Теорія Ймовір. та Матем. Статист.
Вип. 94, 2016

Theor. Probability and Math. Statist.
No. 94, 2017, Pages 185–201
http://dx.doi.org/10.1090/tpms/1017
Article electronically published on August 25, 2017

THE DISTRIBUTION OF THE SUPREMUM OF A γ-REFLECTED STOCHASTIC PROCESS WITH AN INPUT PROCESS BELONGING TO SOME EXPONENTIAL TYPE ORLICZ SPACE

UDC 519.21

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Abstract. The paper is devoted to the study of properties of a γ-reflected process with an input belonging to some exponential type Orlicz space. In particular, sub-Gaussian and φ-sub-Gaussian whose input processes belong to some of the general classes $V(\varphi, \psi)$ are studied. The γ-reflected process is a stochastic process of the form

$$W_\gamma(t) = X(t) - f(t) - \gamma \inf_{s \le t} (X(s) - f(s)),$$

where $f(t)$ is a given function. This kind of process arises in insurance mathematics as a model for risk processes for which the income tax is paid according to the loss-carry-forward scheme where a proportion $\gamma \in [0, 1]$ of incoming premiums is paid when the process is on its maximum. The case of $\gamma < 0$ corresponds to a model with stimulation proportional to the increase of maximum, while the case of $\gamma > 1$ can be interpreted as a corresponding model with inhibition.

Some upper bounds for the ruin probability $\mathsf{P}\{\sup_t W_\gamma(t) > x\}$ are considered in the corresponding risk model for all $\gamma \in \mathbb{R}$. The results obtained in the paper are applied for the case of the sub-Gaussian generalized fractional Brownian motion.

Introduction

The aim of this paper is to study some properties of stochastic processes belonging to exponential Orlicz spaces. The study is motivated by some applied problems arising in insurance mathematics, financial mathematics, and queuing theory. The main object of our studies is the so called γ-reflected process $\{W_\gamma(t), t \in \mathsf{T}\}$ [1] defined on a certain parametric set T. (As a particular case, one can take an interval $[a, b]$ or the semiaxis $\mathbb{R}_+$ as the set T.)

Recall that $\{W_\gamma(t), t \in \mathsf{T}\}$ is called a γ-reflected stochastic process if

$$W_\gamma(t) = X(t) - f(t) - \gamma \inf_{s \in \mathsf{T}\colon s \le t} (X(s) - f(s)), \tag{1}$$

where $\{X(t), t \in \mathsf{T}\}$ is an input process, $\gamma \in [0, 1]$ is the reflection parameter, and $f(t)$ is a given continuous and monotone function.

The process $R_\gamma(t) = u - W_\gamma(t)$ is known in the literature on actuarial mathematics as a risk process with "loss-carry-forward" taxation scheme if $f(t) = ct$, $c > 0$, and $u \ge 0$ is an arbitrary initial reserve. Namely, the process $X(t)$ describes the accumulation of the excess insurance portfolio, which pays taxes according to the scheme mentioned above; that is, a proportion $\gamma \in (0, 1)$ of incoming premiums is paid when the process is on its maximum. The case of $\gamma < 0$ corresponds to a model with stimulation proportional to

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2010 *Mathematics Subject Classification.* Primary 60G07; Secondary 60K25.

Key words and phrases. Generalized fractional Brownian motion, metric entropy, an estimate of the distribution, sub-Gaussian process, Orlicz space.

the increase of maximum, while the case of $\gamma > 1$ can be interpreted as a corresponding model with inhibition. In the queuing theory, the process $W_1(t)$ describes the length of the queue or growth of the queue.

Throughout this paper, we assume that the input $X(t)$ is a stochastic process belonging to a certain exponential type Orlicz space. In particular, $X(t)$ can be a sub-Gaussian or φ-sub-Gaussian process and thus the results hold for Gaussian processes as well. We are going to estimate the ruin probability

$$\Psi_{\gamma,\mathsf{T}}(x) = \mathsf{P}\left\{\sup_{t\in\mathsf{T}} W_\gamma(t) > x\right\}. \tag{2}$$

The fractional Brownian motion is a widely used example of input γ-reflected processes [6], [18]. In particular, the following limit result is obtained for the γ-reflected fractional Brownian motion in [6]: if $\gamma \in (0,1)$ and $\mathsf{T} = [0,T]$ with some $T \in (0,\infty)$, then

$$\lim_{u\to\infty} \frac{\Psi_{\gamma,\mathsf{T}}(x)}{\Psi_{0,\mathsf{T}}(x)} = \mathcal{M}_{H,\gamma,T},$$

where $\mathcal{M}_{H,\gamma,T} = \mathcal{P}_{2H}^{\frac{1-\gamma}{\gamma}}$ if $H < \frac{1}{2}$, $\mathcal{M}_{\frac{1}{2},\gamma,T} = \mathcal{P}_1^{\frac{2-\gamma}{\gamma}}$, and $\mathcal{M}_{H,\gamma,T} = 1$ if $H > \frac{1}{2}$.

The precise value of $\mathcal{P}_{2H}^a$ is known for $H = \frac{1}{2}$ and $H = 1$. (See, for example, [5], [13].) We will apply the results obtained in this paper to the sub-Gaussian fractional Brownian motion $\{X_H(t), t \in \mathsf{T}\}$ with Hurst parameter $H \in (0,1)$; that is, if $X_H(t)$ is a sub-Gaussian stochastic process whose covariance function is given by

$$S\mathsf{P}R_H(t,s) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \qquad t,s \in \mathsf{T}.$$

Recall that ξ is a sub-Gaussian random variable if $\mathsf{E}\,\xi = 0$ and there exists a constant $a > 0$ such that

$$\mathsf{E}\exp\{\lambda\xi\} \le \exp\left\{\frac{a^2\lambda^2}{2}\right\}$$

for all $\lambda \in \mathbb{R}$. In other words, ξ is a sub-Gaussian random variable if its moment generating function is estimated from above by the moment generating function of a Gaussian random variable. Sub-Gaussian random variables are introduced by Kahane [7]. They are intensively studied along with other classes of random variables and stochastic processes of Orlicz spaces including Gaussian random variables and processes. (See the monograph by Buldygin and Kozachenko [4] for a detailed survey of the theory.)

Kozachenko and Ostrovskyĭ [8] introduced Banach spaces of random variables and processes that generalize the spaces of sub-Gaussian random variables.

A random variable ξ is called φ-sub-Gaussian if it is centered and there exists a constant $a > 0$ such that

$$\mathsf{E}\exp\{\lambda\xi\} \le \exp\{\varphi(a\lambda)\}$$

for all $\lambda \in \mathbb{R}$, where $\varphi(x)$, $x \in \mathbb{R}$, is a certain Orlicz N-function.

The collection of φ-sub-Gaussian random variables with finite exponential moments is denoted by $\mathrm{Sub}_\varphi(\Omega)$. Then $\mathrm{Sub}_\varphi(\Omega)$ are subspaces of exponential type Orlicz spaces studied in detail in the books by Buldygin and Kozachenko [4] and Kozachenko, Vasylyk, and Yamnenko [11]. More general are the classes $V(\varphi,\psi)$ of stochastic processes introduced by Kozachenko and Vasylyk [9]. These stochastic processes are studied in [3], [10–12, 14–17] and several others.

The current paper is organized as follows. Section 1 contains necessary notions of the theory of φ-sub-Gaussian random variables and processes. The main results obtained with the help of the metric entropy technique are stated in Section 2 for all $\gamma \in \mathbb{R}$. An example of applications of upper bounds (2) is given in Section 3 for sub-Gaussian processes that, in particular, holds for the fractional Brownian motion.

1. Stochastic processes belonging to the spaces $\mathrm{Sub}_\varphi(\Omega)$ or to classes $V(\varphi,\psi)$

Let $\{\Omega,\mathcal{B},P\}$ be a standard probability space and let T be an arbitrary set of parameters.

1.1. Orlicz N-functions.

Definition 1.1 ([4])**.** A function $U=\{U(x),x\in\mathbb{R}\}$ is called an Orlicz N-function if U is continuous even convex and such that $U(0)=0$, $U(x)$ increases for $x>0$, $U(x)/x\to 0$ as $x\to 0$, and $U(x)/x\to\infty$ as $x\to\infty$.

Lemma 1.1 ([11])**.** *Any Orlicz N-function φ is such that*

1. $\varphi(\alpha x)\le\alpha\varphi(x)$ *for* $x\in\mathbb{R}$ *and* $\alpha\in[0,1]$*;*
2. $\varphi(\alpha x)\ge\alpha\varphi(x)$ *for* $x\in\mathbb{R}$ *and* $\alpha>1$*;*
3. $\varphi(|x|+|y|)\ge\varphi(x)+\varphi(y)$ *for* $x,y\in\mathbb{R}$*;*
4. *there exists a constant* $c>0$ *such that* $\varphi(x)\ge c|x|$ *for* $x>1$*;*
5. *the function* $\zeta(x)=\varphi(x)/x$ *is non-decreasing for* $x>0$*.*

Condition Q. We say that an Orlicz N-function φ admits condition Q if

$$\liminf_{x\to 0}\frac{\varphi(x)}{x^2}=c>0. \tag{3}$$

Definition 1.2. We say that an Orlicz N-function φ_1 is subordinated to an N-function φ_2 (and write $\varphi_1\prec\varphi_2$) if there are some constants $c>0$ and $x_0>0$ such that $\varphi_1(x)<\varphi_2(cx)$ for $x>x_0$. We say that two N-functions φ_1 and φ_2 are equivalent if $\varphi_1\prec\varphi_2$ and $\varphi_2\prec\varphi_1$.

1.2. φ-sub-Gaussian random variables and processes.

Definition 1.3 ([4])**.** Let φ be an Orlicz N-function that admits condition Q. We say that a random variable ξ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$ if $\mathsf{E}\,\xi=0$, $\mathsf{E}\exp\{\lambda\xi\}$ exists, and if, given an arbitrary $\lambda\in\mathbb{R}$, there exists a constant $a>0$ such that

$$\mathsf{E}\exp\{\lambda\xi\}\le\exp\{\varphi(\lambda a)\} \tag{4}$$

for all $\lambda\in\mathbb{R}$.

Theorem 1.1 ([4])**.** *The space* $\mathrm{Sub}_\varphi(\Omega)$ *is a Banach space with respect to the norm*

$$\tau_\varphi(\xi)=\sup_{\lambda>0}\frac{\varphi^{(-1)}(\log\mathsf{E}\exp\{\lambda\xi\})}{\lambda}, \tag{5}$$

where $\varphi^{(-1)}$ *is the inverse function to* φ*. Moreover,*

$$\mathsf{E}\exp\{\lambda\xi\}\le\exp\{\varphi(\lambda\tau_\varphi(\xi))\} \tag{6}$$

for all $\lambda\in\mathbb{R}$*, and there exists a constant* $c>0$ *such that*

$$\left(\mathsf{E}\,\xi^2\right)^{\frac12}\le c\tau_\varphi(\xi). \tag{7}$$

Lemma 1.2 ([2])**.** *Let* $\xi\in\mathrm{Sub}_\varphi(\Omega)$*,* $\tau_\varphi(\xi)>0$*,* $\varepsilon>0$*. Then*

$$\mathsf{P}\{\xi>\varepsilon\}\le\exp\left\{-\varphi^*\left(\frac{\varepsilon}{\tau_\varphi(\xi)}\right)\right\},$$

$$\mathsf{P}\{\xi<-\varepsilon\}\le\exp\left\{-\varphi^*\left(\frac{\varepsilon}{\tau_\varphi(\xi)}\right)\right\},$$

$$\mathsf{P}\{|\xi|>\varepsilon\}\le 2\exp\left\{-\varphi^*\left(\frac{\varepsilon}{\tau_\varphi(\xi)}\right)\right\},$$

where $\varphi^*(x)=\sup_{y\in\mathbb{R}}(xy-\varphi(y))$ *is the Young–Fenchel transform of the function* φ*.*

Example 1.1 ([4]). Centered Gaussian random variables $\xi = N(0, \sigma^2)$ belong to the space $\mathrm{Sub}_{x^2/2}(\Omega)$ and $\tau(\xi) = (\mathsf{E}\,\xi^2)^{\frac{1}{2}}$ in this case.

Example 1.2. Let ξ be a centered bounded random variables, that is $\mathbf{E}\xi = 0$. Assume that there exists a number $c > 0$ such that $|\xi| \le c$ almost surely. Then $\xi \in \mathrm{Sub}(\Omega)$ and $\tau(\xi) \le c$.

Definition 1.4. A stochastic process $X = \{X(t), t \in T\}$ is called φ-sub-Gaussian if random variables $X(t)$, $t \in T$, are φ-sub-Gaussian. (We write $X(t) \in \mathrm{Sub}_\varphi(\Omega)$ in this case.)

If additionally $\varphi(x) = x^2/2$, then the latter processes are called sub-Gaussian. It is obvious that centered Gaussian stochastic processes are sub-Gaussian.

1.3. Stochastic processes of the class $V(\varphi, \psi)$.

Definition 1.5 ([11])**.** Let $\varphi \prec \psi$ be two Orlicz N-functions. We say that a stochastic process $X = \{X(t), t \in T\}$ belongs to the class $V(\varphi, \psi)$ if the random variable $X(t)$ belongs to the space $\mathrm{Sub}_\psi(\Omega)$ for all $t \in T$, and the increments $X(t) - X(s)$ belong to the space $\mathrm{Sub}_\varphi(\Omega)$ for all $s, t \in T$.

Example 1.3 ([4]). Sub-Gaussian processes belong to the class $V(\varphi, \varphi)$, where $\varphi(x) = x^2/2$.

Example 1.4 ([11]). Let $X(t) = \xi_0 + \sum_{k=1}^{\infty} \xi_k f_k(t)$ with a random variable $\xi_0 \in \mathrm{Sub}_\psi(\Omega)$, $\{\xi_k, k = 1, 2, \ldots\} \in \mathrm{Sub}\,\varphi(\Omega)$, and $\sum_{k=1}^{\infty} \tau_\varphi(\xi_k)|f_k(t)| < \infty$. Then the stochastic process $X(t)$ belongs to the class $V(\varphi, \psi)$.

2. An upper bound for the distribution of the supremum of a γ-reflected process of the class $V(\varphi, \psi)$

Let (T, ρ) be a pseudometric (metric) separable space equipped with a pseudometric (metric) ρ.

Assume that the input $\{X(t),\ t \in \mathsf{T}\}$ is a separable stochastic process with φ-sub-Gaussian increments. This property holds, in particular, if the process belongs to the class $V(\varphi, \psi)$, $\varphi \prec \psi$. Assume also that condition Σ holds for this process.

Condition Σ. We say that condition Σ holds for a stochastic process $\{X(t),\ t \in \mathsf{T}\}$ if there exists a continuous increasing function $\sigma = \{\sigma(h), h \ge 0\}$ such that $\sigma(h) \to 0$ for $h \to 0$ and

$$\sup_{\rho(t,s) \le h} \tau_\varphi(X(t) - X(s)) \le \sigma(h). \tag{8}$$

Note that the function

$$\sigma(h) = \sup_{\rho(t,s) \le h} \tau_\varphi(X(t) - X(s))$$

possesses the latter property if the process $X(t)$ is continuous in the metric $\tau_\varphi(\cdot)$.

Condition Δ. We say that condition Δ holds for a function $\{f(t), t \in \mathsf{T}\}$ if f is continuous and such that

$$|f(u) - f(v)| \le \delta(\rho(u, v)), \tag{9}$$

where $\{\delta(u), u \ge 0\}$ is some nonnegative continuous increasing function.

Let B be a compact set, $B \subseteq T$. Denote by $\mathsf{N}(u) = \mathsf{N}_{(B,\rho)}(u)$ the metric massiveness of the space (B, ρ). In other words, $\mathsf{N}(u)$ is the minimal number of balls of radius u that cover the space (B, ρ).

The following result contains an upper bound for the distribution of the supremum of a γ-reflected process $W_\gamma(t)$:

$$\sup_{t\in B} W_\gamma(t) = \sup_{t\in B}\left(X(t) - f(t) - \gamma \inf_{s\in B\colon s\le t}(X(s) - f(s))\right), \qquad \gamma\in\mathbb{R}. \tag{10}$$

Lemma 2.1. *Let the input process $\{X(t), t\in B\}$ be of the class $V(\varphi,\psi)$ and condition Σ hold for the γ-reflected process $W_\gamma(t)$, $\gamma\in\mathbb{R}$. Assume that condition Δ holds for the function $\{f(t), t\in B\}$. Let a sequence $\{q_k\}_{k=1}^\infty$ be such that $q_k > 1$ and*

$$\sum_{k=1}^\infty \frac{1}{q_k} \le 1.$$

Let a sequence $\{\varepsilon_k\}_{k=1}^\infty$ be decreasing and such that $\varepsilon_k > 0$ and $\varepsilon_k \to 0$ as $k\to\infty$. We further assume that $\tau_B = \sup_{t\in B} X(t) < \infty$, $f_B = \sup_{t\in B} f(t) < \infty$, and $\beta > 0$ is a number such that $\beta \le \sigma\left(\inf_{s\in B}\sup_{t\in B}\rho(t,s)\right)$. Then

$$\begin{aligned}\mathsf{E}\exp\left\{\sup_{t\in B}\lambda W_\gamma(t)\right\} \le \prod_{k=2}^\infty \left(\mathsf{L}(\varepsilon_k)\right)^{\frac{1}{q_k}} G_\gamma(\lambda)\\ \times\exp\left\{\sum_{k=2}^\infty\left(\frac{1}{2q_k}\left(\varphi(2q_k\lambda\sigma(\varepsilon_{k-1})) + \varphi(2q_k\gamma\lambda\sigma(\varepsilon_{k-1}))\right)\right.\right.\\ \left.\left. + \lambda(1+|\gamma|)\delta(\varepsilon_{k-1})\right)\right\}\end{aligned} \tag{11}$$

for all $x > 0$ where

$$\begin{aligned}G_\gamma(\lambda) = \Bigg(\sum_{l=0}^{\mathsf{N}(\varepsilon_1)-1}(\mathsf{N}(\varepsilon_1) - l)\\ \times\exp\left\{\varphi(q_1\lambda(\sigma(2\varepsilon_1 l) + |1-\gamma|\tau_B)) + q_1\lambda(\delta(2\varepsilon_1 l) + |1-\gamma| f_B)\right\}\Bigg)^{\frac{1}{q_1}}\end{aligned} \tag{12}$$

and $\mathsf{L}(u) = ((\mathsf{N}(u))^2 + \mathsf{N}(u))/2$.

Proof. Denote by V_{ε_k} the set of all centers of closed balls of radius ε_k that form a minimal covering of the space (B,ρ). The number of points in the set V_{ε_k} is equal to $\mathsf{N}(\varepsilon_k)$. We also put $D_A = \{(u,v)\colon u\le v,\ u,v\in A\}$, where A is an arbitrary set.

Condition Σ and Lemma 1.2 imply

$$\begin{aligned}\mathsf{P}\{|X(t) - X(s)| > \varepsilon\} &\le 2\exp\left\{-\varphi^*\left(\frac{\varepsilon}{\tau_\varphi(X(t)-X(s))}\right)\right\}\\ &\le 2\exp\left\{-\varphi^*\left(\frac{\varepsilon}{\sigma(\rho(t,s))}\right)\right\}\end{aligned}$$

for all $\varepsilon > 0$.

By assumption, the process $X(t)$ is separable. Thus $X(t)$ as well as $Y(t) = X(t) - f(t)$ are continuous in probability. Hence every countable everywhere dense set with respect to the metric ρ is a set of separability for this process. In particular, $V = \bigcup_{k=1}^\infty V_{\varepsilon_k}$ is a set of ρ-separability of the process $Y(t)$ and

$$\sup_{t\in B} Y(t) = \sup_{t\in V} Y(t), \qquad \sup_{(s,t)\in D_B}(Y(t) - \gamma Y(s)) = \sup_{(s,t)\in D_V}(Y(t) - \gamma Y(s))$$

with probability one and

$$\sup_{t\in B} W_\gamma(t) = \sup_{t\in B}\left(Y(t) - \gamma \inf_{s\in B\colon s\le t} Y(s)\right) = \sup_{t\in V}\left(Y(t) - \gamma \inf_{s\in V\colon s\le t} Y(s)\right) = \sup_{t\in V} W_\gamma(t). \tag{13}$$

Consider a mapping $\alpha_n = \{\alpha_n(u), n = 0, 1, \dots\}$ acting from the set V to V_{ε_n}, where $\alpha_n(u)$ is a point of the set V_{ε_n} such that $\rho(u, \alpha_n(u)) \le \varepsilon_n$. If $u \in V_{\varepsilon_n}$, then $\alpha_n(u) = u$. If there are several points of the set V_{ε_n} such that $\rho(u, \alpha_n(u)) \le \varepsilon_n$, then we choose one of them in such a way that $\alpha_n(v) \le \alpha_n(u)$ for all $v \le u$, $v, u \in V$. This point is denoted by $\alpha_n(u)$.

Now Chebyshev's inequality, Theorem 1.1, and condition Σ imply that

$$\begin{aligned}\mathsf{P}\left\{|X(u) - X(\alpha_n(u))| > p^{\frac{n}{2}}\right\} &\le \frac{\mathsf{E}\big(X(u) - X(\alpha_n(u))\big)^2}{p^n}\\ &\le \frac{c^2\tau_\varphi^2\big(X(u) - X(\alpha_n(u))\big)}{p^n}\\ &\le \frac{c^2\sigma^2(\varepsilon_n)}{p^n} = c^2\beta^2 p^n,\end{aligned}$$

where $c = 2e/\big(\varphi^{(-1)}(2)\big)$. This means that

$$\sum_{n=1}^{\infty} \mathsf{P}\left\{|X(u) - X(\alpha_n(u))| > p^{n/2}\right\} < \infty.$$

The Borel–Cantelli lemma yields $X(u) - X(\alpha_n(u)) \to 0$ as $n \to \infty$ with probability one. Since the function f is continuous, $Y(u) - Y(\alpha_n(u)) \to 0$ as $n \to \infty$ with probability one. The set V is countable, whence $X(u) - X(\alpha_n(u)) \to 0$ with probability one as $n \to \infty$ for all u.

Let $(s,t) \in D_V$. Put $s_m = \alpha_m(s)$, $s_{m-1} = \alpha_{m-1}(s_m)$, $\dots$, $s_1 = \alpha_1(s_2)$, and $t_m = \alpha_m(t)$, $t_{m-1} = \alpha_{m-1}(t_m)$, $\dots$, $t_1 = \alpha_1(t_2)$ for all $m \ge 1$. Then

$$\gamma Y(s) = \gamma Y(s_1) + \sum_{k=2}^{m} (\gamma Y(s_k) - \gamma Y(s_{k-1})) + \gamma Y(s) - \gamma Y(\alpha_m(s))$$

and

$$Y(t) = Y(t_1) + \sum_{k=2}^{m} (Y(t_k) - Y(t_{k-1})) + Y(t) - Y(\alpha_m(t))$$

for all $m \ge 2$. Correspondingly,

$$\begin{aligned}Y(t) - \gamma Y(s) \le{}& \max_{(v,u)\in D_{V_{\varepsilon_1}}} (Y(u) - \gamma Y(v))\\ &+ \sum_{k=2}^{m} \max_{(v,u)\in D_{V_{\varepsilon_k}}} \big(Y(u) - \gamma Y(v) - Y(\alpha_{k-1}(u)) + \gamma Y(\alpha_{k-1}(v))\big)\\ &+ Y(t) - \gamma Y(s) - Y(\alpha_m(t)) + \gamma Y(\alpha_m(s)).\end{aligned} \tag{14}$$

This implies that

$$\begin{aligned}\sup_{t\in V} W_\gamma(t) &= \sup_{t\in V}(Y(t)-\gamma \inf_{s\in V:\, s\le t} Y(s)) = \sup_{(s,t)\in D_V}(Y(t)-\gamma Y(s))\\ &\le \liminf_{m\to\infty}\Big(\max_{(v,u)\in D_{V_{\varepsilon_1}}}(Y(u)-\gamma Y(v))\\ &\quad+\sum_{k=2}^{m}\max_{(v,u)\in D_{V_{\varepsilon_k}}}\big(Y(u)-\gamma Y(v)-Y(\alpha_{k-1}(u))+\gamma Y(\alpha_{k-1}(v))\big)\\ &\quad+Y(t)-\gamma Y(s)-Y(\alpha_m(t))+\gamma Y(\alpha_m(s))\Big)\end{aligned}\tag{15}$$

with probability one. Now we apply Hölder's inequalities (15) and Fatou's lemma to conclude that, for all $\lambda>0$,

$$\begin{aligned}\mathsf{E}\exp\Big\{\lambda\sup_{t\in V}W_\gamma(t)\Big\} &= \mathsf{E}\exp\Big\{\lambda\sup_{(s,t)\in D_V}(Y(t)-\gamma Y(s))\Big\}\\ &\le \mathsf{E}\liminf_{m\to\infty}\exp\Big\{\lambda\Big(\max_{(v,u)\in D_{V_{\varepsilon_1}}}(Y(u)-\gamma Y(v))\\ &\quad+\sum_{k=2}^{m}\max_{(v,u)\in D_{V_{\varepsilon_k}}}\big(Y(u)-\gamma Y(v)-Y(\alpha_{k-1}(u))\\ &\quad+\gamma Y(\alpha_{k-1}(v))\big)\Big)\Big\}\\ &\le \liminf_{m\to\infty}\mathsf{E}\exp\Big\{\lambda\Big(\max_{(v,u)\in D_{V_{\varepsilon_1}}}(Y(u)-\gamma Y(v))\\ &\quad+\sum_{k=2}^{m}\max_{(v,u)\in D_{V_{\varepsilon_k}}}\big(Y(u)-\gamma Y(v)-Y(\alpha_{k-1}(u))\\ &\quad+\gamma Y(\alpha_{k-1}(v))\big)\Big)\Big\}\\ &\le \liminf_{m\to\infty}\Big(\mathsf{E}\exp\Big\{q_1\lambda\max_{(v,u)\in D_{V_{\varepsilon_1}}}(Y(u)-\gamma Y(v))\Big\}\Big)^{\frac{1}{q_1}}\\ &\quad\times\prod_{k=2}^{m}\Big(\mathsf{E}\exp\Big\{q_k\lambda\max_{(v,u)\in D_{V_{\varepsilon_k}}}\big(Y(u)-\gamma Y(v)-Y(\alpha_{k-1}(u))\\ &\quad+\gamma Y(\alpha_{k-1}(v))\big)\Big\}\Big)^{\frac{1}{q_k}}\\ &\le \Big(\mathsf{E}\exp\Big\{q_1\lambda\max_{(v,u)\in D_{V_{\varepsilon_1}}}(Y(u)-\gamma Y(v))\Big\}\Big)^{\frac{1}{q_1}}\\ &\quad\times\prod_{k=2}^{\infty}\Big(\mathsf{E}\exp\Big\{q_k\lambda\max_{(v,u)\in D_{V_{\varepsilon_k}}}\big(Y(u)-\gamma Y(v)-Y(\alpha_{k-1}(u))\\ &\quad+\gamma Y(\alpha_{k-1}(v))\big)\Big\}\Big)^{\frac{1}{q_k}}\\ &= (J_1)^{\frac{1}{q_1}}\cdot\prod_{k=2}^{\infty}(J_k)^{\frac{1}{q_k}}.\end{aligned}\tag{16}$$

Every term on the right hand side of (16) is estimated separately. Theorem 1.1 and condition Σ imply

$$\begin{aligned}\mathsf{E}\exp\{q_1\lambda(X(u)-\gamma X(v))\} &\le \exp\{\varphi(q_1\lambda\tau_\varphi(X(u)-\gamma X(v)))\}\\ &= \exp\{\varphi(q_1\lambda\tau_\varphi(X(u)-X(v)+(1-\gamma)X(v)))\}\\ &\le \exp\{\varphi(q_1\lambda\sigma(\rho(u,v))+|1-\gamma|\tau_B)\}.\end{aligned}$$

Using condition Δ we obtain

$$\begin{aligned}J_1 &\le \sum_{(v,u)\in D_{V_{\varepsilon_1}}} \mathsf{E}\exp\{q_1\lambda(X(u)-\gamma X(v))\}\exp\{q_1\lambda(f(v)-\gamma f(u))\}\\ &\le \sum_{(v,u)\in V_{\varepsilon_1}} \exp\{\varphi(q_1\lambda(\sigma(\rho(v,u))+|1-\gamma|\tau_B))+q_1\lambda(\delta(\rho(v,u))+|1-\gamma|f_B)\}\\ &= \sum_{i=1}^{N(\varepsilon_1)}\sum_{j=1}^{i}\exp\big\{\varphi(q_1\lambda\sigma(2\varepsilon_1(i-j))+|1-\gamma|\tau_B)\\ &\qquad\qquad + q_1\lambda(\delta(2\varepsilon_1(i-j))+|1-\gamma|f_B)\big\} \qquad (17)\\ &= \sum_{l=0}^{N(\varepsilon_1)-1}(N(\varepsilon_1)-l)\exp\big\{\varphi(q_1\lambda(\sigma(2\varepsilon_1 l)+|1-\gamma|\tau_B)\\ &\qquad\qquad + q_1\lambda(\delta(2\varepsilon_1 l)+|1-\gamma|f_B)\big\}.\end{aligned}$$

Further, we conclude from Theorem 1.1, Cauchy–Bunyakovskiĭ's inequality, and condition Σ that

$$\begin{aligned}&\mathsf{E}\exp\{q_k\lambda(X(u)-\gamma X(v)-X(\alpha_{k-1}(u))+\gamma X(\alpha_{k-1}(v)))\}\\ &\quad\le \left(\mathsf{E}\exp\{2q_k\lambda(X(u)-X(\alpha_{k-1}(u)))\}\,\mathsf{E}\exp\{2q_k\lambda\gamma(X(\alpha_{k-1}(v))-X(v))\}\right)^{\frac12} \qquad (18)\\ &\quad\le \exp\left\{\frac12\varphi(2q_k\lambda\sigma(\varepsilon_{k-1}))+\frac12\varphi(2q_k\gamma\lambda\sigma(\varepsilon_{k-1}))\right\}.\end{aligned}$$

Then

(19)

$$\begin{aligned}J_k &\le \sum_{(v,u)\in D_{V_{\varepsilon_k}}} \mathsf{E}\exp\big\{q_k\lambda\big(X(u)-X(\alpha_{k-1}(u))-\gamma X(v)+\gamma X(\alpha_{k-1}(v))\big)\big\}\\ &\qquad\qquad\times\exp\big\{q_k\lambda\big(f(u)-f(\alpha_{k-1}(u))\big)\big\}\exp\{q_k\lambda\gamma(f(\alpha_{k-1}(v))-f(v))\}\\ &\le \frac{N(\varepsilon_k)^2+N(\varepsilon_k)}{2}\exp\bigg\{\frac12\varphi(2q_k\lambda\sigma(\varepsilon_{k-1}))+\frac12\varphi(2q_k\lambda\gamma\sigma(\varepsilon_{k-1}))\\ &\qquad + q_k\lambda\max_{u\in V_{\varepsilon_k}}\delta\big(\rho(u,\alpha_{k-1}(u))\big)+q_k|\gamma|\lambda\max_{u\in V_{\varepsilon_k}}\delta\big(\rho(u,\alpha_{k-1}(u))\big)\bigg\}\\ &\le L(\varepsilon_k)\exp\left\{\frac12\varphi\big(2q_k\lambda\sigma(\varepsilon_{k-1})\big)+\frac12\varphi\big(2q_k\lambda\gamma\sigma(\varepsilon_{k-1})\big)+q_k\lambda(1+|\gamma|)\delta(\varepsilon_{k-1})\right\}.\end{aligned}$$

Finally, we complete the proof of the lemma by using inequalities (16)–(19). □

Remark 2.1. The expression J_1 in inequality (17) can be estimated in an another way by obtaining two less precise inequalities for $G_\gamma(\lambda)$ that follow from (12). The first of them is as follows

$$G_\gamma(\lambda) \le \big(L(\varepsilon_1)\big)^{\frac{1}{q_1}}\exp\left\{\frac{1}{q_1}\varphi\big(q_1\gamma\lambda\sigma(2\varepsilon_1(N(\varepsilon_1)-1))\big)+\lambda\gamma\delta(2\varepsilon_1(N(\varepsilon_1)-1))\right\}. \qquad (20)$$

Below is the second estimate:

$$G_\gamma(\lambda)^{q_1} \le \int_0^{\mathsf{N}(\varepsilon_1)} (\mathsf{N}(\varepsilon_1) - x) \exp\{\varphi(q_1\lambda\gamma\sigma(2\varepsilon_1 x)) + q_1\lambda\gamma\delta(2\varepsilon_1 x)\}\, dx. \tag{21}$$

Theorem 2.1. *Let the input process $\{X(t), t \in B\}$ be of the class $V(\varphi,\psi)$ and condition Σ hold for the γ-reflected process $W_\gamma(t)$, $\gamma \in \mathbb{R}$. Assume that condition Δ holds for the function $\{f(t), t \in B\}$, and*

$$\int_0^\beta \zeta_\varphi\left(\ln\left(\mathsf{L}\left(\sigma^{(-1)}(u)\right)\right)\right) du < \infty, \tag{22}$$

where

$$\zeta_\varphi(v) = \frac{v}{\varphi^{(-1)}(v)}$$

and $\mathsf{L}(u) = ((\mathsf{N}(u))^2 + \mathsf{N}(u))/2$. Also let $\tau_B = \sup_{t\in B} X(t) < \infty$, $f_B = \sup_{t\in B} f(t) < \infty$, and $\beta > 0$ be some number such that $\beta \le \sigma\left(\inf_{s\in B}\sup_{t\in B}\rho(t,s)\right)$. Then

$$\begin{aligned}
&\mathsf{P}\Big\{\sup_{t\in B} W_\gamma(t) > x\Big\} \\
&\quad\le \inf_{\lambda>0}\Bigg(G_1(\lambda,\gamma,p) \\
&\qquad\times \exp\Bigg\{\frac{(1+|\gamma|)p}{2}\varphi\left(\frac{2\lambda\beta}{1-p}\right) \\
&\qquad\quad + \frac{2\lambda}{p(1-p)}\int_0^{\beta p^2} \zeta_\varphi\left(\ln\left(\mathsf{L}\left(\sigma^{(-1)}(u)\right)\right)\right) du\Bigg\} \\
&\qquad\times \exp\left\{\lambda(1+|\gamma|)\sum_{k=1}^\infty \delta\left(\sigma^{(-1)}\left(\beta p^k\right)\right) - \lambda x\right\}\Bigg), \\
&\qquad |\gamma| \le 1
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
&\mathsf{P}\Big\{\sup_{t\in B} W_\gamma(t) > x\Big\} \\
&\quad\le \inf_{\lambda>0}\Bigg(G_1(\lambda,\gamma,p) \\
&\qquad\times \exp\Bigg\{\frac{(1+|\gamma|)p}{2|\gamma|}\varphi\left(\frac{2\lambda\gamma\beta}{1-p}\right) \\
&\qquad\quad + \frac{2\lambda|\gamma|}{p(1-p)}\int_0^{\beta p^2} \zeta_\varphi\left(\ln\left(\mathsf{L}\left(\sigma^{(-1)}(u)\right)\right)\right) du\Bigg\} \\
&\qquad\times \exp\left\{\lambda(1+|\gamma|)\sum_{k=1}^\infty \delta\left(\sigma^{(-1)}\left(\beta p^k\right)\right) - \lambda x\right\}\Bigg), \\
&\qquad |\gamma| > 1,
\end{aligned} \tag{24}$$

for all $x > 0$ *and* $p \in (0,1)$, *where*

$$G_1(\lambda,\gamma,p) = \inf_{v \geq \frac{1}{1-p}} \Bigg(\sum_{l=0}^{N(\sigma^{(-1)}(\beta p))-1} \left(N\left(\sigma^{(-1)}(\beta p)\right) - l\right) \times \exp\Big\{ \varphi\Big(\lambda v \left(\sigma\left(2l\sigma^{(-1)}(\beta p)\right) + |1-\gamma_B|\tau_B\right) + \lambda v \left(\delta\left(2l\sigma^{(-1)}(\beta p)\right) + |1-\gamma| f_B\right)\Big\}\Bigg)^{\frac{1}{v}}.$$

Proof. We use the sequences q_k and ε_k introduced in the proof of Lemma 2.1. Here we recall their definition. Let $q_1 = v$, where v is a number such that $v \geq \frac{1}{1-p}$. Then $q_1 > 1$. Further let

$$q_k = \frac{1}{2\lambda\beta p^{k-1}} \varphi^{(-1)}\left(\varphi\left(\frac{2\lambda\beta}{1-p}\right) + \ln(L(\varepsilon_k))\right), \qquad k = 2,3,\ldots, \tag{25}$$

where

$$\varepsilon_k = \sigma^{(-1)}\left(\beta p^k\right), \qquad p \in (0,1),\ k = 0,1,2,\ldots. \tag{26}$$

Then

$$q_k \geq \frac{1}{p^{k-1}(1-p)} > 1 \quad \text{and} \quad \frac{1}{q_k} \leq \frac{2\lambda\beta p^{k-1}}{\varphi^{(-1)}\left(\varphi\left(\frac{2\lambda\beta}{1-p}\right)\right)} = p^{k-1}(1-p),\ k = 2,3,\ldots,$$

that is,

$$\sum_{k=1}^{\infty} \frac{1}{q_k} \leq \sum_{k=1}^{\infty} p^{k-1}(1-p) = 1.$$

We conclude that the sequence introduced above satisfies the assumption of Lemma 2.1. Thus one can use it in inequality (11). First we treat the expression on the right hand side for $|\gamma| \leq 1$:

$$\tilde{Z} = \sum_{k=2}^{\infty} \frac{1}{q_k}\Big(\ln\left(L(\varepsilon_k)\right) + \frac{1}{2}\varphi\big(2\lambda\beta p^{k-1} q_k\big) + \frac{1}{2}\varphi\big(2\lambda\gamma\beta p^{k-1} q_k\big)\Big).$$

Lemma 1.1 implies that $\varphi(\gamma x) \leq |\gamma|\varphi(x)$ for $\gamma \in [-1,1]$, $x \in \mathbb{R}$. Then

$$\begin{aligned}
\tilde{Z} &= \sum_{k=2}^{\infty} \frac{1}{q_k} \ln\big(L(\varepsilon_k)\big) + \sum_{k=2}^{\infty} \frac{1}{q_k} \varphi\left(2\lambda\beta p^{k-1} \frac{\varphi^{(-1)}\left(\varphi\left(\frac{2\lambda\beta}{1-p}\right) + \ln\left(L(\varepsilon_k)\right)\right)}{2\lambda\beta p^{k-1}}\right) \\
&\quad + \sum_{k=2}^{\infty} \frac{1}{q_k} \varphi\left(2\lambda\gamma\beta p^{k-1} \frac{\varphi^{(-1)}\left(\varphi\left(\frac{2\lambda\beta}{1-p}\right) + \ln\left(L(\varepsilon_k)\right)\right)}{2\lambda\beta p^{k-1}}\right) \\
&\leq \sum_{k=2}^{\infty} \frac{1}{q_k} \ln\left(L(\varepsilon_k)\right) + \frac{1+|\gamma|}{2}\varphi\left(\frac{2\lambda\beta}{1-p}\right) \sum_{k=2}^{\infty} \frac{1}{q_k} \qquad (27) \\
&\leq \sum_{k=2}^{\infty} \ln\left(L(\varepsilon_k)\right) \frac{2\lambda\beta p^{k-1}}{\varphi^{(-1)}\left(\ln\left(L(\varepsilon_k)\right)\right)} + \frac{1+|\gamma|}{2}\varphi\left(\frac{2\lambda\beta}{1-p}\right) \sum_{k=2}^{\infty} p^{k-1}(1-p) \\
&= \frac{(1+|\gamma|)p}{2}\varphi\left(\frac{2\lambda\beta}{1-p}\right) + 2\lambda \sum_{k=2}^{\infty} \zeta_{\varphi}\left(\ln\left(L\left(\sigma^{(-1)}\left(\beta p^k\right)\right)\right)\right) \beta p^{k-1}.
\end{aligned}$$

The function $\varphi(x)/x$ increases for $x > 0$ and, as a result, the function

$$\zeta_\varphi(x) = \frac{x}{\varphi^{(-1)}(x)}$$

increases for $x > 0$, too. Thus

$$\int_{\beta p^{k+1}}^{\beta p^k} \zeta_\varphi\left(\ln\left(L\left(\sigma^{(-1)}(u)\right)\right)\right) du \ge \zeta_\varphi\left(\ln\left(L\left(\sigma^{(-1)}\left(\beta p^k\right)\right)\right)\right) \beta p^k(1-p). \tag{28}$$

Now (27) and (28) imply that

$$\tilde{Z} \le \frac{(1+|\gamma|)p}{2}\varphi\left(\frac{2\lambda\beta}{1-p}\right) + \frac{2\lambda}{p(1-p)}\int_0^{\beta p^2} \zeta_\varphi\left(\ln\left(L\left(\sigma^{(-1)}(u)\right)\right)\right) du \tag{29}$$

and inequality (23) follows from (11) and (29).

Next we consider the case of $|\gamma| > 1$. Let

$$q_k = \frac{1}{2\lambda\gamma\beta p^{k-1}}\varphi^{(-1)}\left(\varphi\left(\frac{2\lambda\gamma\beta}{1-p}\right) + \ln(L(\varepsilon_k))\right), \qquad k = 2, 3, \ldots. \tag{30}$$

We see that this sequence satisfies the assumption of Lemma 2.1, as well:

$$q_k \ge \frac{1}{p^{k-1}(1-p)} > 1, \qquad \frac{1}{q_k} \le \frac{2\lambda\beta p^{k-1}}{\varphi^{(-1)}\left(\varphi\left(\frac{2\lambda\beta}{1-p}\right)\right)} = p^{k-1}(1-p), \qquad k = 2, 3, \ldots.$$

Since $\varphi(x/\gamma) \le \varphi(x)/|\gamma|$ for $|\gamma| > 1$, we obtain

$$\begin{aligned}
\tilde{Z}_1 &= \sum_{k=2}^{\infty} \frac{1}{q_k}\ln\left(L(\varepsilon_k)\right) + \sum_{k=2}^{\infty}\frac{1}{q_k}\varphi\left(2\lambda\beta p^{k-1}\frac{\varphi^{(-1)}\left(\varphi\left(\frac{2\lambda\gamma\beta}{1-p}\right) + \ln\left(L(\varepsilon_k)\right)\right)}{2\lambda\gamma\beta p^{k-1}}\right) \\
&\quad + \sum_{k=2}^{\infty}\frac{1}{q_k}\varphi\left(2\lambda\gamma\beta p^{k-1}\frac{\varphi^{(-1)}\left(\varphi\left(\frac{2\lambda\gamma\beta}{1-p}\right) + \ln\left(L(\varepsilon_k)\right)\right)}{2\lambda\gamma\beta p^{k-1}}\right) \\
&\le \sum_{k=2}^{\infty}\frac{1}{q_k}\ln\left(L(\varepsilon_k)\right) + \frac{1+|\gamma|}{2|\gamma|}\varphi\left(\frac{2\lambda\gamma\beta}{1-p}\right)\sum_{k=2}^{\infty}\frac{1}{q_k} \\
&\le \sum_{k=2}^{\infty}\ln\left(L(\varepsilon_k)\right)\frac{2\lambda|\gamma|\beta p^{k-1}}{\varphi^{(-1)}\left(\ln\left(L(\varepsilon_k)\right)\right)} + \frac{1+|\gamma|}{2}\varphi\left(\frac{2\lambda\beta}{1-p}\right)\sum_{k=2}^{\infty}p^{k-1}(1-p) \\
&= \frac{(1+|\gamma|)p}{2|\gamma|}\varphi\left(\frac{2\lambda\gamma\beta}{1-p}\right) + 2\lambda|\gamma|\sum_{k=2}^{\infty}\zeta_\varphi\left(\ln\left(L\left(\sigma^{(-1)}\left(\beta p^k\right)\right)\right)\right)\beta p^{k-1}
\end{aligned} \tag{31}$$

and (24) follows from (11) and (31). □

Using the sequence $q_k = (1-p)^{-1}p^{1-k}$, $k = 1, 2, \ldots$, in inequality (11) of Lemma 2.1 we get a result whose condition are easier to check than those of Theorem 2.1.

Theorem 2.2. *Let the input process $\{X(t), t \in B\}$ be of the class $V(\varphi,\psi)$ and condition Σ hold for the γ-reflected process $W_\gamma(t)$, $\gamma \in \mathbb{R}$. Assume that condition Δ holds for the function $\{f(t), t \in B\}$. Let $r = \{r(u), u \ge 1\}$ be a continuous function such that $r(u) > 0$ for $u > 1$ and $s(t) = r(\exp\{t\})$, $t \ge 0$, is a convex function. Let $\tau_B = \sup_{t\in B} X(t) < \infty$, $f_B = \sup_{t\in B} f(t) < \infty$, and $\beta > 0$ be some number such that*

$$\beta \le \sigma\left(\inf_{s\in B}\sup_{t\in B}\rho(t,s)\right).$$

If

$$\int_0^\beta r\left(\mathsf{L}\left(\sigma^{(-1)}(u)\right)\right)\,du<\infty, \tag{32}$$

where $\mathsf{L}(u)=\frac{(\mathsf{N}(u))^2+\mathsf{N}(u)}{2}$ *for all* $p\in(0;1)$ *and* $x>0$, *then*

$$\begin{aligned}
&\mathsf{P}\left\{\sup_{t\in B}W_\gamma(t)>x\right\}\\
&\quad\le r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p} r\left(\mathsf{L}\left(\sigma^{(-1)}(u)\right)\right)du\right)\inf_{\lambda>0}G_\gamma(\lambda,p)\\
&\qquad\times\exp\left\{\frac{p}{2}\left(\varphi\left(\frac{2\lambda p}{1-p}\right)+\varphi\left(\frac{2\lambda\gamma p}{1-p}\right)\right)\right.\\
&\qquad\qquad\left.+\lambda(1+|\gamma|)\sum_{k=1}^{\infty}\delta\left(\sigma^{(-1)}\left(\beta p^k\right)\right)-\lambda x\right\},
\end{aligned}\tag{33}$$

where

$$\begin{aligned}
G_\gamma(\lambda,p)=&\left(\sum_{l=0}^{\mathsf{N}(\sigma^{(-1)}(\beta p))-1}\left(\mathsf{N}\left(\sigma^{(-1)}(\beta p)\right)-l\right)\right.\\
&\times\exp\left\{\varphi\left(\frac{\lambda}{1-p}(\sigma(2\sigma^{(-1)}(\beta p)l)+|1-\gamma|\tau_B\right)\right.\\
&\qquad\left.\left.+\frac{\lambda}{1-p}\left(\delta\left(2\sigma^{(-1)}(\beta p)l\right)+|1-\gamma|f_B\right)\right\}\right)^{1-p}.
\end{aligned}$$

Proof. It is easy to see that the sequences $q_k=(1-p)^{-1}p^{1-k}$ and $\varepsilon_k=\sigma^{(-1)}(\beta p^k)$ satisfy the assumption of Lemma 2.1. Thus

$$\begin{aligned}
&\mathsf{E}\exp\left\{\lambda\sup_{t\in B}W_\gamma(t)\right\}\\
&\quad\le\exp\left\{\sum_{k=2}^{\infty}\left(\frac{(1-p)p^{k-1}}{2}\left(\varphi\left(\frac{2\lambda p}{1-p}\right)+\varphi\left(\frac{2\lambda\gamma p}{1-p}\right)\right)\right.\right.\\
&\qquad\qquad\left.\left.+\lambda(1+|\gamma|)\delta\left(\sigma^{(-1)}\left(\beta p^{k-1}\right)\right)\right)\right\}\\
&\qquad\times\left(\sum_{l=0}^{\mathsf{N}(\sigma^{(-1)}(\beta p))-1}\left(\mathsf{N}\left(\sigma^{(-1)}(\beta p)\right)-l\right)\right.\\
&\qquad\quad\times\exp\left\{\varphi\left(\frac{\lambda}{1-p}\left(\sigma\left(2\sigma^{(-1)}(\beta p)l\right)+|1-\gamma|\tau_B\right)\right)\right.\\
&\qquad\qquad\left.\left.+\frac{\lambda}{1-p}\left(\delta\left(2\sigma^{(-1)}(\beta p)l\right)+|1-\gamma|f_B\right)\right\}\right)^{1-p}\\
&\qquad\times\exp\left\{\sum_{k=1}^{\infty}(1-p)p^{k-1}\log\mathsf{L}\left(\sigma^{(-1)}\left(\beta p^k\right)\right)\right\}.
\end{aligned}\tag{34}$$

Since

$$
\begin{aligned}
&\exp\left\{\sum_{k=1}^{\infty}(1-p)p^{k-1}\log \mathsf{L}\left(\sigma^{(-1)}(\beta p^k)\right)\right\}\\
&\quad= r^{(-1)}\left(r\left(\exp\left\{\sum_{k=1}^{\infty}(1-p)p^{k-1}\log \mathsf{L}\left(\sigma^{(-1)}(\beta p^k)\right)\right\}\right)\right)\\
&\quad\le r^{(-1)}\left(\sum_{k=1}^{\infty}(1-p)p^{k-1}r\left(\mathsf{L}\left(\sigma^{(-1)}(\beta p^k)\right)\right)\right)\\
&\quad\le r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p} r\left(\mathsf{L}\left(\sigma^{(-1)}(u)\right)\right)du\right),
\end{aligned}
\tag{35}
$$

Corollary 2.2 follows from upper bound (34) and Chebyshev's inequality. □

3. A γ-reflected stochastic process with a sub-Gaussian fractional Brownian input

Assume that $X_H(t) = \{X(t), t \in [a,b]\}$ is a generalized fractional Brownian motion being a sub-Gaussian process. Let $X_H(t)$ be defined in the interval $[a,b]$, $0 \le a < b < \infty$. Then

$$
\tau(X_H(t) - X_H(s)) \le |t-s|^H, \qquad H \in (0,1). \tag{36}
$$

(See [18].) In other words, condition Σ holds for this process with $\sigma(h) = h^H$.

Let $f(t)$ be a linear function defined in the interval $[a,b]$, that is $f(t) = ct$, where $c > 0$ is a constant. Then Corollary 2.2 implies the following upper bound for the ruin probability $\Psi_{\gamma,\mathsf{T}}(x)$.

Theorem 3.1. *Let $\{W_\gamma(t), t \in [a,b]\}$, $\gamma \in \mathbb{R}$, be a γ-reflected process with the input being a fractional Brownian motion and, in addition, a generalized sub-Gaussian stochastic process. Then*

$$
\begin{aligned}
&\mathsf{P}\left\{\sup_{t\in[a,b]} W_\gamma(t) > x\right\}\\
&\quad\le \frac{(b-a)^{2+\frac{2}{H}}(pe)^{\frac{2}{H}}}{2^{1+\frac{2}{H}}}\left(1-\frac{2\alpha}{H}\right)^{-\frac{1}{\alpha}}\\
&\qquad\times\left(\sum_{l=0}^{p^{-\frac{1}{H}}}\left(p^{-\frac{1}{H}}+1-l\right)\right.\\
&\qquad\qquad\left.\times\exp\left\{-\frac{\left(\frac{(1+|\gamma|)c(b-a)p^{1/H}}{2(1-p^{1/H})}+c\left((b-a)p^{1/H}l+|1-\gamma|\right)-x\right)^2}{2\left(\frac{2(1+\gamma^2)p^3}{1-p}+(b-a)^H p l^H+|1-\gamma|b^H\right)}\right\}\right)^{1-p}
\end{aligned}
\tag{37}
$$

for all $p \in \left(0; (2/3)^H\right]$ and $x > 0$.

Proof. We apply Theorem 2.2 for the usual Euclidean metric ρ. Put $\beta = ((b-a)/2)^H$. Consider the terms on the right hand side of (33) separately.

Let $r(u) = u^{\alpha}$, $0 < \alpha < \frac{H}{2}$. It is easy to see that if $p \le (2/3)^H$, then

$$\frac{b-a}{2u^{1/H}} > \frac{3}{2},$$

since $u \le (2/3)^H ((b-a)/2)^H \le p\beta$. Hence

$$\begin{aligned}
&r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p^2} r\left(L\left(\sigma^{(-1)}(u)\right)\right)\,du\right)\\
&\le \left(\frac{1}{\beta p}\int_0^{\beta p^2}\left(\left(\frac{b-a}{2u^{1/H}}+1\right)^2+\frac{b-a}{2u^{1/H}}+1\right)^{\alpha}/2^{\alpha}\,du\right)^{\frac{1}{\alpha}}\\
&\le \frac{1}{2}\left(\frac{1}{\beta p}\int_0^{\beta p^2}\left(\frac{b-a}{2u^{1/H}}+\frac{3}{2}\right)^{2\alpha}du\right)^{\frac{1}{\alpha}} < \frac{1}{2}\left(\frac{1}{\beta p}\int_0^{\beta p}\left(\frac{b-a}{u^{1/H}}\right)^{2\alpha}du\right)^{\frac{1}{\alpha}}\\
&= \frac{(b-a)^2}{2}(\beta p)^{2/H}\left(1-\frac{2\alpha}{H}\right)^{-\frac{1}{\alpha}} = \frac{(b-a)^{2+2/H}p^{2/H}}{2^{1+2/H}}\left(1-\frac{2\alpha}{H}\right)^{-\frac{1}{\alpha}},
\end{aligned}$$

whence

$$r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p^2} r\left(L\left(\sigma^{(-1)}(u)\right)\right)\,du\right) \to \frac{(b-a)^{2+\frac{2}{H}}(pe)^{\frac{2}{H}}}{2^{1+\frac{2}{H}}}\left(1-\frac{2\alpha}{H}\right)^{-\frac{1}{\alpha}}, \quad \alpha\to 0. \tag{38}$$

Further,

$$\sum_{k=1}^{\infty}\delta\left(\sigma^{(-1)}\left(\beta p^k\right)\right) = \sum_{k=1}^{\infty} c\left(\beta p^k\right)^{1/H} = \frac{c\beta^{1/H}p^{1/H}}{1-p^{1/H}} = \frac{c(b-a)p^{1/H}}{2(1-p^{1/H})}$$

and

$$\begin{aligned}
&\exp\left\{\left(\frac{p}{2}\left(\varphi\left(\frac{2\lambda p}{1-p}\right)+\varphi\left(\frac{2\lambda\gamma p}{1-p}\right)\right)+\lambda(1+|\gamma|)\sum_{k=1}^{\infty}\delta\left(\sigma^{(-1)}\left(\beta p^k\right)\right)\right)\right\}\\
&= \exp\left\{\frac{\lambda^2(1+\gamma^2)p^3}{(1-p)^2}+\lambda(1+|\gamma|)\frac{c(b-a)p^{1/H}}{2(1-p^{1/H})}\right\}.
\end{aligned} \tag{39}$$

Using (39) and a bound for the metric massiveness $N(u) \le \frac{b-a}{2u} + 1$ we get

$$
\begin{aligned}
&\inf_{\lambda>0} \exp\left\{ \frac{p}{2}\left(\varphi\left(\frac{2\lambda p}{1-p}\right) + \varphi\left(\frac{2\lambda\gamma p}{1-p}\right)\right) + \lambda(1+|\gamma|)\sum_{k=1}^{\infty} \delta\left(\sigma^{(-1)}\left(\beta p^k\right)\right) - \lambda x \right\} G_\gamma(\lambda, p) \\
&\quad = \inf_{\lambda>0} \exp\left\{ \frac{\lambda^2(1+\gamma^2)p^3}{(1-p)^2} + \frac{\lambda(1+|\gamma|)c(b-a)p^{1/H}}{2(1-p^{1/H})} - \lambda x \right\} \\
&\qquad \times \Bigg(\sum_{l=0}^{N\left((\beta p)^{1/H}\right)-1} \left(\frac{b-a}{2(\beta p)^{1/H}} + 1 - l \right) \\
&\qquad\qquad \times \exp\Bigg\{ \frac{1}{2}\left(\frac{\lambda}{1-p}\left(2^H \beta p l^H + |1-\gamma| b^H\right)\right)^2 \\
&\qquad\qquad\qquad + \frac{\lambda}{1-p}\left(c\left(2(\beta p)^{1/H} l\right) + |1-\gamma| c\right)\Bigg\}\Bigg)^{1-p} \\
&\quad \le \Bigg(\sum_{l=0}^{N\left((b-a)p^{1/H}/2\right)-1} \left(p^{-1/H} + 1 - l \right) \\
&\qquad \times \inf_{\lambda>0} \exp\Bigg\{ \frac{\lambda^2(1+\gamma^2)p^3}{(1-p)^3} + \frac{\lambda(1+|\gamma|)c(b-a)p^{1/H}}{2(1-p^{1/H})(1-p)} \\
&\qquad\qquad + \frac{1}{2}\left(\frac{\lambda}{1-p}\right)^2 \left((b-a)^H p l^H + |1-\gamma| b^H\right)^2 \\
&\qquad\qquad\qquad + \frac{\lambda}{1-p}\left(c\left((b-a)p^{1/H} l\right) + |1-\gamma| c\right)\Bigg\}\Bigg)^{1-p} \\
&\quad \le \Bigg(\sum_{l=0}^{p^{-1/H}} \left(p^{-1/H} + 1 - l \right) \\
&\qquad \times \inf_{\lambda>0} \exp\Bigg\{ \frac{\lambda^2}{(1-p)^2}\left(\frac{(1+\gamma^2)p^3}{1-p} + \frac{1}{2}\left((b-a)^H p l^H + |1-\gamma| b^H\right)\right) \\
&\qquad\qquad + \frac{\lambda}{1-p}\Bigg(\frac{(1+|\gamma|)c(b-a)p^{1/H}}{2(1-p^{1/H})} \\
&\qquad\qquad\qquad + c\left((b-a)p^{1/H} l + |1-\gamma|\right) - x\Bigg)\Bigg\}\Bigg)^{1-p} \\
&\quad = \Bigg(\sum_{l=0}^{p^{-1/H}} \left(p^{-1/H} + 1 - l \right) \\
&\qquad \times \exp\left\{ - \frac{\left(\frac{(1+|\gamma|)c(b-a)p^{1/H}}{2(1-p^{1/H})} + c\left((b-a)p^{1/H} l + |1-\gamma|\right) - x\right)^2}{2\left(\frac{2(1+\gamma^2)p^3}{1-p} + (b-a)^H p l^H + |1-\gamma| b^H\right)} \right\}\Bigg)^{1-p}.
\end{aligned}
$$

The latter inequality and (38) complete the proof of Theorem 3.1. □

Remark 3.1. It is easily seen that if $p \in \left(1/2^H, 2^H/3^H\right]$, then the sum in (37) consists of a single term, that is

$$\mathsf{P}\left\{\sup_{t\in[a,b]} W_\gamma(t) > x\right\}$$
$$\le \frac{(b-a)^{2+\frac{2}{H}}(pe)^{2/H}}{2^{1+\frac{2}{H}}}\left(1-\frac{2\alpha}{H}\right)^{-\frac{1}{\alpha}}\left(p^{-1/H}+1\right)^{1-p}$$
$$\times \exp\left\{-\frac{(1-p)\left(\frac{(1+|\gamma|)c(b-a)p^{1/H}}{2(1-p^{1/H})}+c\left((b-a)p^{1/H}l+|1-\gamma|\right)-x\right)^2}{2\left(\frac{2(1+\gamma^2)p^3}{1-p}+(b-a)^H p l^H+|1-\gamma|b^H\right)}\right\}.$$

If $p \in \left(3^{-H}, 2^{-H}\right]$, then the sum in (37) consists of two terms and so on.

Concluding remarks

Some properties of stochastic processes with φ-sub-Gaussian increments are studied in this paper. In particular, we consider processes belonging to a general class $V(\varphi,\psi)$ such that $\varphi \prec \psi$. Upper bounds are obtained for the distribution of the supremum of the corresponding γ-reflected process

$$\sup_{t\in B} W_\gamma(t) = \sup_{t\in B}\left(X(t)-f(t)-\gamma\inf_{s\in B}(X(s)-f(s))\right), \qquad \gamma\in[0,1],$$

where $f(t)$ is a continuous increasing function. The results obtained in the paper can be applied to a wide class of stochastic processes including Gaussian processes. A corresponding bound is given for the case of a γ-reflected process being a fractional Brownian motion and generalized sub-Gaussian process.

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Received 21/MAR/2016

Translated by S. KVASKO