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UPPER BOUNDS FOR SUPREMUMS OF THE NORMS OF THE DEVIATION BETWEEN A HOMOGENEOUS ISOTROPIC RANDOM FIELD AND ITS MODEL

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Abstract. Some estimates are obtained for the norm of the deviation between a homogeneous isotropic random field and its model.

1. Introduction

One of the most important problems of the theory of stochastic processes and random fields is the problem of modeling and approximating the processes and fields. Several methods for constructing models of stochastic processes are known in the literature. The most popular method of modeling for stationary processes is the method of splitting and randomization of the spectrum developed by Mikhaĭlov et al. (see [9]–[13]). M. I. Yadrenko et al. used a different method (see [16]–[18]). When constructing models of stochastic processes or random fields, it is important to know how close are the approximating models and corresponding processes or fields in some metrics. A number of papers by Yu. V. Kozachenko and his collaborators are devoted to constructing models of random fields with given reliability and accuracy (see [3]–[6], [8]).

A Gaussian mean square continuous real-valued homogeneous and isotropic random field on $\mathbb{R}^2$ is considered in this paper. A modified method of splitting and randomization of the spectrum developed in the papers [8, 14, 15] is used here to construct a new model for such a field. The models of random fields constructed with the help of the modified method of splitting and randomization of the spectrum are sub-Gaussian. This is an advantage of our method, since the covariance function of the model coincides with that of the field itself, while this property does not hold for the majority of other methods. The representation of a homogeneous isotropic random field obtained by Yadrenko [17] is used in our paper. In the problems of modeling random fields, it is important to estimate the probability of the deviation in the uniform metric between the field and its model on some compact set $\mathbb{T}$; namely, to estimate the probability

$$\mathsf{P}\left\{\sup_{t\in\mathbb{T}} |X(t,x) - \hat{X}(t,x)| > \varepsilon\right\},$$

where $X(t,x)$ is a field and $\hat{X}(t,x)$ is its model. Note that the above probability is the tail of the distribution of the norm in the space $C(T)$ of the deviation between the field and its model. Since $X(t,x) - \hat{X}(t,x) \in \mathrm{Sub}(\Omega)$, an estimate of the reliability and accuracy of the model in the space $C(T)$ relies on an estimate of the supremum of norms of the deviations between the field and its model. The latter is the problem

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which the current paper is devoted to. An estimate for the distribution of the norm of the deviation between the field and its model in the space $C(T)$ will be obtained in a forthcoming paper.

It is worth mentioning that estimates of the type described above for the deviation between fields and their models are easy to obtain in $\mathbb{R}^n$, $n > 2$; one only needs to overcome some minor technical difficulties.

The paper is organized as follows. Section 1 is the introduction. Section 2 contains necessary definitions and preliminary results of the theory of sub-Gaussian random variables. Section 3 is the main part of the paper containing new results on estimates for the supremum of norms of the deviations between a random field and its model. Also Section 3 contains the proofs. Section 4 is a collection of concluding remarks.

2. Preliminary results

Definition 2.1 ([1]). A random variable χ is called sub-Gaussian if there exists a number $a \geq 0$ such that

$$\mathsf{E}\exp\{\lambda\chi\} \leq \exp\left\{\frac{a^2\lambda^2}{2}\right\}$$

for all $\lambda \in \mathbb{R}$.

The family of all sub-Gaussian random variables defined on a standard probability space $\{\Omega, \mathbf{B}, \mathsf{P}\}$ is denoted by $\mathrm{Sub}(\Omega)$. The family $\mathrm{Sub}(\Omega)$ is a Banach space with respect to the norm

$$\tau(\chi) = \sup_{\lambda\neq 0}\left[\frac{2\ln \mathsf{E}\exp\{\lambda\chi\}}{\lambda^2}\right]^{\frac{1}{2}}.$$

Definition 2.2 ([1]). A stochastic process $\xi = \{\xi(t), t \in T\}$ is called sub-Gaussian if $\xi(t) \in \mathrm{Sub}(\Omega)$ and $\sup_{t\in T}\tau(\xi(t)) < \infty$ for all $t \in T$.

Lemma 2.1 ([1]). *Let $\xi_1, \xi_2, \dots, \xi_n$ be independent sub-Gaussian random variables. Then*

$$\tau^2\left(\sum_{k=1}^{n}\xi_k\right) \leq \sum_{k=1}^{n}\tau^2(\xi_k).$$

Corollary 2.1 ([2]). *Let $\xi_1, \xi_2, \dots,$ be independent sub-Gaussian random variables. Then*

$$\tau^2\left(\sum_{k=1}^{\infty}\xi_k\right) \leq \sum_{k=1}^{\infty}\tau^2(\xi_k).$$

Lemma 2.2 ([1]). *Let ξ be a centered random variable such that $\mathsf{E}\,\xi^{2k+1} = 0$ and*

$$\theta(\xi) = \sup_{k\geq 1}\left[\frac{2^k k!}{(2k)!}\,\mathsf{E}\,\xi^{2k}\right]^{\frac{1}{2k}} < \infty.$$

Then $\xi \in \mathrm{Sub}(\Omega)$ and $\tau(\xi) \leq \theta(\xi)$.

Let

$$J_k(u) = \frac{1}{\pi}\int_0^{\pi}\cos(k\varphi - u\sin\varphi)\,d\varphi, \qquad k = 1, 2, \dots,$$

be the integral representation of the Bessel functions of the first kind.

Lemma 2.3 ([14]). *For all $0 < \alpha \leq 1$,*

$$|J_k(u)| \leq 2^{1-\alpha}|u|^{\alpha}\pi^{\alpha}\frac{1}{k^{\alpha}}.$$

Lemma 2.4 ([14]). *For all* $0 < \alpha \le 1$,

$$|J_k(t\lambda) - J_k(tu)| \le 4^{1-\alpha} t^\alpha |\lambda - u|^\alpha \pi^\alpha \cdot \frac{1}{k^\alpha}\left(1 + \frac{t^\alpha|\lambda+u|^\alpha}{2^\alpha}\right).$$

Definition 2.3 ([17]). A random field $X = \{X(t), t \in \mathbb{R}^2\}$ is called homogeneous in the wide sense in $\mathbb{R}^2$ if $\mathsf{E}\, X(t) = \text{const}$, $t \in \mathbb{R}^2$, and

$$\mathsf{E}\, X(t)X(s) = B(t-s) = \int_{\mathbb{R}^2} e^{i(\lambda, t-s)}\, dF(\lambda), \qquad t, s \in \mathbb{R}^2.$$

Definition 2.4 ([17]). Let $SO(2)$ be the group of all rotations about the origin of the space $\mathbb{R}^2$. A homogeneous random field $X(t)$, $t \in \mathbb{R}^2$, is called isotropic if

$$\mathsf{E}\, X(t)X(s) = \mathsf{E}\, X(gt)X(gs)$$

for all elements g of the group $SO(2)$ and for all $t, s \in \mathbb{R}^2$.

3. Mainstream

Let $X = \{X(t,x), t \in \mathbb{R}, x \in [0, 2\pi]\}$ be a real-valued mean square continuous homogeneous and isotropic Gaussian random field on $\mathbb{R}^2$. The following representation is obtained similarly to [17] where complex-valued fields are considered:

$$X(t,x) = \sum_{k=1}^{\infty} \cos(kx) \int_0^\infty J_k(t\lambda)\, d\eta_{1,k}(\lambda) + \sum_{k=1}^{\infty} \sin(kx) \int_0^\infty J_k(t\lambda)\, d\eta_{2,k}(\lambda), \tag{1}$$

where $\eta_{i,k}(\lambda)$, $i = 1, 2$, $k = 1, 2, \dots$, are independent Gaussian processes with independent increments, $\mathsf{E}\, \eta_{i,k}(\lambda) = 0$, $\mathsf{E}(\eta_{i,k}(b) - \eta_{i,k}(c))^2 = F(b) - F(c)$, $b > c$, $F(\lambda)$ is the spectral function of the field.

Consider a partition $L = \{\lambda_0, \dots, \lambda_N\}$ of the set $[0, \infty)$ such that $\lambda_0 = 0$, $\lambda_l < \lambda_{l+1}$, $\lambda_{N-1} = \Lambda$, $\lambda_N = \infty$, and $C = \max_{0<l\le N-2} \frac{\lambda_{l+1}}{\lambda_l} < \infty$.

The process

$$\hat{X}(t,x) = \sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \eta_{1,k,l} J_k(t\zeta_l) + \sum_{k=1}^{M} \sin(kx) \sum_{l=0}^{N-1} \eta_{2,k,l} J_k(t\zeta_l)$$

is viewed as a model of the field $X(t,x)$ where $\eta_{i,k,l}$, $i = 1, 2$, are independent Gaussian random variables, $\eta_{i,k,l} = \int_{\lambda_l}^{\lambda_{l+1}} d\eta_{i,k}(\lambda)$ are such that $\mathsf{E}\, \eta_{i,k,l} = 0$ and

$$\mathsf{E}\, \eta_{i,k,l}^2 = F(\lambda_{l+1}) - F(\lambda_l) = b_l^2.$$

Here ζ_l, $l = 0, \dots, N-2$, are independent random variables being independent of $\eta_{i,k,l}$ and assuming values in the intervals $[\lambda_l, \lambda_{l+1}]$, $\zeta_{N-1} = \Lambda$, and such that

$$F_l(\lambda) = \mathsf{P}\{\zeta_l < \lambda\} = \frac{F(\lambda) - F(\lambda_l)}{F(\lambda_{l+1}) - F(\lambda_l)}.$$

If $b_l^2 = 0$, then $\zeta_l = 0$ with probability 1. For the sake of simplicity assume that $b_l^2 > 0$, $l = 0, 1, \dots, N-1$.

Thus $\hat{X}(t,x)$ is written as follows:

$$\begin{aligned} \hat{X}(t,x) = {} & \sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} J_k(t\zeta_l) d\eta_{1,k}(\lambda) \\ & + \sum_{k=1}^{M} \sin(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} J_k(t\zeta_l)\, d\eta_{2,k}(\lambda). \end{aligned} \tag{2}$$

Note that $X(t,x)$ admits the following representation:

$$X(t,x) = \sum_{k=1}^{\infty} \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} J_k(t\lambda)\, d\eta_{1,k}(\lambda) + \sum_{k=1}^{\infty} \sin(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} J_k(t\lambda)\, d\eta_{2,k}(\lambda).$$

Consider the deviation $X(t,x) - \hat{X}(t,x)$ and put

$$\begin{aligned} \chi_M(t,x) &= X(t,x) - \hat{X}(t,x) \\ &= \left(\sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l))\, d\eta_{1,k}(\lambda) \right. \\ &\qquad \left. + \sum_{k=M+1}^{\infty} \cos(kx) \int_0^{\infty} J_k(t\lambda) d\eta_{1,k}(\lambda) \right) \\ &\quad + \left(\sum_{k=1}^{M} \sin(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l))\, d\eta_{2,k}(\lambda) \right. \\ &\qquad \left. + \sum_{k=M+1}^{\infty} \sin(kx) \int_0^{\infty} J_k(t\lambda)\, d\eta_{2,k}(\lambda) \right). \end{aligned} \tag{3}$$

Denote the two terms on the right-hand side of (3) by $\chi_{M,1}(t,x)$ and $\chi_{M,2}(t,x)$. Consider the difference

$$\chi_M(t,x) - \chi_M(s,y) = (\chi_{M,1}(t,x) - \chi_{M,1}(s,y)) + (\chi_{M,2}(t,x) - \chi_{M,2}(s,y)). \tag{4}$$

It is clear that

$$\begin{aligned} \chi_{M,1}(t,x) - \chi_{M,1}(s,y) &= \sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l))\, d\eta_{1,k}(\lambda) \\ &\quad + \sum_{k=M+1}^{\infty} \cos(kx) \int_0^{\infty} J_k(t\lambda)\, d\eta_{1,k}(\lambda) \\ &\quad - \sum_{k=1}^{M} \cos(ky) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(s\lambda) - J_k(s\zeta_l))\, d\eta_{1,k}(\lambda) \\ &\quad - \sum_{k=M+1}^{\infty} \cos(ky) \int_0^{\infty} J_k(s\lambda)\, d\eta_{1,k}(\lambda) \\ &= \sum_{k=1}^{M} \left(\cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l))\, d\eta_{1,k}(\lambda) \right. \\ &\qquad \left. - \cos(ky) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(s\lambda) - J_k(s\zeta_l))\, d\eta_{1,k}(\lambda) \right) \\ &\quad + \sum_{k=M+1}^{\infty} \left(\cos(kx) \int_0^{\infty} J_k(t\lambda)\, d\eta_{1,k}(\lambda) \right. \\ &\qquad \left. - \cos(ky) \int_0^{\infty} J_k(s\lambda)\, d\eta_{1,k}(\lambda) \right) \end{aligned}$$

and therefore

$$\begin{aligned}
&\chi_{M,1}(t,x) - \chi_{M,1}(s,y) \\
&\quad = \sum_{k=1}^{M}\Bigg(\cos(kx)\sum_{l=0}^{N-1}\int_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda) - J_k(t\zeta_l) - J_k(s\lambda) + J_k(s\zeta_l))\,d\eta_{1,k}(\lambda) \\
&\qquad\qquad + (\cos(kx) - \cos(ky))\sum_{l=0}^{N-1}\int_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda) - J_k(s\zeta_l))\,d\eta_{1,k}(\lambda)\Bigg) \\
&\qquad + \sum_{k=M+1}^{\infty}\Bigg(\cos(kx)\int_0^{\infty}(J_k(t\lambda) - J_k(s\lambda))\,d\eta_{1,k}(\lambda) \\
&\qquad\qquad + (\cos(kx) - \cos(ky))\int_0^{\infty}J_k(s\lambda)\,d\eta_{1,k}(\lambda)\Bigg)
\end{aligned}$$

for all $t, s \in [0, T]$ and $x, y \in [0, 2\pi]$.

Similarly,

$$\begin{aligned}
&\chi_{M,2}(t,x) - \chi_{M,2}(s,y) \\
&\quad = \sum_{k=1}^{M}\Bigg(\sin(kx)\sum_{l=0}^{N-1}\int_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda) - J_k(t\zeta_l) - J_k(s\lambda) + J_k(s\zeta_l))\,d\eta_{2,k}(\lambda) \\
&\qquad\qquad + (\sin(kx) - \sin(ky))\sum_{l=0}^{N-1}\int_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda) - J_k(s\zeta_l))\,d\eta_{2,k}(\lambda)\Bigg) \\
&\qquad + \sum_{k=M+1}^{\infty}\Bigg(\sin(kx)\int_0^{\infty}(J_k(t\lambda) - J_k(s\lambda))\,d\eta_{2,k}(\lambda) \\
&\qquad\qquad + (\sin(kx) - \sin(ky))\int_0^{\infty}J_k(s\lambda)\,d\eta_{2,k}(\lambda)\Bigg).
\end{aligned}$$

Then

$$\tau(\chi_M(t,x) - \chi_M(s,y)) \le \tau(\chi_{M,1}(t,x) - \chi_{M,1}(s,y)) + \tau(\chi_{M,2}(t,x) - \chi_{M,2}(s,y))$$

and

$$\begin{aligned}
&\tau^2(\chi_{M,1}(t,x) - \chi_{M,1}(s,y)) \\
&\quad \le 4\tau^2\left(\sum_{k=1}^{M}\cos(kx)\sum_{l=0}^{N-1}\int_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda) - J_k(t\zeta_l) - J_k(s\lambda) + J_k(s\zeta_l))\,d\eta_{1,k}(\lambda)\right) \\
&\qquad + 4\tau^2\left(\sum_{k=1}^{M}(\cos(kx) - \cos(ky))\sum_{l=0}^{N-1}\int_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda) - J_k(s\zeta_l))\,d\eta_{1,k}(\lambda)\right) \\
&\qquad + 4\tau^2\left(\sum_{k=M+1}^{\infty}\cos(kx)\int_0^{\infty}(J_k(t\lambda) - J_k(s\lambda))\,d\eta_{1,k}(\lambda)\right) \\
&\qquad + 4\tau^2\left(\sum_{k=M+1}^{\infty}(\cos(kx) - \cos(ky))\int_0^{\infty}J_k(s\lambda)\,d\eta_{1,k}(\lambda)\right),
\end{aligned}$$

whence

$$
\begin{aligned}
&\tau^2(\chi_{M,2}(t,x) - \chi_{M,2}(s,y)) \\
&\quad \le 4\tau^2 \left(\sum_{k=1}^{M} \sin(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l) - J_k(s\lambda) + J_k(s\zeta_l))\, d\eta_{2,k}(\lambda) \right) \\
&\qquad + 4\tau^2 \left(\sum_{k=1}^{M} (\sin(kx) - \sin(ky)) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(s\lambda) - J_k(s\zeta_l))\, d\eta_{2,k}(\lambda) \right) \\
&\qquad + 4\tau^2 \left(\sum_{k=M+1}^{\infty} \sin(kx) \int_0^{\infty} (J_k(t\lambda) - J_k(s\lambda))\, d\eta_{2,k}(\lambda) \right) \\
&\qquad + 4\tau^2 \left(\sum_{k=M+1}^{\infty} (\sin(kx) - \sin(ky)) \int_0^{\infty} J_k(s\lambda)\, d\eta_{2,k}(\lambda) \right).
\end{aligned}
$$

Let $\sigma_0 = \sup_{0 \le t \le T} \tau(\chi_M(t,x))$ and

$$
\sigma(h) = \sup_{\substack{|t-s| \le h \\ |x-y| \le h}} \tau(\chi_M(t,x) - \chi_M(s,y)).
$$

The following auxiliary results are needed to find a bound for σ_0. The proofs of these results can be found in [14].

Lemma 3.1. *For all* $\frac{1}{2} < \alpha \le 1$,

$$
\begin{aligned}
&\tau^2 \left(\sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l))\, d\eta_{1,k}(\lambda) \right) \\
&\quad \le \frac{1}{2\alpha - 1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) 2 \cdot 4^{2(1-\alpha)} \pi^{2\alpha} t^{2\alpha} \\
&\qquad \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{t(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha}\, dF(\lambda) \right) \\
&\qquad + 4M(F(+\infty) - F(\Lambda)), \\
&\tau^2 \left(\sum_{k=1}^{M} \sin(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l))\, d\eta_{2,k}(\lambda) \right) \\
&\quad \le \frac{1}{2\alpha - 1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) 2 \cdot 4^{2(1-\alpha)} \pi^{2\alpha} t^{2\alpha} \\
&\qquad \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{t(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha}\, dF(\lambda) \right) \\
&\qquad + 4M(F(+\infty) - F(\Lambda)),
\end{aligned}
$$

where $C = \max_{0 < l \le N-2} \lambda_{l+1}/\lambda_l$.

Lemma 3.2. *Let*

$$
\int_0^{\infty} \lambda^{2\alpha}\, dF(\lambda) < \infty
$$

for $\frac{1}{2} < \alpha \leq 1$. *Then*

$$\tau^2 \left(\sum_{k=M+1}^{\infty} \cos(kx) \int_0^{\infty} J_k(t\lambda)\, d\eta_{1,k}(\lambda) \right) \leq 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \frac{1}{(2\alpha-1)M^{2\alpha-1}} \left(\int_0^{\infty} \lambda^{2\alpha}\, dF(\lambda) \right)$$

and

$$\tau^2 \left(\sum_{k=M+1}^{\infty} \sin(kx) \int_0^{\infty} J_k(t\lambda)\, d\eta_{2,k}(\lambda) \right) \leq 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \frac{1}{(2\alpha-1)M^{2\alpha-1}} \left(\int_0^{\infty} \lambda^{2\alpha}\, dF(\lambda) \right).$$

Theorem 3.1. *Let* $X(t,x)$ *and* $\hat{X}(t,x)$ *be defined by* (1) *and* (2), *respectively. Assume that* $\frac{1}{2} < \alpha \leq 1$ *and*

$$\int_0^{\infty} \lambda^{2\alpha}\, dF(\lambda) < \infty.$$

Then

$$\sigma_0 \leq \Bigg(\frac{2}{2\alpha-1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) 2 \cdot 4^{2(1-\alpha)} T^{2\alpha} \pi^{2\alpha} \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{T(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha}\, dF(\lambda) \right) + 8M(F(+\infty) - F(\Lambda)) + 2^{2(1-\alpha)} T^{2\alpha} \pi^{2\alpha} \frac{2}{(2\alpha-1)M^{2\alpha-1}} \int_0^{\infty} \lambda^{2\alpha}\, dF(\lambda) \Bigg)^{1/2},$$

where

$$C = \max_{0<l\leq N-2} \frac{\lambda_{l+1}}{\lambda_l}.$$

Proof. Since

$$\tau^2(\chi_M(t,x)) = [\tau(\chi_{M,1}(t,x)) + \tau(\chi_{M,2}(t,x))]^2 \leq 2\left[\tau^2(\chi_{M,1}(t,x)) + \tau^2(\chi_{M,2}(t,x)) \right],$$

Lemmas 3.1 and 3.2 imply that

$$\tau^2(\chi_M(t,x)) \leq 2\Bigg[\tau^2 \left(\sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l))\, d\eta_{1,k}(\lambda) \right) + \tau^2 \left(\sum_{k=1}^{M} \sin(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l))\, d\eta_{2,k}(\lambda) \right) + \tau^2 \left(\sum_{k=M+1}^{\infty} \cos(kx) \int_0^{\infty} J_k(t\lambda)\, d\eta_{1,k}(\lambda) \right) + \tau^2 \left(\sum_{k=M+1}^{\infty} \sin(kx) \int_0^{\infty} J_k(t\lambda)\, d\eta_{2,k}(\lambda) \right) \Bigg]$$

and thus

$$\tau^2(\chi_M(t,x)) \le 2\Bigg[2\cdot 4^{2(1-\alpha)} t^{2\alpha}\pi^{2\alpha}\sum_{k=1}^{M}\frac{1}{k^{2\alpha}}\left(\cos^2(kx)+\sin^2(kx)\right)$$

$$\times\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left(\frac{t(1+C)}{2}\right)^{2\alpha}\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}\,dF(\lambda)\right)$$

$$+4\sum_{k=1}^{M}\left(\cos^2(kx)+\sin^2(kx)\right)(F(+\infty)-F(\Lambda))+2^{2(1-\alpha)}t^{2\alpha}\pi^{2\alpha}$$

$$\times\sum_{k=M+1}^{\infty}\frac{1}{k^{2\alpha}}\left(\cos^2(kx)+\sin^2(kx)\right)\times\left(\int_0^\infty\lambda^{2\alpha}\,dF(\lambda)\right)\Bigg]$$

$$\le\frac{2}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)2\cdot 4^{2(1-\alpha)}t^{2\alpha}\pi^{2\alpha}$$

$$\times\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left(\frac{t(1+C)}{2}\right)^{2\alpha}\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}\,dF(\lambda)\right)$$

$$+8M(F(+\infty)-F(\Lambda))+2^{2(1-\alpha)}t^{2\alpha}\pi^{2\alpha}\frac{2}{(2\alpha-1)M^{2\alpha-1}}\int_0^\infty\lambda^{2\alpha}\,dF(\lambda).$$

This yields

$$\sigma_0=\sup_{0\le t\le T}\tau(\chi_M(t,x))$$

$$\le\Bigg[\frac{2}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)2\cdot 4^{2(1-\alpha)}T^{2\alpha}\pi^{2\alpha}$$

$$\times\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left(\frac{T(1+C)}{2}\right)^{2\alpha}\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}\,dF(\lambda)\right)$$

$$+8M(F(+\infty)-F(\Lambda))+2^{2(1-\alpha)}T^{2\alpha}\pi^{2\alpha}\frac{2}{(2\alpha-1)M^{2\alpha-1}}\int_0^\infty\lambda^{2\alpha}\,dF(\lambda)\Bigg]^{1/2}. \square$$

Corollary 3.1. *Let a partition $L=\{\lambda_0,\dots,\lambda_N\}$ of the set $[0,\infty)$ be such that $\lambda_l<\lambda_{l+1}$ and $\lambda_{l+1}-\lambda_l=\frac{\Lambda}{N-1}$, $l=0,\dots,N-2$. If all assumptions of Theorem* 3.1 *hold, then*

$$\sigma_0\le\Bigg[\frac{4^{2(1-\alpha)+1}T^{2\alpha}\pi^{2\alpha}}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)\left(\frac{\Lambda}{N-1}\right)^{2\alpha}$$

$$\times\left(F(\Lambda)+\left(\frac{3T}{2}\right)^{2\alpha}\int_0^\Lambda\lambda^{2\alpha}\,dF(\lambda)\right)$$

$$+8M(F(+\infty)-F(\Lambda))+\frac{2^{2(1-\alpha)+1}T^{2\alpha}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}}\int_0^\infty\lambda^{2\alpha}\,dF(\lambda)\Bigg]^{1/2}.$$

Lemma 3.3. *For all $0<\alpha\le 1$ and $0<\beta\le 1$,*

$$|J_k(t\lambda)-J_k(s\lambda)|\le 4^{1-\alpha}\pi^\alpha\cdot\frac{1}{k^\alpha}\left(\lambda^\alpha|s-t|^\alpha+\lambda^{\alpha+\beta}|s-t|^\beta|s+t|^\alpha\right).$$

Proof. It is shown in [14] that

$$I_1 = -\frac{1}{4}\int_{-\pi}^{\pi} \cos(k\varphi)\cos\left(u\sin\left(\varphi+\frac{\pi}{k}\right)\right)\,d\varphi + \frac{1}{4}\int_{-\pi}^{\pi}\cos(k\varphi)\cos(u\sin\varphi)\,d\varphi,$$

$$I_2 = -\frac{1}{4}\int_{-\pi}^{\pi} \sin(k\varphi)\sin\left(u\sin\left(\varphi+\frac{\pi}{k}\right)\right)\,d\varphi + \frac{1}{4}\int_{-\pi}^{\pi}\sin(k\varphi)\sin(u\sin\varphi)\,d\varphi.$$

Substituting the expressions for the integrals I_1 and I_2,

$$\begin{aligned}
|J_k(t\lambda) - J_k(s\lambda)| = \frac{1}{\pi}\Bigg|\Bigg(&-\frac{1}{4}\int_{-\pi}^{\pi}\cos(k\varphi)\cos\left(t\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)\,d\varphi \\
&+\frac{1}{4}\int_{-\pi}^{\pi}\cos(k\varphi)\cos(t\lambda\sin\varphi)\,d\varphi \\
&+\frac{1}{4}\int_{-\pi}^{\pi}\cos(k\varphi)\cos\left(s\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)\,d\varphi \\
&-\frac{1}{4}\int_{-\pi}^{\pi}\cos(k\varphi)\cos(s\lambda\sin\varphi)\,d\varphi\Bigg) \\
+\Bigg(&-\frac{1}{4}\int_{-\pi}^{\pi}\sin(k\varphi)\sin\left(t\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)\,d\varphi \\
&+\frac{1}{4}\int_{-\pi}^{\pi}\sin(k\varphi)\sin(t\lambda\sin\varphi)\,d\varphi \\
&+\frac{1}{4}\int_{-\pi}^{\pi}\sin(k\varphi)\sin\left(s\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)\,d\varphi \\
&-\frac{1}{4}\int_{-\pi}^{\pi}\sin(k\varphi)\sin(s\lambda\sin\varphi)\,d\varphi\Bigg)\Bigg| \\
= \frac{1}{\pi}|S_1+S_2| \le \frac{1}{\pi}\left(|S_1|+|S_2|\right).
\end{aligned}$$

Now we estimate $|S_1|$:

$$\begin{aligned}
|S_1| = \frac{1}{4}\Bigg|\int_{-\pi}^{\pi}\cos(k\varphi)\Big[&(\cos(t\lambda\sin\varphi)-\cos(s\lambda\sin\varphi)) \\
&-\left(\cos\left(t\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)-\cos\left(s\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)\right)\Big]\,d\varphi\Bigg| \\
\le \frac{1}{2}\int_{-\pi}^{\pi}|\cos(k\varphi)|\Bigg|&\sin\frac{\lambda(s+t)\sin\varphi}{2}\sin\frac{\lambda(s-t)\sin\varphi}{2} \\
&-\sin\frac{\lambda(s+t)\sin(\varphi+\frac{\pi}{k})}{2}\sin\frac{\lambda(s-t)\sin(\varphi+\frac{\pi}{k})}{2}\Bigg|\,d\varphi \\
= \frac{1}{2}\int_{-\pi}^{\pi}|\cos(k\varphi)|\Bigg|&\sin\frac{\lambda(s+t)\sin\varphi}{2}\left(\sin\frac{\lambda(s-t)\sin\varphi}{2}-\sin\frac{\lambda(s-t)\sin(\varphi+\frac{\pi}{k})}{2}\right) \\
&+\sin\frac{\lambda(s-t)\sin(\varphi+\frac{\pi}{k})}{2} \\
&\times\left(\sin\frac{\lambda(s+t)\sin\varphi}{2}-\sin\frac{\lambda(s+t)\sin(\varphi+\frac{\pi}{k})}{2}\right)\Bigg|\,d\varphi
\end{aligned}$$

$$= \int_{-\pi}^{\pi} |\cos(k\varphi)| \Bigg| \sin\frac{\lambda(s+t)\sin\varphi}{2}\cos\frac{\lambda(s-t)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}$$

$$\times \sin\frac{\lambda(s-t)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4}$$

$$+ \cos\frac{\lambda(s+t)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}\sin\frac{\lambda(s+t)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4}$$

$$\times \sin\frac{\lambda(s-t)\sin(\varphi+\frac{\pi}{k})}{2}\Bigg| \, d\varphi$$

$$\leq \int_{-\pi}^{\pi} \Bigg(\left| \sin\frac{\lambda(s-t)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4} \right|$$

$$+ \left| \sin\frac{\lambda(s-t)\sin(\varphi+\frac{\pi}{k})}{2}\sin\frac{\lambda(s+t)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4} \right| \Bigg) \, d\varphi$$

$$\leq 2\pi \left(\frac{\lambda^\alpha |s-t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha} + \frac{\lambda^\beta |s-t|^\beta \lambda^\alpha |s+t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\beta \cdot 2^\alpha} \right).$$

A bound for $|S_2|$ is obtained similarly. Then

$$|J_k(t\lambda) - J_k(s\lambda)| \leq \frac{1}{\pi} \Bigg[2\pi \left(\frac{\lambda^\alpha |s-t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha} + \frac{\lambda^\beta |s-t|^\beta \lambda^\alpha |s+t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\beta \cdot 2^\alpha} \right)$$

$$+ 2\pi \left(\frac{\lambda^\alpha |s-t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha} + \frac{\lambda^\beta |s-t|^\beta \lambda^\alpha |s+t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\beta \cdot 2^\alpha} \right) \Bigg]$$

$$= 4^{1-\alpha}\pi^\alpha \frac{1}{k^\alpha} \left(\lambda^\alpha |s-t|^\alpha + \lambda^{\alpha+\beta} |s-t|^\beta |s+t|^\alpha \right). \qquad \square$$

Lemma 3.4. *For all* $0 < \alpha \leq 1$,

$$|J_k(t\lambda) - J_k(tu) - J_k(s\lambda) + J_k(su)|$$

$$\leq 2 \cdot 4^{1-\alpha} |\lambda - u|^\alpha |s-t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha$$

$$\times \left(1 + \frac{|\lambda+u|^\alpha |s-t|^\alpha}{4^\alpha} + \frac{|t+s|^\alpha (\lambda^\alpha + 2u^\alpha)}{2^\alpha} + \frac{|t+s|^{2\alpha} u^\alpha |\lambda+u|^\alpha}{4^\alpha \cdot 2^\alpha} \right).$$

Proof. Since

$$J_k(t\lambda) = \frac{1}{\pi} \Bigg(-\frac{1}{4} \int_{-\pi}^{\pi} \cos(k\varphi) \cos\left(t\lambda \sin\left(\varphi + \frac{\pi}{k}\right)\right) \, d\varphi$$

$$+ \frac{1}{4} \int_{-\pi}^{\pi} \cos(k\varphi) \cos(t\lambda \sin\varphi) \, d\varphi$$

$$- \frac{1}{4} \int_{-\pi}^{\pi} \sin(k\varphi) \sin\left(t\lambda \sin\left(\varphi + \frac{\pi}{k}\right)\right) \, d\varphi \tag{5}$$

$$+ \frac{1}{4} \int_{-\pi}^{\pi} \sin(k\varphi) \sin(t\lambda \sin\varphi) \, d\varphi \Bigg),$$

we conclude that

$$
\begin{aligned}
&|J_k(t\lambda) - J_k(tu) - J_k(s\lambda) + J_k(su)| \\
&\quad = \frac{1}{\pi}\Bigg|\frac{1}{4}\int_{-\pi}^{\pi} \cos(k\varphi)\Bigg(\cos\left(tu\sin\left(\varphi+\frac{\pi}{k}\right)\right) - \cos\left(t\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right) \\
&\qquad - \cos(tu\sin\varphi) + \cos(t\lambda\sin\varphi) - \cos(s\lambda\sin\varphi) \\
&\qquad + \cos\left(s\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right) \\
&\qquad + \cos(su\sin\varphi) - \cos\left(su\sin\left(\varphi+\frac{\pi}{k}\right)\right)\Bigg)\,d\varphi \\
&\quad + \frac{1}{4}\int_{-\pi}^{\pi} \sin(k\varphi)\Bigg(\sin\left(tu\sin\left(\varphi+\frac{\pi}{k}\right)\right) - \sin\left(t\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right) \\
&\qquad - \sin(tu\sin\varphi) + \sin(t\lambda\sin\varphi) - \sin(s\lambda\sin\varphi) \\
&\qquad + \sin\left(s\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right) + \sin(su\sin\varphi) \\
&\qquad - \sin\left(su\sin\left(\varphi+\frac{\pi}{k}\right)\right)\Bigg)\,d\varphi\Bigg| \\
&\quad = \frac{1}{\pi}|K_1 + K_2| \le \frac{1}{\pi}\left(|K_1| + |K_2|\right).
\end{aligned}
$$

To estimate $|K_1|$ and $|K_2|$ one needs some trigonometric identities similar to those used to estimate $|S_1|$ and $|S_2|$. More precisely,

$$
\begin{aligned}
&|K_1| \le 2\int_{-\pi}^{\pi} |\cos(k\varphi)| \\
&\quad \times \Bigg[\Bigg|\sin\frac{\lambda(t+s)\sin\varphi}{2}\cos\frac{(\lambda-u)(s-t)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8} \\
&\qquad \times \sin\frac{(\lambda-u)(s-t)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{4}\cos\frac{(\lambda+u)(s-t)\sin\varphi}{4}\Bigg| \\
&\quad + \Bigg|\sin\frac{(\lambda-u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4}\sin\frac{(\lambda+u)(s-t)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8} \\
&\qquad \times \sin\frac{\lambda(t+s)\sin\varphi}{2}\sin\frac{(\lambda+u)(s-t)\cos(\varphi+\frac{\pi}{2k})\sin\frac{\pi}{2k}}{4}\Bigg| \qquad (6) \\
&\quad + \Bigg|\sin\frac{(\lambda-u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4}\cos\frac{(\lambda+u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4} \\
&\qquad \times \cos\frac{\lambda(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}\sin\frac{\lambda(t+s)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{2}\Bigg| \\
&\quad + \Bigg|\sin\frac{u(s-t)\sin\varphi}{2}\cos\frac{(\lambda-u)(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8} \\
&\qquad \times \sin\frac{(\lambda-u)(t+s)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{4}\cos\frac{(\lambda+u)(t+s)\sin\varphi}{4}\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg|\sin\frac{(\lambda-u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\sin\frac{u(s-t)\sin\varphi}{2}\\
&\quad\times\sin\frac{(\lambda+u)(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\\
&\qquad\times\sin\frac{(\lambda+u)(t+s)\cos(\varphi+\frac{\pi}{2k})\sin\frac{\pi}{2k}}{4}\Bigg|\\
&+\Bigg|\sin\frac{(\lambda-u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\cos\frac{(\lambda+u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\\
&\quad\times\cos\frac{u(s-t)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}\\
&\qquad\times\sin\frac{u(s-t)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{2}\Bigg|\Bigg]\,d\varphi\\
&\le 4^{1-\alpha}\pi|\lambda-u|^\alpha|s-t|^\alpha\\
&\quad\times\left(\frac{\pi}{2k}\right)^\alpha\left(1+\frac{|\lambda+u|^\alpha|s-t|^\alpha}{4^\alpha}+\frac{|t+s|^\alpha(\lambda^\alpha+2u^\alpha)}{2^\alpha}+\frac{|t+s|^{2\alpha}u^\alpha|\lambda+u|^\alpha}{4^\alpha\cdot 2^\alpha}\right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|K_2|&\le 4^{1-\alpha}\pi|\lambda-u|^\alpha|s-t|^\alpha\\
&\quad\times\left(\frac{\pi}{2k}\right)^\alpha\left(1+\frac{|\lambda+u|^\alpha|s-t|^\alpha}{4^\alpha}+\frac{|t+s|^\alpha(\lambda^\alpha+2u^\alpha)}{2^\alpha}+\frac{|t+s|^{2\alpha}u^\alpha|\lambda+u|^\alpha}{4^\alpha\cdot 2^\alpha}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
&|J_k(t\lambda)-J_k(tu)-J_k(s\lambda)+J_k(su)|\\
&\quad\le 2\cdot 4^{1-\alpha}|\lambda-u|^\alpha|s-t|^\alpha\\
&\qquad\times\left(\frac{\pi}{2k}\right)^\alpha\left(1+\frac{|\lambda+u|^\alpha|s-t|^\alpha}{4^\alpha}+\frac{|t+s|^\alpha(\lambda^\alpha+2u^\alpha)}{2^\alpha}+\frac{|t+s|^{2\alpha}u^\alpha|\lambda+u|^\alpha}{4^\alpha\cdot 2^\alpha}\right).
\end{aligned}
$$

□

Lemma 3.5. *Let* $\frac{1}{2}<\alpha\le 1$ *and*

$$\int_0^\infty \lambda^{2\alpha}\,dF(\lambda)<\infty.$$

Then

$$
\begin{aligned}
&\tau^2\left(\sum_{k=1}^{M}\cos(kx)\sum_{l=0}^{N-1}\int_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l)-J_k(s\lambda)+J_k(s\zeta_l))\,d\eta_{1,k}(\lambda)\right)\\
&\quad\le 4^{2(2-\alpha)}|s-t|^{2\alpha}\left(\frac{\pi}{2}\right)^{2\alpha}M\left(\sum_{k=1}^{M}\cos^2(kx)\cdot\frac{1}{k^{2\alpha}}\right)\\
&\qquad\times\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\Bigg(b_l^2+\Bigg[\left(\frac{|s-t|(1+C)}{4}\right)^{2\alpha}+\left(\frac{|t+s|}{2}\right)^{2\alpha}(1+2C^\alpha)\\
&\qquad\qquad+\left(\frac{|t+s|^2\lambda_{l+1}(1+C)}{8}\right)^{2\alpha}\Bigg]\\
&\qquad\qquad\times\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}\,dF(\lambda)\Bigg)
\end{aligned}
$$

$$+ 18 \cdot 4^{3-2\alpha}|s-t|^{2\alpha}$$
$$\times M \sum_{k=1}^{M} \cos^2(kx) \left(\int_{\Lambda}^{\infty} |\lambda - \Lambda|^{2\alpha} dF(\lambda) + 2^{2\alpha}\Lambda^{2\alpha}(F(+\infty) - F(\Lambda)) \right)$$

and

$$\tau^2 \left(\sum_{k=1}^{M} \sin(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l) - J_k(s\lambda) + J_k(s\zeta_l))\, d\eta_{2,k}(\lambda) \right)$$
$$\le 4^{2(2-\alpha)}|s-t|^{2\alpha} \left(\frac{\pi}{2}\right)^{2\alpha} M \left(\sum_{k=1}^{M} \sin^2(kx) \cdot \frac{1}{k^{2\alpha}} \right)$$
$$\times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha}$$
$$\times \left(b_l^2 + \left[\left(\frac{|s-t|(1+C)}{4} \right)^{2\alpha} + \left(\frac{|t+s|}{2} \right)^{2\alpha} (1 + 2C^{\alpha}) \right.\right.$$
$$\left.\left. + \left(\frac{|t+s|^2 \lambda_{l+1}(1+C)}{8} \right)^{2\alpha} \right] \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha}\, dF(\lambda) \right)$$
$$+ 18 \cdot 4^{3-2\alpha}|s-t|^{2\alpha}$$
$$\times M \sum_{k=1}^{M} \sin^2(kx) \left(\int_{\Lambda}^{\infty} |\lambda - \Lambda|^{2\alpha}\, dF(\lambda) + 2^{2\alpha}\Lambda^{2\alpha}(F(+\infty) - F(\Lambda)) \right),$$

where

$$C = \max_{0<l\le N-2} \frac{\lambda_{l+1}}{\lambda_l}.$$

Proof. Since

$$(a_1 + a_2 + \cdots + a_n)^2 \le n \left(a_1^2 + a_2^2 + \cdots + a_n^2 \right)$$

for all real $a_1, a_2, \ldots, a_n$, we derive from Lemmas 2.1 and 2.2 that

$$\tau^2 \left(\sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l) - J_k(s\lambda) + J_k(s\zeta_l)) d\eta_{1,k}(\lambda) \right)$$
$$\le M \sum_{k=1}^{M} \cos^2(kx) \sum_{l=0}^{N-1} \tau^2 \left(\int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l) - J_k(s\lambda) + J_k(s\zeta_l)) d\eta_{1,k}(\lambda) \right)$$
$$\le M \sum_{k=1}^{M} \cos^2(kx) \sum_{l=0}^{N-1} \theta^2 \left(\int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l) - J_k(s\lambda) + J_k(s\zeta_l)) d\eta_{1,k}(\lambda) \right)$$
$$= M \sum_{k=1}^{M} \cos^2(kx) \sum_{l=0}^{N-1} \sup_{m\ge 1} \left[\frac{2^m \cdot m!}{(2m)!} \mathsf{E} \left(\int_{\lambda_l}^{\lambda_{l+1}} \big(J_k(t\lambda) - J_k(t\zeta_l) \right.\right.$$
$$\left.\left. - J_k(s\lambda) + J_k(s\zeta_l) \big)\, d\eta_{1,k}(\lambda) \right)^{2m} \right]^{\frac{1}{m}}.$$

Since

$$\mathsf{E}\,\xi = 0, \qquad \mathsf{E}\,\xi^{2k+1} = 0, \qquad \mathsf{E}\,\xi^{2k} = \frac{(2k)!}{2^k \cdot k!}\sigma^{2k}$$

for a centered Gaussian random variable ξ and since the random variables ζ_l do not depend on $\eta_{i,k}(\lambda)$, $i=1,2$, Fubini's theorem, Cauchy–Bunyakovskiĭ inequality, and Lemma 3.4 with $l \le N-2$ imply that

$$\mathsf{E}\left(\int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda)-J_k(t\zeta_l)-J_k(s\lambda)+J_k(s\zeta_l))\,d\eta_{1,k}(\lambda)\right)^{2m}$$

$$\le \frac{(2m)!}{2^m\cdot m!}\,\mathsf{E}\left(\int_{\lambda_l}^{\lambda_{l+1}} |J_k(t\lambda)-J_k(t\zeta_l)-J_k(s\lambda)+J_k(s\zeta_l)|^2\,dF(\lambda)\right)^m$$

$$\le \frac{(2m)!}{2^m\cdot m!}\,\mathsf{E}\Bigg(\int_{\lambda_l}^{\lambda_{l+1}} \Bigg(4\cdot 4^{2(1-\alpha)}|\lambda-\zeta_l|^{2\alpha}|s-t|^{2\alpha}\left(\frac{\pi}{2k}\right)^{2\alpha} \times\Bigg(1+\frac{|\lambda+\zeta_l|^{\alpha}|s-t|^{\alpha}}{4^{\alpha}} + \frac{|t+s|^{\alpha}(\lambda^{\alpha}+2\zeta_l^{\alpha})}{2^{\alpha}} + \frac{|t+s|^{2\alpha}\zeta_l^{\alpha}|\lambda+\zeta_l|^{\alpha}}{4^{\alpha}\cdot 2^{\alpha}}\Bigg)^2\Bigg)\,dF(\lambda)\Bigg)^m$$

$$= \frac{(2m)!}{2^m\cdot m!}4^m 4^{2m(1-\alpha)}\left(\frac{\pi}{2k}\right)^{2m\alpha}|s-t|^{2m\alpha} \times\mathsf{E}\Bigg(\int_{\lambda_l}^{\lambda_{l+1}} |\lambda-\zeta_l|^{2\alpha}\Bigg(1+\frac{|\lambda+\zeta_l|^{\alpha}|s-t|^{\alpha}}{4^{\alpha}} + \frac{|t+s|^{\alpha}(\lambda^{\alpha}+2\zeta_l^{\alpha})}{2^{\alpha}} + \frac{|t+s|^{2\alpha}\zeta_l^{\alpha}|\lambda+\zeta_l|^{\alpha}}{4^{\alpha}\cdot 2^{\alpha}}\Bigg)^2\,dF(\lambda)\Bigg)^m$$

$$= \frac{(2m)!}{2^m\cdot m!}4^m 4^{2m(1-\alpha)}\left(\frac{\pi}{2k}\right)^{2m\alpha}|s-t|^{2m\alpha} \times\int_{\lambda_l}^{\lambda_{l+1}}\Bigg(\int_{\lambda_l}^{\lambda_{l+1}} |\lambda-u|^{2\alpha}\Bigg(1+\frac{|\lambda+u|^{\alpha}|s-t|^{\alpha}}{4^{\alpha}} + \frac{|t+s|^{\alpha}(\lambda^{\alpha}+2u^{\alpha})}{2^{\alpha}} + \frac{|t+s|^{2\alpha}u^{\alpha}|\lambda+u|^{\alpha}}{4^{\alpha}\cdot 2^{\alpha}}\Bigg)^2\,dF(\lambda)\Bigg)^m\,dF_l(u)$$

$$\le \frac{(2m)!}{2^m\cdot m!}4^m\cdot 4^{2m(1-\alpha)}\left(\frac{\pi}{2k}\right)^{2m\alpha}|s-t|^{2m\alpha}|\lambda_{l+1}-\lambda_l|^{2m\alpha} \times\int_{\lambda_l}^{\lambda_{l+1}}\Bigg(\int_{\lambda_l}^{\lambda_{l+1}}\Bigg(1+\frac{\lambda^{\alpha}\left(1+\frac{u}{\lambda}\right)^{\alpha}|s-t|^{\alpha}}{4^{\alpha}} + \frac{|t+s|^{\alpha}\lambda^{\alpha}(1+2\left(\frac{u}{\lambda}\right)^{\alpha})}{2^{\alpha}} + \frac{|t+s|^{2\alpha}u^{\alpha}\lambda^{\alpha}\left(1+\frac{u}{\lambda}\right)^{\alpha}}{4^{\alpha}\cdot 2^{\alpha}}\Bigg)^2\,dF(\lambda)\Bigg)^m\,dF_l(u)$$

$$
\begin{aligned}
&\leq \frac{(2m)!}{2^m \cdot m!} 4^m \cdot 4^{2m(1-\alpha)} \left(\frac{\pi}{2k}\right)^{2m\alpha} |s-t|^{2m\alpha} |\lambda_{l+1} - \lambda_l|^{2m\alpha} \\
&\times \left(\int_{\lambda_l}^{\lambda_{l+1}} \left(1 + \frac{\lambda^\alpha \left(1 + \frac{\lambda_{l+1}}{\lambda_l}\right)^\alpha |s-t|^\alpha}{4^\alpha} \right.\right. \\
&\qquad + \frac{|t+s|^\alpha \lambda^\alpha \left(1 + 2\left(\frac{\lambda_{l+1}}{\lambda_l}\right)^\alpha\right)}{2^\alpha} \\
&\qquad \left.\left. + \frac{|t+s|^{2\alpha} \lambda_{l+1}^\alpha \lambda^\alpha \left(1 + \frac{\lambda_{l+1}}{\lambda_l}\right)^\alpha}{4^\alpha \cdot 2^\alpha} \right)^2 dF(\lambda) \right)^m \\
&\leq \frac{(2m)!}{2^m \cdot m!} 4^m \cdot 4^{2m(1-\alpha)} \left(\frac{\pi}{2k}\right)^{2m\alpha} |s-t|^{2m\alpha} |\lambda_{l+1} - \lambda_l|^{2m\alpha} \\
&\times \left(4 \int_{\lambda_l}^{\lambda_{l+1}} \left(1 + \frac{\lambda^{2\alpha}(1+C)^{2\alpha}|s-t|^{2\alpha}}{4^{2\alpha}} + \frac{|t+s|^{2\alpha}\lambda^{2\alpha}(1+2C^\alpha)^2}{2^{2\alpha}} \right.\right. \\
&\qquad \left.\left. + \frac{|t+s|^{4\alpha}\lambda_{l+1}^{2\alpha}\lambda^{2\alpha}(1+C)^{2\alpha}}{4^{2\alpha} \cdot 2^{2\alpha}} \right) dF(\lambda) \right)^m \\
&= \frac{(2m)!}{2^m \cdot m!} 4^m \cdot 4^{2m(1-\alpha)} \left(\frac{\pi}{2k}\right)^{2m\alpha} |s-t|^{2m\alpha} |\lambda_{l+1} - \lambda_l|^{2m\alpha} 4^m \\
&\times \left[b_l^2 + \left(\frac{|s-t|(1+C)}{4}\right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha}\, dF(\lambda) \right. \\
&\qquad + \left(\frac{|t+s|}{2}\right)^{2\alpha} (1+2C^\alpha)^2 \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha}\, dF(\lambda) \\
&\qquad \left. + \left(\frac{|t+s|^2 \lambda_{l+1}(1+C)}{8}\right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha}\, dF(\lambda) \right]^m .
\end{aligned}
$$

Consider the case of $l = N-1$. Applying the inequality $|\sin x| \leq x^\alpha$, $\frac{1}{2} < \alpha \leq 1$, to those terms in (6) that do not contain $|\lambda + u|$ and $|t+s|$ and bounding the remaining ones with sin and cos by 1 we obtain

$$
\begin{aligned}
&\mathsf{E} \left(\int_\Lambda^\infty (J_k(t\lambda) - J_k(t\zeta_l) - J_k(s\lambda) + J_k(s\zeta_l)) d\eta_{1,k}(\lambda) \right)^{2m} \\
&\quad \leq \frac{(2m)!}{2^m \cdot m!} \left(\int_\Lambda^\infty |J_k(t\lambda) - J_k(t\Lambda) - J_k(s\lambda) + J_k(s\Lambda)|^2\, dF(\lambda) \right)^m \\
&\quad \leq \frac{(2m)!}{2^m \cdot m!} \left(64 \int_\Lambda^\infty \left(3\frac{|\lambda - \Lambda|^\alpha |s-t|^\alpha}{4^\alpha} + 3\frac{\Lambda^\alpha |s-t|^\alpha}{2^\alpha} \right)^2 dF(\lambda) \right)^m \\
&\quad = \frac{(2m)!}{2^m \cdot m!} 4^{(3-2\alpha)m} 9^m |s-t|^{2m\alpha} \left(\int_\Lambda^\infty \left(|\lambda - \Lambda|^\alpha + 2^\alpha \Lambda^\alpha\right)^2\, dF(\lambda) \right)^m \\
&\quad \leq \frac{(2m)!}{2^m \cdot m!} 4^{(3-2\alpha)m} 18^m |s-t|^{2m\alpha} \left(\int_\Lambda^\infty \left(|\lambda - \Lambda|^{2\alpha} + 2^{2\alpha}\Lambda^{2\alpha}\right)\, dF(\lambda) \right)^m \\
&\quad = \frac{(2m)!}{2^m \cdot m!} 4^{(3-2\alpha)m} 18^m |s-t|^{2m\alpha} \left(\int_\Lambda^\infty |\lambda - \Lambda|^{2\alpha}\, dF(\lambda) + \int_\Lambda^\infty 2^{2\alpha}\Lambda^{2\alpha}\, dF(\lambda) \right)^m
\end{aligned}
$$

$$= \frac{(2m)!}{2^m \cdot m!} 4^{(3-2\alpha)m} 18^m |s-t|^{2m\alpha} \times \left(\int_\Lambda^\infty |\lambda - \Lambda|^{2\alpha} \, dF(\lambda) + 2^{2\alpha} \Lambda^{2\alpha} (F(+\infty) - F(\Lambda)) \right)^m .$$

Thus

$$\tau^2 \left(\sum_{k=1}^M \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l) - J_k(s\lambda) + J_k(s\zeta_l)) d\eta_{1,k}(\lambda) \right)$$

$$\le M \sum_{k=1}^M \cos^2(kx) \times \Bigg[4^{2(2-\alpha)} |s-t|^{2\alpha} \left(\frac{\pi}{2k}\right)^{2\alpha} \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \times \Bigg(b_l^2 + \Bigg(\left(\left(\frac{|s-t|(1+C)}{4} \right)^{2\alpha} + \left(\frac{|t+s|}{2} \right)^{2\alpha} (1 + 2C^\alpha) + \left(\frac{|t+s|^2 \lambda_{l+1}(1+C)}{8} \right)^{2\alpha} \right) \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} \, dF(\lambda) \Bigg) + 18 \cdot 4^{3-2\alpha} |s-t|^{2\alpha} \left(\int_\Lambda^\infty |\lambda - \Lambda|^{2\alpha} \, dF(\lambda) + 2^{2\alpha} \Lambda^{2\alpha} (F(+\infty) - F(\Lambda)) \right) \Bigg]$$

$$= 4^{2(2-\alpha)} |s-t|^{2\alpha} \left(\frac{\pi}{2} \right)^{2\alpha} M \sum_{k=1}^M \left(\cos^2(kx) \cdot \frac{1}{k^{2\alpha}} \right) \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \Bigg(b_l^2 + \Bigg[\left(\frac{|s-t|(1+C)}{4} \right)^{2\alpha} + \left(\frac{|t+s|}{2} \right)^{2\alpha} (1 + 2C^\alpha) + \left(\frac{|t+s|^2 \lambda_{l+1}(1+C)}{8} \right)^{2\alpha} \Bigg] \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} \, dF(\lambda) \Bigg)$$

$$+ 18 \cdot 4^{3-2\alpha} |s-t|^{2\alpha} \times M \sum_{k=1}^M \cos^2(kx) \left(\int_\Lambda^\infty |\lambda - \Lambda|^{2\alpha} dF(\lambda) + 2^{2\alpha} \Lambda^{2\alpha} (F(+\infty) - F(\Lambda)) \right).$$

The proof of the second inequality is the same. □

Lemma 3.6. *Let the integral* $\int_0^\infty \lambda^{2\alpha} \, dF(\lambda)$ *converge for all* $\frac{1}{2} < \alpha \le 1$. *Then*

$$\tau^2 \left(\sum_{k=1}^M (\cos(kx) - \cos(ky)) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(s\lambda) - J_k(s\zeta_l)) \, d\eta_{1,k}(\lambda) \right)$$

$$\le 2 \cdot 4^{2(1-\alpha)} s^{2\alpha} \pi^{2\alpha} M \sum_{k=1}^M (\cos(kx) - \cos(ky))^2 \frac{1}{k^{2\alpha}} \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{s(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} \, dF(\lambda) \right)$$

$$+ 4M \sum_{k=1}^M (\cos(kx) - \cos(ky))^2 (F(+\infty) - F(\Lambda))$$

and

$$\tau^2\left(\sum_{k=1}^{M}(\sin(kx)-\sin(ky))\sum_{l=0}^{N-1}\int_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-J_k(s\zeta_l))\,d\eta_{2,k}(\lambda)\right)$$
$$\leq 2\cdot 4^{2(1-\alpha)}s^{2\alpha}\pi^{2\alpha}M$$
$$\times\sum_{k=1}^{M}(\sin(kx)-\sin(ky))^2\frac{1}{k^{2\alpha}}$$
$$\times\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left(\frac{s(1+C)}{2}\right)^{2\alpha}\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}\,dF(\lambda)\right)$$
$$+4M\sum_{k=1}^{M}(\sin(kx)-\sin(ky))^2(F(+\infty)-F(\Lambda)).$$

Proof. Lemmas 2.1 and 2.2 imply

$$\tau^2\left(\sum_{k=1}^{M}(\cos(kx)-\cos(ky))\sum_{l=0}^{N-1}\int_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-J_k(s\zeta_l))\,d\eta_{1,k}(\lambda)\right)$$
$$\leq M\sum_{k=1}^{M}(\cos(kx)-\cos(ky))^2\sum_{l=0}^{N-1}\tau^2\left(\int_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-J_k(s\zeta_l))\,d\eta_{1,k}(\lambda)\right)$$
$$\leq M\sum_{k=1}^{M}(\cos(kx)-\cos(ky))^2\sum_{l=0}^{N-1}\theta^2\left(\int_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-J_k(s\zeta_l))\,d\eta_{1,k}(\lambda)\right)$$
$$=M\sum_{k=1}^{M}(\cos(kx)-\cos(ky))^2$$
$$\times\sum_{l=0}^{N-1}\sup_{m\geq 1}\left[\frac{2^m\cdot m!}{(2m)!}\mathsf{E}\left(\int_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-J_k(s\zeta_l))\,d\eta_{1,k}(\lambda)\right)^{2m}\right]^{\frac{1}{m}}.$$

We use Lemma 2.4 and a reasoning similar to that in the proof of Lemma 3.1 to estimate the expression

$$\left[\frac{2^m\cdot m!}{(2m)!}\mathsf{E}\left(\int_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-J_k(s\zeta_l))\,d\eta_{1,k}(\lambda)\right)^{2m}\right]^{\frac{1}{m}}.$$

With this estimate, we obtain

$$\tau^2\left(\sum_{k=1}^{M}(\cos(kx)-\cos(ky))\sum_{l=0}^{N-1}\int_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-J_k(s\zeta_l))\,d\eta_{1,k}(\lambda)\right)$$
$$\leq 2\cdot 4^{2(1-\alpha)}s^{2\alpha}\pi^{2\alpha}M\sum_{k=1}^{M}(\cos(kx)-\cos(ky))^2$$
$$\times\frac{1}{k^{2\alpha}}\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left(\frac{s(1+C)}{2}\right)^{2\alpha}\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}\,dF(\lambda)\right)$$
$$+4M\sum_{k=1}^{M}(\cos(kx)-\cos(ky))^2(F(+\infty)-F(\Lambda)).$$

The second inequality is proved similarly. □

Lemma 3.7. *Let the integral $\int_0^\infty \lambda^{2\nu}\, dF(\lambda)$ converge for $\nu > \frac{1}{2}$. Then*

$$\tau^2\left(\sum_{k=M+1}^{\infty}\cos(kx)\int_0^\infty (J_k(t\lambda)-J_k(s\lambda))\,d\eta_{1,k}(\lambda)\right)$$
$$\leq \frac{2\cdot 4^{2(1-\alpha)}\pi^{2\alpha}}{\left(\ln\left(1+\frac{1}{|s-t|}\right)\right)^{2\delta}}\sum_{k=M+1}^{\infty}\left(\cos^2(kx)\cdot\frac{1}{k^{2\alpha}}\right)$$
$$\times\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_0^\infty \lambda^{2\alpha}\,dF(\lambda)+\left(\frac{\delta}{\beta}\right)^{2\delta}|s+t|^{2\alpha}\int_0^\infty \lambda^{2\nu}\,dF(\lambda)\right)$$

and

$$\tau^2\left(\sum_{k=M+1}^{\infty}\sin(kx)\int_0^\infty (J_k(t\lambda)-J_k(s\lambda))d\eta_{2,k}(\lambda)\right)$$
$$\leq \frac{2\cdot 4^{2(1-\alpha)}\pi^{2\alpha}}{\left(\ln\left(1+\frac{1}{|s-t|}\right)\right)^{2\delta}}\sum_{k=M+1}^{\infty}\left(\sin^2(kx)\cdot\frac{1}{k^{2\alpha}}\right)$$
$$\times\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_0^\infty \lambda^{2\alpha}\,dF(\lambda)+\left(\frac{\delta}{\beta}\right)^{2\delta}|s+t|^{2\alpha}\int_0^\infty \lambda^{2\nu}\,dF(\lambda)\right),$$

where $\frac{\alpha}{\delta} \leq 1$, $\frac{1}{2} < \alpha \leq 1$, $\delta > 0$, and $0 < \beta \leq 1$.

Proof. Lemmas 2.1 and 2.2 imply that

$$\tau^2\left(\sum_{k=M+1}^{\infty}\cos(kx)\int_0^\infty (J_k(t\lambda)-J_k(s\lambda))\,d\eta_{1,k}(\lambda)\right)$$
$$\leq \sum_{k=M+1}^{\infty}\cos^2(kx)\tau^2\left(\int_0^\infty (J_k(t\lambda)-J_k(s\lambda))\,d\eta_{1,k}(\lambda)\right)$$
$$\leq \sum_{k=M+1}^{\infty}\cos^2(kx)\theta^2\left(\int_0^\infty (J_k(t\lambda)-J_k(s\lambda))\,d\eta_{1,k}(\lambda)\right)$$
$$= \sum_{k=M+1}^{\infty}\cos^2(kx)\sup_{m\geq 1}\left[\frac{2^m\cdot m!}{(2m)!}\mathsf{E}\left(\int_0^\infty (J_k(t\lambda)-J_k(s\lambda))\,d\eta_{1,k}(\lambda)\right)^{2m}\right]^{\frac{1}{m}}.$$

Given $h > 0$ and $0 < \gamma \leq 1$ we obtain

$$\ln\left(1+\frac{1}{h}\right)=\frac{1}{\gamma}\ln\left(1+\frac{1}{h}\right)^\gamma \leq \frac{1}{\gamma}\ln\left(1+\left(\frac{1}{h}\right)^\gamma\right)\leq \frac{1}{h^\gamma\cdot\gamma},$$

whence $h^\gamma \leq \frac{1}{\gamma\ln\left(1+\frac{1}{h}\right)}$. Thus

$$h^{\gamma\delta} \leq \frac{1}{\gamma^\delta\left(\ln\left(1+\frac{1}{h}\right)\right)^\delta}, \qquad \delta > 0.$$

If $\gamma\cdot\delta = \alpha$, then

$$h^\alpha \leq \frac{1}{\left(\frac{\alpha}{\delta}\right)^\delta\left(\ln\left(1+\frac{1}{h}\right)\right)^\delta}. \tag{7}$$

We conclude from Lemma 3.3 and inequality (7) that

$$
\begin{aligned}
&\mathsf{E}\left(\int_0^\infty (J_k(t\lambda) - J_k(s\lambda))\, d\eta_{1,k}(\lambda)\right)^{2m} \\
&\quad \le \frac{(2m)!}{2^m \cdot m!}\left(\int_0^\infty (J_k(t\lambda) - J_k(s\lambda))\, dF(\lambda)\right)^m \\
&\quad \le \frac{(2m)!}{2^m \cdot m!}\left(\int_0^\infty \left(4^{1-\alpha}\pi^\alpha \frac{1}{k^\alpha}(\lambda^\alpha |s-t|^\alpha + \lambda^{\alpha+\beta}|s-t|^\beta \cdot |s+t|^\alpha)\right)^2 dF(\lambda)\right)^m \\
&\quad \le \frac{(2m)!}{2^m \cdot m!}\frac{2\cdot 4^{2m(1-\alpha)}\pi^{2m\alpha}}{k^{2m\alpha}} \\
&\qquad \times \left(|s-t|^{2\alpha}\int_0^\infty \lambda^{2\alpha}\, dF(\lambda) + |s-t|^{2\beta}\cdot |s+t|^{2\alpha}\int_0^\infty \lambda^{2(\alpha+\beta)}\, dF(\lambda)\right)^m \\
&\quad \le \frac{(2m)!}{2^m \cdot m!}\frac{2\cdot 4^{2m(1-\alpha)}\pi^{2m\alpha}}{k^{2m\alpha}} \frac{1}{\left(\ln\left(1+\frac{1}{|s-t|}\right)\right)^{2m\delta}} \\
&\qquad \times \left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_0^\infty \lambda^{2\alpha}\, dF(\lambda) + \left(\frac{\delta}{\beta}\right)^{2\delta}\cdot |s+t|^{2\alpha}\int_0^\infty \lambda^{2(\alpha+\beta)}\, dF(\lambda)\right)^m .
\end{aligned}
$$

We introduce the numbers α and β as follows $\alpha = \frac{1}{2}+\beta$, $\beta = \frac{1}{2}(\nu - \frac{1}{2})$, where $\frac{1}{2} < \alpha \le 1$, $0 < \beta \le 1$, and $\nu > \frac{1}{2}$. Then

$$
\begin{aligned}
&\tau^2\left(\sum_{k=M+1}^\infty \cos(kx)\int_0^\infty (J_k(t\lambda) - J_k(s\lambda))\, d\eta_{1,k}(\lambda)\right) \\
&\quad \le \frac{2\cdot 4^{2(1-\alpha)}\pi^{2\alpha}}{\left(\ln\left(1+\frac{1}{|s-t|}\right)\right)^{2\delta}} \sum_{k=M+1}^\infty \left(\cos^2(kx)\cdot \frac{1}{k^{2\alpha}}\right) \\
&\qquad \times \left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_0^\infty \lambda^{2\alpha}\, dF(\lambda) + \left(\frac{\delta}{\beta}\right)^{2\delta}|s+t|^{2\alpha}\int_0^\infty \lambda^{2\nu}\, dF(\lambda)\right).
\end{aligned}
$$

The proof of the second inequality is the same. □

Lemma 3.8. *Let the integral $\int_0^\infty \lambda^{2\alpha}\, dF(\lambda)$ converge for $\frac{1}{2} < \alpha \le 1$. Then*

$$
\begin{aligned}
&\tau^2\left(\sum_{k=M+1}^\infty (\cos(kx) - \cos(ky))\int_0^\infty J_k(s\lambda)\, d\eta_{1,k}(\lambda)\right) \\
&\quad \le 2^{2(1-\alpha)}s^{2\alpha}\pi^{2\alpha}\sum_{k=M+1}^\infty (\cos(kx) - \cos(ky))^2 \cdot \frac{1}{k^{2\alpha}}\int_0^\infty \lambda^{2\alpha}\, dF(\lambda)
\end{aligned}
$$

and

$$
\begin{aligned}
&\tau^2\left(\sum_{k=M+1}^\infty (\sin(kx) - \sin(ky))\int_0^\infty J_k(s\lambda) d\eta_{2,k}(\lambda)\right) \\
&\quad \le 2^{2(1-\alpha)}s^{2\alpha}\pi^{2\alpha}\sum_{k=M+1}^\infty (\sin(kx) - \sin(ky))^2 \cdot \frac{1}{k^{2\alpha}}\int_0^\infty \lambda^{2\alpha}\, dF(\lambda).
\end{aligned}
$$

Proof. Lemma 3.8 follows from Lemma 2.3. □

Theorem 3.2. *Let $X(t,x)$ and $\hat{X}(t,x)$ be defined by (1) and (2), respectively, and*

$$\sigma(h) = \sup_{\substack{|t-s|\le h \\ |x-y|\le h}} \tau(\chi_M(t,x) - \chi_M(s,y)),$$

where $\chi_M(t,x)$ is defined by (3). Assume that

$$\int_0^\infty \lambda^{2\nu}\, dF(\lambda) < \infty$$

for $\nu > \frac{1}{2}$. Then

$$\begin{aligned}
\sigma(h) \le{} & \frac{1}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{\delta}} \\
& \times \Bigg[2\cdot 4^{2(2-\alpha)} \left(\frac{\delta}{\alpha}\right)^{2\delta} \left(\frac{\pi}{2}\right)^{2\alpha} \frac{M}{2\alpha-1}\left(2\alpha - \frac{1}{M^{2\alpha-1}}\right) \sum_{l=0}^{N-2} |\lambda_{l+1}-\lambda_l|^{2\alpha} \\
& \qquad \times \left(b_l^2 + \left(\left(\frac{T(1+C)}{4}\right)^{2\alpha} + T^{2\alpha}(1+2C^{\alpha}) \right.\right. \\
& \qquad\qquad \left.\left. + \left(\frac{T^2\Lambda(1+C)}{2}\right)^{2\alpha}\right) \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha}\, dF(\lambda) \right) \\
& \qquad + 9\cdot 4^{4-2\alpha}\cdot M^2 \left(\frac{\delta}{\alpha}\right)^{2\delta} \left(\int_\Lambda^\infty |\lambda-\Lambda|^{2\alpha}\, dF(\lambda) + 2^{2\alpha}\Lambda^{2\alpha}(F(+\infty)-F(\Lambda)) \right) \\
& \qquad + 4^{4-2\alpha} T^{2\alpha}\pi^{2\alpha} M \sum_{k=1}^{M} \frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}} \\
& \qquad\qquad \times \sum_{l=0}^{N-2} |\lambda_{l+1}-\lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{T(1+C)}{2}\right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha}\, dF(\lambda) \right) \\
& \qquad + 16M(F(+\infty)-F(\Lambda)) \sum_{k=1}^{M} \left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta} \\
& \qquad + \frac{4^{3-2\alpha}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}} \\
& \qquad\quad \times \left(\left(\frac{\delta}{\alpha}\right)^{2\delta} \int_0^\infty \lambda^{2\alpha}\, dF(\lambda) + (2T)^{2\alpha} \left(\frac{\delta}{\beta}\right)^{2\delta} \int_0^\infty \lambda^{2\nu}\, dF(\lambda) \right) \\
& \qquad\qquad + 2^{2(2-\alpha)} T^{2\alpha}\pi^{2\alpha} \int_0^\infty \lambda^{2\alpha}\, dF(\lambda) \sum_{k=M+1}^{\infty} \frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}} \Bigg]^{\frac{1}{2}},
\end{aligned}$$

where

$$C = \max_{0<l\le N-2} \frac{\lambda_{l+1}}{\lambda_l}, \qquad \frac{1}{2} < \alpha \le 1, \quad \frac{\alpha}{\delta} \le 1, \quad \delta > 0, \quad 0 < \beta \le 1.$$

Proof. Lemmas 3.5–3.8 imply

$$
\begin{aligned}
&\tau^2(\chi_M(t,x)-\chi_M(s,y)) \\
&\quad\le [\tau(\chi_{M,1}(t,x)-\chi_{M,1}(s,y))+\tau(\chi_{M,2}(t,x)-\chi_{M,2}(s,y))]^2 \\
&\quad\le 2\tau^2(\chi_{M,1}(t,x)-\chi_{M,1}(s,y))+2\tau^2(\chi_{M,2}(t,x)-\chi_{M,2}(s,y)) \\
&\quad\le 2\cdot 4^{2(2-\alpha)}|s-t|^{2\alpha}\left(\frac{\pi}{2}\right)^{2\alpha} M \\
&\qquad\times\sum_{k=1}^{M}\frac{1}{k^{2\alpha}}\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha} \\
&\qquad\times\left(b_l^2+\left[\left(\frac{|s-t|(1+C)}{4}\right)^{2\alpha}+\left(\frac{|t+s|}{2}\right)^{2\alpha}(1+2C^\alpha)\right.\right. \\
&\qquad\qquad\left.\left.+\left(\frac{|t+s|^2\lambda_{l+1}(1+C)}{8}\right)^{2\alpha}\right]\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}\,dF(\lambda)\right) \\
&\qquad+9\cdot 4^{4-2\alpha}|s-t|^{2\alpha}M^2\left(\int_\Lambda^\infty|\lambda-\Lambda|^{2\alpha}\,dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}(F(+\infty)-F(\Lambda))\right) \\
&\qquad+4^{3-2\alpha}s^{2\alpha}\pi^{2\alpha}M\left(\sum_{k=1}^{M}\left((\cos(kx)-\cos(ky))^2+(\sin(kx)-\sin(ky))^2\right)\frac{1}{k^{2\alpha}}\right) \\
&\qquad\quad\times\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left(\frac{s(1+C)}{2}\right)^{2\alpha}\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}\,dF(\lambda)\right) \\
&\qquad+8M(F(+\infty)-F(\Lambda))\sum_{k=1}^{M}\left((\cos(kx)-\cos(ky))^2+(\sin(kx)-\sin(ky))^2\right) \\
&\qquad+\frac{4^{3-2\alpha}\pi^{2\alpha}}{\left(\ln\left(1+\frac{1}{|s-t|}\right)\right)^{2\delta}} \\
&\qquad\quad\times\sum_{k=M+1}^{\infty}\left(\frac{1}{k^{2\alpha}}\right)\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_0^\infty\lambda^{2\alpha}\,dF(\lambda)+\left(\frac{\delta}{\beta}\right)^{2\delta}|s+t|^{2\alpha}\int_0^\infty\lambda^{2\nu}\,dF(\lambda)\right) \\
&\qquad+2^{3-2\alpha}s^{2\alpha}\pi^{2\alpha}\int_0^\infty\lambda^{2\alpha}\,dF(\lambda) \\
&\qquad\quad\times\sum_{k=M+1}^{\infty}\frac{1}{k^{2\alpha}}\left((\cos(kx)-\cos(ky))^2++(\sin(kx)-\sin(ky))^2\right).
\end{aligned}
$$

Now we apply the inequality

$$
|\cos(kx)-\cos(ky)|\le\frac{\left(\ln\left(k^2+e^\delta\right)\right)^\delta}{\left(\ln\left(\frac{1}{|x-y|}+e^\delta\right)\right)^\delta}
$$

for some $\delta>0$ (this is inequality (10) in [7]). Since

$$\sum_{k=1}^{M}\frac{1}{k^{2\alpha}} \le 1+\sum_{k=2}^{M}\int_{k-1}^{k}\frac{1}{x^{2\alpha}}\,dx = 1+\int_{1}^{M}\frac{1}{x^{2\alpha}}\,dx$$
$$= 1+\frac{x^{1-2\alpha}}{1-2\alpha}\Big|_{1}^{M} = \frac{2\alpha}{2\alpha-1}-\frac{1}{(2\alpha-1)M^{2\alpha-1}}$$

for all $\frac{1}{2}<\alpha\le 1$ and

$$\sum_{k=M+1}^{\infty}\frac{1}{k^{2\alpha}} \le \sum_{k=M+1}^{\infty}\int_{k-1}^{k}\frac{1}{x^{2\alpha}}\,dx = \int_{M}^{\infty}\frac{1}{x^{2\alpha}}\,dx = \frac{x^{1-2\alpha}}{1-2\alpha}\Big|_{M}^{\infty} = \frac{1}{(2\alpha-1)M^{2\alpha-1}},$$

we have

$$\begin{aligned}
&\tau^2(\chi_M(t,x)-\chi_M(s,y)) \\
&\quad \le 2\cdot 4^{2(2-\alpha)}|s-t|^{2\alpha}\left(\frac{\pi}{2}\right)^{2\alpha}\frac{M}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right) \\
&\qquad \times \sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\Bigg(b_l^2+\Bigg[\left(\frac{|s-t|(1+C)}{4}\right)^{2\alpha}+\left(\frac{|t+s|}{2}\right)^{2\alpha}(1+2C^{\alpha}) \\
&\qquad\qquad +\left(\frac{|t+s|^2\lambda_{l+1}(1+C)}{8}\right)^{2\alpha}\Bigg]\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}\,dF(\lambda)\Bigg) \\
&\qquad + 9\cdot 4^{4-2\alpha}|s-t|^{2\alpha}\cdot M^2\left(\int_{\Lambda}^{\infty}|\lambda-\Lambda|^{2\alpha}\,dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}(F(+\infty)-F(\Lambda))\right) \\
&\qquad + \frac{4^{4-2\alpha}s^{2\alpha}\pi^{2\alpha}M}{\left(\ln\left(\frac{1}{|x-y|}+e^{\delta}\right)\right)^{2\delta}}\sum_{k=1}^{M}\frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}} \\
&\qquad \times \sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left(\frac{s(1+C)}{2}\right)^{2\alpha}\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}\,dF(\lambda)\right) \\
&\qquad + \frac{16M(F(+\infty)-F(\Lambda))}{\left(\ln\left(\frac{1}{|x-y|}+e^{\delta}\right)\right)^{2\delta}}\sum_{k=1}^{M}\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta} \\
&\qquad + \frac{4^{3-2\alpha}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}}\frac{1}{\left(\ln\left(1+\frac{1}{|s-t|}\right)\right)^{2\delta}} \\
&\qquad \times \left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_0^{\infty}\lambda^{2\alpha}\,dF(\lambda)+\left(\frac{\delta}{\beta}\right)^{2\delta}|s+t|^{2\alpha}\int_0^{\infty}\lambda^{2\alpha}\,dF(\lambda)\right) \\
&\qquad + \frac{2^{2(2-\alpha)}s^{2\alpha}\pi^{2\alpha}}{\left(\ln\left(\frac{1}{|x-y|}+e^{\delta}\right)\right)^{2\delta}}\int_0^{\infty}\lambda^{2\alpha}\,dF(\lambda)\sum_{k=M+1}^{\infty}\frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}}.
\end{aligned}$$

Therefore

$$
\begin{aligned}
&\sup_{\substack{|t-s|\le h\\|x-y|\le h}} \tau(\chi_M(t,x)-\chi_M(s,y))\\
&\quad\le \Bigg[2\cdot 4^{2(2-\alpha)}h^{2\alpha}\left(\frac{\pi}{2}\right)^{2\alpha}\frac{M}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\\
&\qquad\times\Bigg(b_l^2+\left[\left(\frac{T(1+C)}{4}\right)^{2\alpha}+T^{2\alpha}(1+2C^\alpha)+\left(\frac{T^2\Lambda(1+C)}{2}\right)^{2\alpha}\right]\\
&\qquad\qquad\times\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}\,dF(\lambda)\Bigg)\\
&\qquad+9\cdot 4^{4-2\alpha}h^{2\alpha}\cdot M^2\left(\int_\Lambda^\infty|\lambda-\Lambda|^{2\alpha}\,dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}(F(+\infty)-F(\Lambda))\right)\\
&\qquad+\frac{4^{4-2\alpha}T^{2\alpha}\pi^{2\alpha}M}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\left(\sum_{k=1}^{M}\frac{\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}}{k^{2\alpha}}\right)\\
&\qquad\times\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left(\frac{T(1+C)}{2}\right)^{2\alpha}\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}\,dF(\lambda)\right)\\
&\qquad+\frac{16M(F(+\infty)-F(\Lambda))}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\sum_{k=1}^{M}\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}\\
&\qquad+\frac{4^{3-2\alpha}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}}\frac{1}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\\
&\qquad\times\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_0^\infty\lambda^{2\alpha}\,dF(\lambda)+(2T)^{2\alpha}\left(\frac{\delta}{\beta}\right)^{2\delta}\int_0^\infty\lambda^{2\nu}\,dF(\lambda)\right)\\
&\qquad\qquad+\frac{2^{2(2-\alpha)}T^{2\alpha}\pi^{2\alpha}}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\int_0^\infty\lambda^{2\alpha}\,dF(\lambda)\sum_{k=M+1}^{\infty}\frac{\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}}{k^{2\alpha}}\Bigg]^{\frac{1}{2}}.
\end{aligned}
$$

Now inequality (7) implies

$$
\begin{aligned}
&\sup_{\substack{|t-s|\le h\\|x-y|\le h}} \tau(\chi_M(t,x)-\chi_M(s,y))\\
&\quad\le \Bigg[2\cdot 4^{2(2-\alpha)}\frac{1}{\left(\frac{\alpha}{\delta}\right)^{2\delta}\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\left(\frac{\pi}{2}\right)^{2\alpha}\frac{M}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)\\
&\qquad\times\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\\
&\qquad\qquad\times\Bigg(b_l^2+\left[\left(\frac{T(1+C)}{4}\right)^{2\alpha}+T^{2\alpha}(1+2C^\alpha)+\left(\frac{T^2\Lambda(1+C)}{2}\right)^{2\alpha}\right]\\
&\qquad\qquad\qquad\times\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}\,dF(\lambda)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{9 \cdot 4^{4-2\alpha} \cdot M^2}{\left(\frac{\alpha}{\delta}\right)^{2\delta} \left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}} \left(\int_{\Lambda}^{\infty} |\lambda - \Lambda|^{2\alpha}\, dF(\lambda) + 2^{2\alpha} \Lambda^{2\alpha} (F(+\infty) - F(\Lambda)) \right) \\
&+ \frac{4^{4-2\alpha} T^{2\alpha} \pi^{2\alpha} M}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}} \left(\sum_{k=1}^{M} \frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}} \right) \\
&\quad \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{T(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha}\, dF(\lambda) \right) \\
&+ \frac{16 M (F(+\infty) - F(\Lambda))}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}} \sum_{k=1}^{M} \left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta} \\
&+ \frac{4^{3-2\alpha} \pi^{2\alpha}}{(2\alpha-1) M^{2\alpha-1}} \cdot \frac{1}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}} \\
&\quad \times \left(\left(\frac{\delta}{\alpha} \right)^{2\delta} \int_0^{\infty} \lambda^{2\alpha}\, dF(\lambda) + (2T)^{2\alpha} \left(\frac{\delta}{\beta} \right)^{2\delta} \int_0^{\infty} \lambda^{2\nu}\, dF(\lambda) \right) \\
&+ \left. \frac{2^{2(2-\alpha)} T^{2\alpha} \pi^{2\alpha}}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}} \int_0^{\infty} \lambda^{2\alpha}\, dF(\lambda) \sum_{k=M+1}^{\infty} \frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}} \right]^{\frac{1}{2}},
\end{aligned}
$$

whence

$$
\begin{aligned}
\sigma(h) \le{}& \frac{1}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{\delta}} \\
&\times \Bigg[2 \cdot 4^{2(2-\alpha)} \left(\frac{\delta}{\alpha} \right)^{2\delta} \left(\frac{\pi}{2} \right)^{2\alpha} \frac{M}{2\alpha-1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) \\
&\quad \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \\
&\qquad \times \left(b_l^2 + \left[\left(\frac{T(1+C)}{4} \right)^{2\alpha} + T^{2\alpha}(1+2C^{\alpha}) + \left(\frac{T^2 \Lambda (1+C)}{2} \right)^{2\alpha} \right] \right. \\
&\qquad\qquad \left. \times \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha}\, dF(\lambda) \right) \\
&+ 9 \cdot 4^{4-2\alpha} \cdot M^2 \left(\frac{\delta}{\alpha} \right)^{2\delta} \left(\int_{\Lambda}^{\infty} |\lambda - \Lambda|^{2\alpha}\, dF(\lambda) + 2^{2\alpha} \Lambda^{2\alpha} (F(+\infty) - F(\Lambda)) \right) \\
&+ 4^{4-2\alpha} T^{2\alpha} \pi^{2\alpha} M \left(\sum_{k=1}^{M} \frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}} \right) \\
&\quad \times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{T(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha}\, dF(\lambda) \right) \\
&+ 16 M (F(+\infty) - F(\Lambda)) \sum_{k=1}^{M} \left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}
\end{aligned}
$$

$$+\frac{4^{3-2\alpha)}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}}$$
$$\times\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_0^\infty \lambda^{2\alpha}\,dF(\lambda)+(2T)^{2\alpha}\left(\frac{\delta}{\beta}\right)^{2\delta}\int_0^\infty \lambda^{2\nu}\,dF(\lambda)\right)$$
$$\left.+2^{2(2-\alpha)}T^{2\alpha}\pi^{2\alpha}\int_0^\infty \lambda^{2\alpha}\,dF(\lambda)\sum_{k=M+1}^{\infty}\frac{\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}}{k^{2\alpha}}\right]^{\frac{1}{2}}.\square$$

Corollary 3.2. *Let a partition* $L=\{\lambda_0,\dots,\lambda_N\}$ *of the set* $[0,\infty)$ *be such that*

$$\lambda_l<\lambda_{l+1}$$

and

$$\lambda_{l+1}-\lambda_l=\frac{\Lambda}{N-1}.$$

Let all the assumptions of Theorem 3.2 *hold. Then*

$$\sigma(h)\le\frac{C_1}{\left(\ln\left(\frac{1}{h}+1\right)\right)^\delta},$$

where

$$C_1=\left[2\cdot 4^{2(2-\alpha)}\left(\frac{\delta}{\alpha}\right)^{2\delta}\left(\frac{\pi}{2}\right)^{2\alpha}\frac{M}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)\left(\frac{\Lambda}{N-1}\right)^{2\alpha}\right.$$
$$\times\left(F(\Lambda)+\left[\left(\frac{3T}{4}\right)^{2\alpha}+(1+2^{\alpha+1})T^{2\alpha}+\left(\frac{3T^2\Lambda}{2}\right)^{2\alpha}\right]\int_0^\Lambda \lambda^{2\alpha}\,dF(\lambda)\right)$$
$$+9\cdot 4^{4-2\alpha}M^2\left(\frac{\delta}{\alpha}\right)^{2\delta}\left(\int_\Lambda^\infty|\lambda-\Lambda|^{2\alpha}\,dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}(F(+\infty)-F(\Lambda))\right)$$
$$+4^{4-2\alpha}T^{2\alpha}\pi^{2\alpha}M\left(\sum_{k=1}^{M}\frac{\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}}{k^{2\alpha}}\right)\left(\frac{\Lambda}{N-1}\right)^{2\alpha}$$
$$\times\left(F(\Lambda)+\left(\frac{3T}{2}\right)^{2\alpha}\int_0^\Lambda \lambda^{2\alpha}\,dF(\lambda)\right)$$
$$+16M(F(+\infty)-F(\Lambda))\sum_{k=1}^{M}\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}$$
$$+\frac{4^{3-2\alpha)}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}}\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_0^\infty \lambda^{2\alpha}\,dF(\lambda)+(2T)^{2\alpha}\left(\frac{\delta}{\beta}\right)^{2\delta}\int_0^\infty \lambda^{2\nu}\,dF(\lambda)\right)$$
$$\left.+2^{4-\alpha}T^{2\alpha}\pi^{2\alpha}\int_0^\infty \lambda^{2\alpha}\,dF(\lambda)\sum_{k=M+1}^{\infty}\frac{\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}}{k^{2\alpha}}\right]^{\frac{1}{2}}$$

with $\frac{1}{2}<\alpha\le 1$, $\frac{\alpha}{\delta}\le 1$, $\delta>0$, *and* $0<\beta\le 1$.

4. Concluding remarks

A model constructed in this paper for a Gaussian homogeneous isotropic random field uses a modified method of splitting and randomization of the spectrum. Some upper bounds for the norms of supremums of deviations between a homogeneous isotropic random fields and its model are obtained. These estimates are helpful for studies of the reliability and accuracy of the model in the space $C(T)$.

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