcal{D}_v^u$ *satisfy the condition of minimality* (21) *and let the function* $h_f(f^0)$ *calculated from equality* (32) *be bounded.*

Then the spectral density $f^0(\lambda)$ *is the least favorable in the class* $\mathcal{D}_v^u$ *for the optimal interpolation of the functional* $A_T\xi$ *from observations of the stochastic process* $\xi(t)$ *for* $t \in \mathbb{R} \setminus [0;T]$ *if*

$$f^0(\lambda) = \max\left\{v(\lambda), \min\left\{u(\lambda), \frac{|\lambda|^n \left| \int_0^{T+\tau n} \big(((\mathbf{F}_T^\tau)^0)^{-1} [D_T^\tau \mathbf{a}]_{+\tau n}\big)(t) e^{i\lambda t}\, dt \right|}{\beta |1 - e^{i\lambda\tau}|^n |1 + i\lambda|^n}\right\}\right\}$$

determines a solution of extremum problem (24). *The minimax spectral characteristic* $h^0 = h_\tau^\xi(f^0)$ *of the optimal estimate* $\widehat{A}_T\xi$ *is obtained from* (19).

7. Concluding remarks

The problem of the optimal in the mean square sense linear estimation from observations of the stochastic process $\xi(t) + \eta(t)$ for $t \in \mathbb{R} \setminus [0,T]$ is studied in the paper for the functional $A_T\xi = \int_0^T a(t)\xi(t)dt$ that depends on unknown values of a stochastic process $\xi(t)$ with stationary increments of order n in the case of a stationary noise $\eta(t)$ uncorrelated with the process $\xi(t)$. We apply the classical and minimax (robust) methods of estimation for the case of spectral certainty and spectral uncertainty, respectively.

In particular, formulas are found for the spectral characteristic and mean square error of the optimal estimate. If spectral densities are unknown, but a class of admissible spectral densities is specified, we obtain relations that determine the least favorable spectral densities and minimax spectral characteristics.

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