Теорія Ймовір. та Матем. Статист.
Вип. 96, 2017

Theor. Probability and Math. Statist.
No. 96, 2018, Pages 27–43
https://doi.org/10.1090/tpms/1032
Article electronically published on October 5, 2018

THE STUDY OF BASIC RISK PROCESSES BY DISCRETE-TIME NON-HOMOGENEOUS MARKOV PROCESSES

UDC 519.21

G. D'AMICO, F. GISMONDI, J. JANSSEN, R. MANCA, F. PETRONI, AND E. VOLPE DI PRIGNANO

Abstract. This paper elaborates how it is possible to calculate precisely the aggregate claim amount and the claim number by means of Markov reward models in a non-homogeneous time setting. More precisely, evolution equations of the non-homogeneous Markov reward processes are presented in a discounted environment for the calculation of the aggregate claim amount and in a non-discounted case for the calculation of the claim number. The underlying Markov process has a denumerable number of states. In the last section, an application of the proposed models is presented using real data obtained by merging databases of two small insurance companies. The results highlight the importance of the insured's age in the calculation of the actuarial quantities.

1. Introduction

It is well known that the three basic insurance processes, namely,

1) the claim number process,
2) the aggregated claim amount process,
3) the premium process

were first studied in the seminal Lundberg papers [19, 20]. The former papers formed the basis of the renewal processes but in the simpler Poisson process environment. More precisely, Lundberg's papers proposed to model the claim number and the aggregate claim amount by two Poisson processes. Later Cramér [8, 9] proposed a generalization of Lundberg's model, which considered a Poisson process for the study of the claim number process and a "general process" for the claim amount, namely, a renewal process with a distribution that fits well the observed data. Many authors gave important contributions generalizing Lundberg's results. We refer to the papers by de Finetti [13], Cramér [9], and Andersen [1]. More recently the main results were given in [3, 17, 21, 22, 25].

Markov chains are well-known tools, which are applied in almost all scientific fields. For an easy introduction, see [7] and for applications [6, 16].

In risk theory, Markov processes were applied first by Howard and Matheson [14]. In that paper Markov reward processes were used for an introduction to Markov decision processes that were the main tools. Many papers followed this seminal work (see, for example, [4, 5]) presenting applications not in the insurance field. The risk insurance approach by means of a Markov environment was presented for the first time in Janssen [15]. More precisely, contributions based on semi-Markov models, which are strong generalizations of Markov processes, were given in [15, 18]. In those papers the

2010 *Mathematics Subject Classification.* Primary 60J10; Secondary 62P05.

Key words and phrases. Aggregate claim amount process, claim number process, Markov chains, reward processes, non-homogeneity.

states of the process stood for the different types of claims. Following Janssen, Asmussen [2] applied the Markov process in an actuarial environment. Asmussen's paper ignited interest in this direction, and now many researchers have turned their attention towards this approach. Many references on this approach are given in [3].

In the present paper, the Markov approach is given from a non-homogeneous point of view, which is totally new and takes into account the age as a time variable but not the calendar time. For the sake of precision, the non-homogeneous setting is not usually applied. Furthermore, the age as a time variable has never been used in the non-homogeneous environment. To the author's knowledge, the eight evolution equations of non-homogenous Markov reward processes are demonstrated only in [16]. Furthermore, there are 300 different non-homogeneous Markov reward processes: 30 of them are in non-discounted cases, and 270 are in discounted ones.

Indeed, we present a non-homogeneous Markov model that permits the study of the aggregate claim amount risk process and the claim number process in a simple and effective way. The state sets are denumerable and represent the number of claims in both the aggregate claim amount and the claim number processes.

To compute the mean accumulated claim values and the mean number of claims, it is necessary to introduce the non-homogeneous Markov reward processes, which are tools that have never been applied to the calculation of these quantities. To this end, the paper describes both cases, namely, a discounted one for the aggregate claim amount process and a non-discounted one for the claim number process (see [10, 16]).

The present paper generalizes D'Amico et al.'s [11] homogeneous version of this model. Indeed, the introduction of non-homogeneity raises the possibility of considering the different behaviors of insured people of different ages, which is of relevant importance in an insurance setting.

If the non-homogeneous variable is the calendar time, then forecasting of future events is a function of the past data, thus conjuring a biting its tail scenario. In an insurance setting three types of variables are considered: calendar time, age, and seniority. In our opinion, the main time variable is the insured's age. This is evident in life and pension insurance contracts. Nevertheless, in non-life insurance the insured's age also has significant relevance.

The insured's seniority is also of great importance in pension schemes. In age and seniority the non-homogeneity can be applied without the problem of lapsed calendar time because, for example, it is possible to consider that a person of age k will have the same behavior as a person of the same age ten years later. The same could hold true for two workers of the same seniority at different times; i.e., the past data could be used to forecast the future. However, if the non-homogeneity is the calendar time, most of the phenomena change and forecasting of the non-homogeneous models becomes more difficult.

The paper is organized as follows. In Section 2, the Markov reward environment is introduced briefly. In Section 3, the Markov aggregate claim amount process is presented. In Section 4, the claim number process is given. Section 5 presents a numerical example. In Section 6, some conclusive remarks are given.

2. The non-homogeneous discrete-state space and discrete-time Markov processes

2.1. Introduction to discrete-state space and discrete-time non-homogeneous Markov processes. First we want to explore the different ways of generalizing the Markov process in a non-homogeneous environment (see [16]). Let us consider a probability space $(\Omega, \Im, \mathsf{P})$ and a random variable $J_n \colon \Omega \to E$, where E is a discrete set. Here

J_n represents the state of the system at the nth transition. We suppose that T_n is the time of the nth transition and $T_n = n$. The transition at time n will be ruled by the non-homogeneous transition matrix $\mathbf{P}(n)$ with elements

$$p_{ij}(n+1) = \mathsf{P}\left[J_{n+1} = j \mid J_n = i, J_{n-1}, \ldots, J_1, J_0\right] = \mathsf{P}\left[J_{n+1} = j \mid J_n = i\right].$$

Let $\mathbf{\Phi}(s,t) = \big(\phi_{i,j}(s,t)\big)_{i,j\in E}$, $s,t \in \mathbb{N}$, be the matrix function with elements given by $\phi_{i,j}(s,t) = \mathsf{P}\left[J_t = j | J_s = i\right]$. Then we have

$$\prod_{k=s+1}^{t} \mathbf{P}(k) = \mathbf{\Phi}(s,t), \qquad s,t \in \mathbb{N},\ 0 \le s < t.$$

Furthermore, given the vector of the initial probability distribution $\pi(s,s)$ at time s, the related distribution probability vector $\pi(s,t)$ at time t is given by

$$\pi(s,t) = \pi(s,s) * \mathbf{\Phi}(s,t), \qquad s,t \in \mathbb{N},\ 0 \le s < t.$$

Moreover, the Chapman-Kolmogorov equation holds:

$$\mathbf{\Phi}(s,t) * \mathbf{\Phi}(t,k) = \mathbf{\Phi}(s,k), \qquad s \le t \le k, \quad \text{and} \quad \mathbf{\Phi}(t,t) = \mathbf{I},$$

where $\mathbf{I}$ represents the identity matrix.

2.2. **Discrete-state set of non-homogeneous Markov reward process.** Markov reward processes, as specified in [16], can be seen as a class of stochastic processes. Indeed, there are many different evolution equations depending on the problem that should be solved.

Rewards can be measured in amounts of money or differently. In the first case, it is natural to have discounted reward processes. In the second case, discounting does not make sense, and we have non-discounted processes.

In an insurance environment, the rewards that represent money assume great relevance, but, as we will see, in some cases the non-discounted evolution equations can be useful. We will present a general formula with the aim of explaining the meaning of the reward structure. The interested reader is referred to [16] for further reading.

In general, it is possible to have permanence rewards or transition rewards (sometimes also called rate and impulse rewards, respectively; see [23]). The first one is paid or received because of the permanence inside a state, the second one because of a transition.

Rewards can be constant or variable in time. In this second case, they can change as a function of the current time t or depending on the initial time s when the evaluation of the system started. In this last case, we speak of time non-homogeneous rewards.

Let $\psi = \big(\psi_i\big)_{i\in E}$ be the column vector denoting the permanence reward constant in time paid for the occupancy at a given time of state $i \in E$. Moreover, let $\mathbf{\Gamma} = \big(\gamma_{i,j}\big)_{i,j\in E}$ be the matrix of transition rewards constant in time paid for the transition at a given time from state i into state j. Whenever these rewards depend on the current time t, we adopt the corresponding notation $\psi(t) = \big(\psi_i(t)\big)_{i\in E}$ and $\mathbf{\Gamma}(t) = \big(\gamma_{i,j}(t)\big)_{i,j\in E}$. Whenever these rewards are non-homogeneous (i.e., depend on the calendar time t and on the initial time s), we adopt the corresponding notation $\psi(s,t) = \big(\psi_i(s,t)\big)_{i\in E}$ and $\mathbf{\Gamma}(s,t) = \big(\gamma_{i,j}(s,t)\big)_{i,j\in E}$.

In (1)–(3), we give the non-homogeneous Markov reward evolution equations with permanence and transition rewards in the *immediate* case (the permanence rewards are paid at the end of the period). These relations allow us to calculate the reward present values that are paid during the first year starting at time s. More precisely, (1) corresponds to the case where the permanence and transition rewards are functions of the states only. In (2), the permanence and transition rewards are functions of the states and the calendar time. At last, in (3), the rewards are functions of states and, in this case, the starting and ending times are to be considered. Thus, we have

$$V_i(s,s+1)=\nu\left(\psi_i+\sum_{j\in E}p_{ij}(s+1)\gamma_{ij}\right), \tag{1}$$

$$\bar{V}_i(s,s+1)=\nu\left(\psi_i(s+1)+\sum_{j\in E}p_{ij}(s+1)\gamma_{ij}(s+1)\right), \tag{2}$$

$$\bar{\bar{V}}_i(s,s+1)=\nu\left(\psi_i(s,s+1)+\sum_{j\in E}p_{ij}(s+1)\gamma_{ij}(s,s+1)\right). \tag{3}$$

Here ν is the discount factor for one period subject to the interest rate that does not change during the time (flat interest rate structure), the permanence reward is a function of the state in which the system is at time s, while the transition reward is paid at the end of the period because it is possible to know if there has been a transition reward at the end of the period once the transition occurred. Henceforth, the above marked symbol means the variation in the function of the spending time, and the above double marked symbol corresponds to the non-homogeneous case.

In equations (4)–(6), we give the general formulas with $t > s$. We would like to point out that working in a non-homogeneous setting, we can have a homogeneous interest rate term structure (interest rate that changes as a function of calendar time) and a non-homogeneous interest rate term structure where the interest rate changes depending on starting and ending times. Thus, we have

$$\begin{aligned}V_i(s,t)&=\sum_{\tau=s+1}^{t-1}\nu^{\tau-s}\sum_{k\in E}\phi_{ik}(s,\tau-1)\left(\psi_k+\sum_{j\in E}p_{kj}(\tau)\gamma_{kj}\right)\\&\quad+\nu^{t-s}\sum_{k\in E}\phi_{ik}(s,t-1)\left(\psi_k+\sum_{j\in E}p_{kj}(t)\gamma_{kj}\right)\\&=V_i(s,t-1)+\nu^{t-s}\sum_{k\in E}\phi_{ik}(s,t-1)\left(\psi_k+\sum_{j\in E}p_{kj}(t)\gamma_{kj}\right),\end{aligned} \tag{4}$$

$$\begin{aligned}\bar{V}_i(s,t)&=\sum_{\tau=s+1}^{t-1}\nu^{\tau-s}\sum_{k\in E}\phi_{ik}(s,\tau-1)\left(\psi_k(\tau)+\sum_{j\in E}p_{ij}(\tau)\gamma_{ij}(\tau)\right)\\&\quad+\nu^{t-s}\sum_{k\in E}\phi_{ik}(s,t-1)\left(\psi_k(t)+\sum_{j\in E}p_{kj}(t)\gamma_{kj}(t)\right)\\&=\bar{V}_i(s,t-1)+\nu^{t-s}\sum_{k\in E}\phi_{ik}(s,t-1)\left(\psi_k(t-1)+\sum_{j\in E}p_{kj}(t)\gamma_{kj}(t)\right),\end{aligned} \tag{5}$$

$$\begin{aligned}\bar{\bar{V}}_i(s,t)&=\sum_{\tau=s+1}^{t-1}\nu^{\tau-s}\sum_{k\in E}\phi_{ik}(s,\tau-1)\left(\psi_k(s,\tau)+\sum_{j\in E}p_{ij}(\tau)\gamma_{ij}(s,\tau)\right)\\&\quad+\nu^{t-s}\sum_{k\in E}\phi_{ik}(s,t-1)\left(\psi_k(s,t)+\sum_{j\in E}p_{kj}(t)\gamma_{kj}(s,t)\right)\\&=\bar{\bar{V}}_i(s,t-1)+\nu^{t-s}\sum_{k\in E}\phi_{ik}(s,t-1)\left(\psi_k(s,t)+\sum_{j\in E}p_{kj}(t)\gamma_{kj}(s,t)\right),\end{aligned} \tag{6}$$

where $V_i(s,t)$, $\bar{V}_i(s,t)$, $\bar{\bar{V}}_i(s,t)$ represent the mean present values of all the rewards (RMPV) that have been paid and/or received up to time t starting from the state i at time s. Here ν^t is the discount factor for t periods in the three considered cases and $\mathbf{\Phi}(s,s) = \mathbf{I}$ is the identity matrix [17]. The evolution equations are iterative; i. e., the subsequent step is obtained by adding the present value of the rewards paid in the last period to the previous steps. The previous relations are simple to understand; i. e., $\phi_{ik}(s,t)$ is the probability to remain at time t in the state k, given that at time s the system was in the state i.

Under the same hypotheses the evolution equations in the *due* case (the rate rewards are paid at beginning of each period and the impulse rewards at the end) are presented in (7)–(9) provided that one year has passed after the beginning time s. In (10)–(12), we give the general relations with $t > s$. So we have

$$\ddot{V}_i(s,s+1) = \left(\psi_i + \nu \sum_{j\in E} p_{ij}(s+1)\gamma_{ij}\right), \tag{7}$$

$$\ddot{\bar{V}}_i(s,s+1) = \left(\psi_i(s) + \nu \sum_{j\in E} p_{ij}(s+1)\gamma_{ij}(s+1)\right), \tag{8}$$

$$\ddot{\bar{\bar{V}}}_i(s,s+1) = \left(\psi_i(s,s) + \nu \sum_{j\in E} p_{ij}(s+1)\gamma_{ij}(s,s+1)\right), \tag{9}$$

$$\begin{aligned}
\ddot{V}_i(s,t) &= \left(\psi_i + v \sum_{j\in E} p_{ij}(s,s+1)\gamma_{ij}\right) \\
&\quad + \sum_{\tau=s+2}^{t-1} \nu^{\tau-s-1} \sum_{k\in E} \phi_{ik}(s,\tau-1)\left(\psi_k + v \sum_{j\in E} p_{kj}(\tau)\gamma_{kj}\right) \\
&\quad + \nu^{t-s-1} \sum_{k\in E} \phi_{ik}(s,t-1)\left(\psi_k + \nu \sum_{j\in E} p_{kj}(t)\gamma_{kj}\right) \\
&= \ddot{V}_i(s,t-1) + \nu^{t-s-1} \sum_{k\in E} \phi_{ik}(s,t-1)\left(\psi_k + \nu \sum_{j\in E} p_{kj}(t)\gamma_{kj}\right),
\end{aligned} \tag{10}$$

$$\begin{aligned}
\ddot{\bar{V}}_i(s,t) &= \left(\psi_i(s) + \nu \sum_{j\in E} p_{ij}(s,\tau)\gamma_{ij}(\tau)\right) \\
&\quad + \sum_{\tau=s+2}^{t-1} \nu^{\tau-s-1} \sum_{k\in E} \phi_{ik}(s,\tau-1)\left(\psi_k(\tau) + \nu \sum_{j\in E} p_{kj}(\tau)\gamma_{kj}(\tau)\right) \\
&\quad + \nu^{t-s-1} \sum_{k\in E} \phi_{ik}(s,t-1)\left(\psi_{ik}(t-1) + \nu \sum_{j\in E} p_{kj}(t)\gamma_{kj}(t)\right) \\
&= \ddot{\bar{V}}_i(s,t-1) + \nu^{t-s-1} \sum_{k\in E} \phi_{ik}(s,t-1)\left(\psi_k(t-1) + \nu \sum_{j\in E} p_{kj}(t)\gamma_{kj}(t)\right),
\end{aligned} \tag{11}$$

$$
\begin{aligned}
\ddot{\bar{\bar{V}}}_i(s,t) &= \Bigg(\psi_i(s,s) + \nu \sum_{j\in E} p_{ij}(s,s+1)\gamma_{ij}(s,s+1)\Bigg) \\
&\quad + \sum_{\tau=s+2}^{t-1} \nu^{\tau-s-1} \sum_{k\in E} \phi_{ik}(s,\tau-1) \\
&\qquad \times \Bigg(\psi_k(s,\tau-1) + \nu \sum_{j\in E} p_{kj}(\tau)\gamma_{kj}(s,\tau)\Bigg) \\
&\quad + \nu^{t-s-1} \sum_{k\in E} \phi_{ik}(s,t-1)\Bigg(\psi_k(s,t-1) + \nu \sum_{j\in E} p_{kj}(t)\gamma_{kj}(s,t)\Bigg) \\
&= \ddot{\bar{\bar{V}}}_i(s,t-1) \\
&\quad + \nu^{t-s-1} \sum_{k\in E} \phi_{ik}(s,t-1)\Bigg(\psi_k(s,t-1) + \nu \sum_{j\in E} p_{kj}(t)\gamma_{kj}(s,t)\Bigg).
\end{aligned} \tag{12}
$$

It is assumed that the transition reward is always paid at the end of the period. By this reason, in the due case, it is necessary to consider one more period of discounting. The dieresis means that the relations are referred to the due cases.

Before defining the matrix evolution equation, it is necessary to define the following matrix product (see [10]).

Definition 1. For two matrices $\mathbf{A}$ and $\mathbf{B}$ with m rows and n columns, we define the following operation:

$$\mathbf{v} = \mathbf{A} \circ \mathbf{B}, \qquad v_i = \sum_{j=1}^{n} a_{ij} \cdot b_{ij} = \mathbf{a}_{i*} * \mathbf{b}_{i*}, \qquad i = 1, \dots, m \quad \text{and} \quad \mathbf{a}_{i*}, \mathbf{b}_{i*} \in \mathbb{R}^n.$$

We have $\mathbf{c} = \mathbf{A} \circ \mathbf{B}$ where $c(i) = \sum_{j\in E} a_{ij} b_{ij} = \mathbf{a}_{i*} * \mathbf{b}_{i*}$ and $\mathbf{a}_{i*}, \mathbf{b}_{i*} \in \mathbb{R}^n$ if E is finite or $\mathbf{a}_{i*}, \mathbf{b}_{i*} \in \mathbb{R}^{\mathbb{N}}$ if E is denumerable. The relations (4)–(6) were written for a flat interest rate. Now we rewrite those equations in matrix form. The equations (13) and (14) correspond to a spot interest rate structure, and (15) is written for the case of a non-homogeneous interest rate structure. So we have

$$
\begin{aligned}
\mathbf{V}(s,t) &= \nu(s,s+1)\mathbf{I} * \big(\psi + \mathbf{P}(s+1) \circ \mathbf{\Gamma}\big) + \dots \\
&\quad + \nu(s,t)\big(\mathbf{\Phi}(s,t-1) * \psi + \mathbf{\Phi}(s,t) \circ \mathbf{\Gamma}\big),
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\bar{\mathbf{V}}(s,t) &= \nu(s,s+1)\mathbf{I} * \big(\psi(s+1) + \mathbf{P}(s+1) \circ \mathbf{\Gamma}(s+1)\big) + \dots \\
&\quad + \nu(s,t)\big(\mathbf{\Phi}(s,t-1) * \psi(t) + \mathbf{\Phi}(s,t) \circ \mathbf{\Gamma}(t)\big),
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\bar{\bar{\mathbf{V}}}(s,t) &= \dot{\bar{\nu}}(s,s+1)\mathbf{I} * \big(\psi(s,s+1) + \mathbf{P}(s+1) \circ \mathbf{\Gamma}(s,s+1)\big) + \dots \\
&\quad + \dot{\bar{\nu}}(s,t)\big(\mathbf{\Phi}(s,t-1) * \psi(s,t) + \mathbf{\Phi}(s,t) \circ \mathbf{\Gamma}(s,t)\big),
\end{aligned} \tag{15}
$$

where

$$v(s,t) = \prod_{h=s+1}^{t} (1 + r(h))^{-1},$$

$r(h)$ is the interest rate at time h,

$$\dot{\bar{v}}(s,t) = \prod_{h=s+1}^{t} (1 + r(s,h))^{-1},$$

and $r(s,h)$ is the non-homogeneous interest rate from the time s up to the time t. In addition, let $\dot{v}(s,h) = (1+r(s,h))^{-1}$. Furthermore, $*$ represents the usual row-column matrix product.

Formulas (10)–(12) can be rewritten more compactly. Formulas (16) and (17) below correspond to the case of a spot interest rate structure, and (18) is written for the case of a non-homogeneous interest rate structure:

$$\begin{aligned}\ddot{\mathbf{V}}(s,t) &= \big(\psi + v(s,s+1)\mathbf{P}(s+1)\circ\mathbf{\Gamma}\big)\\ &\quad+\sum_{\tau=s+2}^{t-1}\nu(s,\tau-1)\big(\mathbf{\Phi}(s,\tau-1)*\psi+\nu(\tau)\mathbf{\Phi}(s,\tau)\circ\mathbf{\Gamma}\big)\\ &\quad+\nu(s,t-1)\big(\mathbf{\Phi}(s,t-1)*\psi+\nu(t)\mathbf{\Phi}(s,t)\circ\mathbf{\Gamma}\big),\end{aligned}\tag{16}$$

$$\begin{aligned}\ddot{\ddot{\mathbf{V}}}(s,t) &= \psi(s) + v(s,s+1)\mathbf{P}(s+1)\circ\mathbf{\Gamma}(s+1)\\ &\quad+\sum_{\tau=s+2}^{t-1}\nu(s,\tau-1)\big(\mathbf{\Phi}(s,\tau-1)*\psi(\tau-1)+\nu(\tau)\mathbf{\Phi}(s,\tau)\circ\mathbf{\Gamma}(\tau)\big)\\ &\quad+\nu(s,t-1)\big(\mathbf{\Phi}(s,t-1)*\psi(t-1)+\nu(t)\mathbf{\Phi}(s,t)\circ\mathbf{\Gamma}(t)\big),\end{aligned}\tag{17}$$

$$\begin{aligned}\ddot{\ddot{\ddot{\mathbf{V}}}}(s,t) &= \big(\psi(s,s) + v(s,s+1)\mathbf{P}(s+1)\circ\mathbf{\Gamma}(s,s+1)\big)\\ &\quad+\sum_{\tau=s+2}^{t-1}\nu(s,\tau-1)\big(\mathbf{\Phi}(s,\tau-1)*\ \psi(s,\tau-1)+\dot{\nu}(s,\tau)\mathbf{\Phi}(s,\tau)\circ\mathbf{\Gamma}(s,\tau)\big)\\ &\quad+\nu(s,t-1)\big(\mathbf{\Phi}(s,t-1)*\psi(s,t-1)+\dot{\nu}(s,t)\mathbf{\Phi}(s,t)\circ\mathbf{\Gamma}(s,t)\big).\end{aligned}\tag{18}$$

Remark 1. If $E=\mathbb{N}$, then $\mathbf{V}(s,t)\left(\ddot{\mathbf{V}}(s,t)\right)$ is obtained as a sum of infinite vectors whose elements are in series. Indeed, at each time step we will have two infinite vectors of series. In addition, nobody can assure the related convergence for each element of the vectors. Furthermore, there are infinite elements in each vector. In our opinion, the only way to solve these evolution equations is to apply the truncation method [24] as shown in (19)–(21) for the three cases that we consider now.

For given s and t, we calculate

$$s^i_{m'_k}(s,t) = \sum_{j=1}^{m'_k}\sum_{\tau=s+1}^{t}\phi_{ij}(s,\tau)\gamma_{ij},\qquad k\in\mathbb{N},$$

and

$$s^i_{m'_k+1}(s,t) = \sum_{j=1}^{m'_k+1}\sum_{\tau=s}^{t}\phi_{ij}(s,\tau)\gamma_{ij},$$

stopping the calculation when

$$\left|s^i_{m'_k+1}(s,t) - s^i_{m'_k}(s,t)\right| < \varepsilon;\tag{19}$$

$$s^i_{m''_k}(s,t) = \sum_{j=1}^{m''_k}\sum_{\tau=s+1}^{t}\phi_{ij}(s,\tau)\gamma_{ij}(\tau)$$

and

$$s^i_{m''_k+1}(s,t) = \sum_{j=1}^{m''_k+1}\sum_{\tau=s+1}^{t}\phi_{ij}(s,\tau)\gamma_{ij}(\tau),$$

stopping the calculation when

$$\left| s^i_{m''_k+1}(s,t) - s^i_{m''_k}(s,t) \right| < \varepsilon; \tag{20}$$

$$s^i_{m'''_k}(s,t) = \sum_{j=1}^{m'''_k} \sum_{\tau=s+1}^{t} \phi_{ij}(s,\tau)\gamma_{ij}(s,\tau)$$

and

$$s^i_{m'''_k+1}(s,t) = \sum_{j=1}^{m'''_k+1} \sum_{\tau=s+1}^{t} \phi_{ij}(s,\tau)\gamma_{ij}(s,\tau),$$

stopping the calculation when

$$\left| s^i_{m'''_k+1}(s,t) - s^i_{m'''_k}(s,t) \right| < \varepsilon, \tag{21}$$

where ε is fixed at the beginning. In this way we have fixed the maximum order of the transition matrices of the non-homogeneous model. It should be remarked that the same approach has to be applied in order to control the convergence of the permanence rewards.

Remark 2. Our model is described by a block matrix. The order of inner matrices can be infinite as shown in Remark 1. When we need to compute $\mathbf{V}(s,T)$ for large values of T, we should activate another numerical process in the following way. Once the order of the block matrices is decided, we can begin with a start time s and an arrival time t. Then we calculate

$$\sum_{k=0}^{t} \|\mathbf{V}(k,t) - \mathbf{V}(k,t+1)\| < \varepsilon, \qquad k = 0,\dots,t-1 \wedge \|\mathbf{V}(t,t+1)\|\,,\ T = t,$$

where $\|*\|$ represents the Euclidean norm of a vector. This strategy works well whenever the rate of increasing reward is lower than the interest rate.

3. The non-homogeneous Markov aggregate claim amount model

In the proposed model the state of the Markov process J_n represents the number of claims reported up to the nth period, i.e., $J_n \in \mathbb{N}$. Each year represents the discrete time period, as it naturally occurs, for example, in car insurance contracts. The matrix equation can be solved applying the truncation method (see [24]) described in Remarks 1 and 2. We now study the aggregate claim process and the Markov chains in a non-homogeneous case.

Indeed, we study an accumulation process, and J_n represents the total number of claims up to the nth period. Under this hypothesis, it follows that $J_{n-1}(\omega) \le J_n(\omega)$ for all $\omega \in \Omega$.

The non-homogeneous Markov chain will be the following:

$$\mathbf{P}(s) = \begin{bmatrix} p_{0,0}(s) & p_{0,1}(s) & \cdots & p_{0,k}(s) & p_{0,k+1}(s) & \cdots \\ 0 & p_{1,1}(s) & \cdots & p_{1,k}(s) & p_{1,k+1}(s) & \cdots \\ 0 & 0 & \cdots & p_{2,k}(s) & p_{2,k+1}(s) & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & p_{k,k}(s) & p_{k,k+1}(s) & \cdots \\ 0 & 0 & \cdots & 0 & p_{k+1,k+1}(s) & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{22}$$

In this case, $p_{ij}(s)$ represents the probability of going from the number of claims i to the number of claims j at time s. In particular, $p_{ii}(s)$ represents the probability that at time s there were no claims (probability of a virtual transition of the state i at time s) given that at the time s the number of reported claims was i.

It is well known that the probability to have 0 claims in one year is the highest of the other possible events. The transition matrix (22) is triangular, and the main diagonal has all the elements greater than 0, and its determinant is not equal to 0. This is the condition for having the solution to a countable system with countable unknowns [24]. Furthermore, the hypothesis that the interest rate is greater than the rate of reward increasing implies that the formulas (19)–(21) tend to 0.

In the general cases, the transition matrix of the Markov process is denumerable for all $s \in \mathbb{N}$. The treatment of denumerable homogeneous Markov chains is well known. In constrast, the non-homogeneous case is less studied. In the aggregate claim amount, there are no permanence rewards. The relations (4)–(6) become

$$V_i(s,t) = V_i(s,t-1) + \nu^{t-s} \sum_{k\in E} \phi_{ik}(s,t-1) \sum_{j\in E} p_{kj}(t)\gamma_{kj},$$

$$\bar{V}_i(s,t) = \bar{V}_i(s,t-1) + \nu^{t-s} \sum_{k\in E} \phi_{ik}(s,t-1) \sum_{j\in E} p_{kj}(t)\gamma_{kj}(t),$$

$$\bar{\bar{V}}_i(s,t) = \bar{\bar{V}}_i(s,t-1) + \nu^{t-s} \sum_{k\in E} \phi_{ik}(s,t-1) \sum_{j\in E} p_{kj}(t)\gamma_{kj}(s,t).$$

In the case of the absence of permanence rewards, the due case does not make sense because a transition reward cannot be paid at the beginning of the period. Indeed, it is impossible to know if the transition will happen or not, and we will suppose that transition rewards will be paid only at the end of the periods.

It is clear that it is not possible that an insured can have an infinite number of claims. Having a good database, it is possible to check the maximum number of claims that each insured in the database has had in one year. Suppose that this number is k; then we can suppose that the maximum number of accidents in one year is k. This hypothesis implies that each row of the Markov matrix will have k non-zero elements. Now we will show how the product of n upper triangular matrices $\mathbf{P}$ with only the main diagonal and the following $k-1$-element diagonal, which may be non-null, behaves.

Proposition 1. *For $h = 1, \dots, n$, let*

$$\mathbf{P}(h) = \begin{bmatrix} p_{1,1}(h) & p_{1,2}(h) & \cdots & p_{1,k}(h) & 0 & 0 & \cdots \\ 0 & p_{2,2}(h) & \cdots & p_{2,k}(h) & p_{2,k+1}(h) & 0 & \cdots \\ 0 & 0 & \cdots & p_{3,k}(h) & p_{3,k+1}(h) & p_{3,k+2}(h) & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & p_{k,k}(h) & p_{k,k+1}(h) & p_{k,k+2}(h) & \cdots \\ 0 & 0 & \cdots & 0 & p_{k+1,k+1}(h) & p_{k+1,k+2}(h) & \cdots \\ 0 & 0 & \cdots & 0 & 0 & p_{k+2,k+2}(h) & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{23}$$

be an upper triangular transition probability matrix with the first k diagonal that may be not equal to 0. *Letting $\mathbf{\Phi}(0,n) = \prod_{h=1}^{n} \mathbf{P}(h)$ gives*

$$\phi_{ij}(0,n) = 0 \quad \textit{if } i < j \quad \textit{or} \quad j > i + nk - (n-1).$$

The steps of the proof are the same as in [11].

Remark 3. It is evident that the nth product of the matrices will have in each row at most $nk - (n-1)$ non-zero coefficients.

Remark 4. The elements of the matrix (23) give the probability to make $j-i$ claims in one period given that there were i claims before. In his car's lifetime, a given driver made i claims if he is in the state i. Therefore, in this case, it is impossible that a person could have more than k claims. The value of k can be obtained by the data set as explained previously. For example, we have two small databases in which in total there are about $262,000$ records, and the maximum number of claims in one year is 4. Furthermore, we could consider the matrices of (23) as square matrices of order m; i.e., we suppose that in the driving life the maximum number of claims of an insured will be less than or equal to m. The matrix (23) becomes

(24)

$$\mathbf{P}(h) = \begin{bmatrix} p_{0,0}(h) & p_{0,1}(h) & \cdots & p_{0,k}(h) & 0 & \cdots & 0 & 0 \\ 0 & p_{1,1}(h) & \cdots & p_{1,k}(h) & p_{1,k+1}(h) & \cdots & 0 & 0 \\ 0 & 0 & \cdots & p_{2,k}(h) & p_{2,k+1}(h) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{k-1,k}(h) & p_{k-1,k+1}(h) & \cdots & 0 & 0 \\ 0 & 0 & \cdots & p_{k,k}(h) & p_{k,k+1}(h) & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & p_{k+1,k+1}(h) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & p_{m-1,m-1}(h) & p_{m-1,m}(h) \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & p_{m,m}(h) \end{bmatrix}.$$

The model with matrix (24) overcomes the second problem of Remark 1.

4. The Markov claim number processes

By means of small changes in the non-homogeneous Markov reward processes, it is possible to construct a useful model for the calculation of mean claim numbers. In this case, it is necessary to note that reward processes can be discounted or non-discounted (for a complete classification of reward processes see [16]). In the previous sections, we gave the evolution equation for discounted cases. In the non-discounted cases, it does not make sense to give the immediate and due distinction. By setting $v(n)=1$ for all $n \in \mathbb{N}$, relations (4)–(6) become

$$V_i(s,t) = V_i(s,t-1) + \sum_{k\in E} \phi_{ik}(s,t-1)\left(\sum_{j\in E} p_{kj}(t)\right),$$

$$\bar{V}_i(s,t) = \bar{V}_i(s,t-1) + \sum_{k\in E} \phi_{ik}(s,t-1)\left(\sum_{j\in E} p_{kj}(t)\right),$$

$$\bar{\bar{V}}_i(s,t) = \bar{\bar{V}}_i(s,t-1) + \sum_{k\in E} \phi_{ik}(s,t-1)\left(\sum_{j\in E} p_{kj}(t)\gamma_{kj}(s,t)\right),$$

which in matrix form can be written as

$$\begin{aligned} \mathbf{V}(s,t) &= \mathbf{V}(s,t-1) + \mathbf{\Phi}(s,t-1) * \big(\mathbf{P}(t)\big), \\ \bar{\mathbf{V}}(s,t) &= \bar{\mathbf{V}}(s,t-1) + \mathbf{\Phi}(s,t-1) * \big(\mathbf{P}(t)\circ \mathbf{\Gamma}(t)\big), \\ \bar{\bar{\mathbf{V}}}(s,t) &= \bar{\bar{\mathbf{V}}}(s,t-1) + \mathbf{\Phi}(s,t-1) * \big(\mathbf{P}(t)\circ \mathbf{\Gamma}(s,t)\big). \end{aligned} \tag{25}$$

The Markov claim number model does not have the permanence rewards, and the transition rewards are constant. Under these hypotheses, relation (25) becomes

$$\mathbf{V}(s,t) = \mathbf{V}(s,t-1) + \mathbf{\Phi}(s,t-1) * \big(\mathbf{P}(t)\big).$$

Relation (26) gives the reward matrix of the Markov claim number process. In γ_{ij}, $i-1$ represents the number of claims previously reported by an insured and $j-i$ the claims reported in the last studied period. For example, row 1 represents an insured that has never reported claims, and the element 2, which is the coefficient $\gamma_{1,3}$ in the matrix, represents the number of claims that could have been reported in the last period. Continuing the example, the third row represents the insured who has already reported 2 claims, and the 2 in the position $\gamma_{3,5}$ gives the number of claims that could have been reported in the last period. Thus, we have

$$\mathbf{\Gamma} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & \cdots \\ 0 & 0 & 1 & 2 & 3 & \cdots \\ 0 & 0 & 0 & 1 & 2 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{26}$$

Under this hypothesis, for each starting state i, the relations (25) will give the mean number of claims that will be reported by an insured for each time along the studied horizon.

5. An applicative example using an almost real data set

5.1. **The aggregate claim amount.** The construction of an applicative example implies that the data set that will be used is collected taking into account the model to which it will be applied. Fortunately, we had data sets with 156 428 and 105 627 records. The first of them was the complete history (up to 1998) of a small Italian car insurance company. The second one was the three years' history of an Italian car's insurance. We decided to merge the two files, appending the records of the second set to the first one. In this way, we had an artificial data set coming by real data. Moreover, the mean costs of 1, 2, 3, and 4 claims were obtained by the more reliable dataset. In the evaluation of the Markov transition matrix, the virtual transitions assume great relevance, because a great percentage of insured people do not crash and consequently do not report claims. In our model, if an insured person does not report a claim, then she/he will remain in the same state and will have a virtual transition. In the first data set, we had the dates of insurance contract signings. We can compute all the virtual transitions that an insured had during her/his life. As outlined previously, the maximum number of claims per period k has been set to 4. Furthermore, the maximum reported claims over the driving lifetime of each insured was 10. By this reasoning, the Markov matrix of our example has the order 10. In Table 1, we give the frequency matrix of the number of transitions that we obtain with our data. Altogether we had 2 915 729 transitions. In Figure 1, we present the Markov transition matrix obtained by Table 1.

It is possible to observe that state 10 is absorbing and about 90% of the insured will remain in state 1 next time. From the data, we evaluated the mean cost of 1 claim, 2 claims, 3 claims, and 4 claims that happened in one year. The initial reward matrix is shown in Table 2.

Table 1. Number of transitions among the states; the outside column contains the state; the element i, j is the number of transitions from state i to state j.

State	1	2	3	4	5	6	7	8	9	10
1	1365340	160175	15775	202	58	0	0	0	0	0
2	0	815827	216110	2905	164	8	0	0	0	0
3	0	0	190041	35494	2958	76	16	0	0	0
4	0	0	0	54206	18917	921	82	0	0	0
5	0	0	0	0	18849	5139	680	48	0	0
6	0	0	0	0	0	5863	2481	104	16	0
7	0	0	0	0	0	0	1846	568	140	0
8	0	0	0	0	0	0	0	540	116	2
9	0	0	0	0	0	0	0	0	44	15
10	0	0	0	0	0	0	0	0	0	3

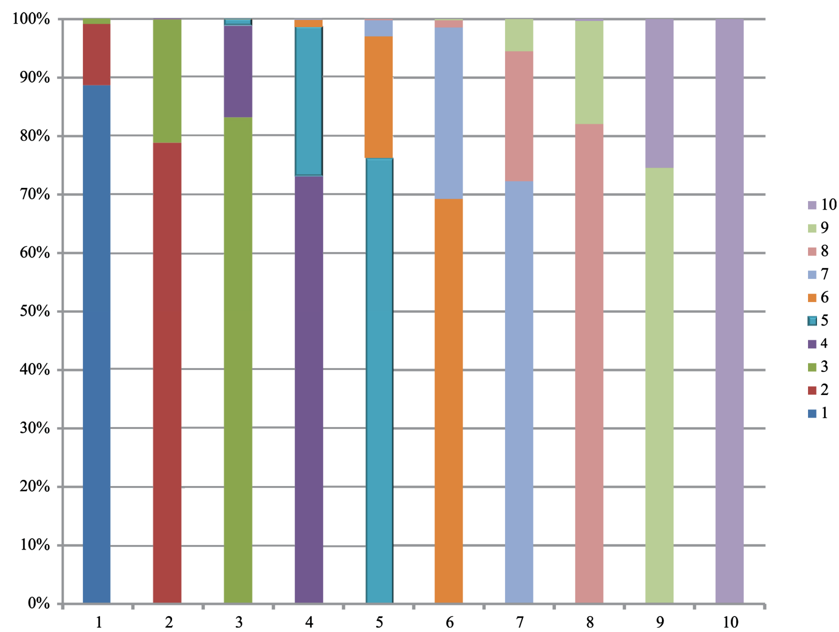

Figure 1. Discrete-time homogeneous Markov chain.

The amount 2487.7 euro represents the mean cost of 1 claim. The amount 4259.163 is the mean sum paid in one year if 2 claims are reported. The amount 5019.9 is the mean expense that the company should pay to an insured in the case that she/he reports 3 claims. At last, 7001.9 is the mean cost sustained by the company in one year if 4 claims are reported. The sums given in Table 2 represent the values paid in the first year. We suppose that the values will increase at a yearly rate of 1%. We follow the evolution of our phenomenon for 30 years. In Table 3, the rewards in the last year are given.

In Table 4, we present the mean expenses sustained for each time and each state. The values are discounted at time 0 with 3% interest rate. The element 216.14, which is in the second place of column 25, represents the mean cost that the insurance company should pay to an insured who made 1 claim at the start time, i.e., $V_2(0, 24) - V_2(0, 23)$.

TABLE 2. Mean costs of different numbers of claims at the first year.

State	1	2	3	4	5	6	7	8	9	10
1	0	2487.7	4259.2	5019.9	7001.9	0	0	0	0	0
2	0	0	2487.7	4259.2	5019.9	7001.9	0	0	0	0
3	0	0	0	2487.7	4259.2	5019.9	7001.9	0	0	0
4	0	0	0	0	2487.7	4259.2	5019.9	7001.9	0	0
5	0	0	0	0	0	2487.7	4259.2	5019.9	7001.9	0
6	0	0	0	0	0	0	2487.7	4259.2	5019.9	7001.9
7	0	0	0	0	0	0	0	2487.7	4259.2	5019.9
8	0	0	0	0	0	0	0	0	2487.7	4259.2
9	0	0	0	0	0	0	0	0	0	2487.7
10	0	0	0	0	0	0	0	0	0	0

TABLE 3. Mean costs of different numbers of claims in the last year.

State	1	2	3	4	5	6	7	8	9	10
1	0	3352.9	5740.7	6766.2	9437.6	0	0	0	0	0
2	0	0	3352.9	5740.7	6766.2	9437.6	0	0	0	0
3	0	0	0	3352.9	5740.7	6766.2	9437.6	0	0	0
4	0	0	0	0	3352.9	5740.7	6766.2	9437.6	0	0
5	0	0	0	0	0	3352.9	5740.7	6766.2	9437.6	0
6	0	0	0	0	0	0	3352.9	5740.7	6766.2	9437.6
7	0	0	0	0	0	0	0	3352.9	5740.7	6766.2
8	0	0	0	0	0	0	0	0	3352.9	5740.7
9	0	0	0	0	0	0	0	0	0	5740.7
10	0	0	0	0	0	0	0	0	0	0

TABLE 4. The matrix element i, j gives the mean costs of claims paid for each five years.

	Years						
State	1	5	10	15	20	25	30
1	294.16	345.35	354.23	345.83	312.22	256.86	192.94
2	516.72	457.40	444.22	396.55	311.48	216.14	134.59
3	430.63	510.17	478.60	382.12	263.51	160.57	88.39
4	673.13	601.50	476.11	313.31	176.88	88.22	39.89
5	625.41	601.01	411.73	235.30	116.77	43.69	21.26
6	767.98	546.24	307.78	148.19	62.76	19.92	13.31
7	763.80	427.46	212.41	90.03	52.22	10.01	6.54
8	438.35	326.76	141.10	52.51	34.42	3.96	2.03
9	614.03	125.85	25.04	4.98	18.29	90.10	0.04
10	0	0	0	0	0	0	0

In Table 5, the aggregate claim amounts for each year of the time horizon are shown. In addition, these values are discounted at time 0.

The elements of Table 5 represent the sequence of the mean total cost of insured customers. For example, 9739 in the second place of column 25 represents the mean total cost that was paid within 25 years for an insured who at time 0 was in state 2 (1 claim), i.e., $V_2(0, 24)$.

Table 5. Aggregate claim amounts for each year of the horizon time.

State	Years						
	1	5	10	15	20	25	30
1	294	1647	3760	5161	6799	8201	9332
2	517	2382	5074	6731	8469	9739	10616
3	431	2436	5387	7046	8600	9598	10215
4	673	3164	6270	7718	8857	9457	9768
5	625	3167	5991	7128	7925	8296	8472
6	768	3287	5544	6302	6762	6947	7025
7	764	2817	4453	4933	5197	5294	5333
8	438	2005	3158	3449	3596	3646	3665
9	614	1485	1768	1797	1812	1815	1815
10	0	0	0	0	0	0	0

Table 6. Number of reported claims in one year.

State	1	2	3	4	5	6	7	8	9	10
1	0	1	2	3	4	0	0	0	0	0
2	0	0	1	2	3	4	0	0	0	0
3	0	0	0	1	2	3	4	0	0	0
4	0	0	0	0	1	2	3	4	0	0
5	0	0	0	0	0	1	2	3	4	0
6	0	0	0	0	0	0	1	2	3	4
7	0	0	0	0	0	0	0	1	2	3
8	0	0	0	0	0	0	0	0	1	2
9	0	0	0	0	0	0	0	0	0	1
10	0	0	0	0	0	0	0	0	0	0

5.2. **The mean claim number.** In this part, we present the results that we obtained for the total mean claim number. As we already pointed out, in this case we applied the non-discounted Markov reward model. In Table 6, the matrix of the rewards is shown.

Each reward gives the number of claims that are reported. For example, 3 in the second place of the sixth column represents the number of claims reported by an insured who was at time 0 in state 2 (1 claim).

In Table 7, the mean number of claims reported for each year of the horizon time is given.

For example, 0.14 in the third place of column 25 gives the mean number of claims reported by an insured who was at time 0 in state 3 (2 reported claims).

Table 8 gives the total mean number of claims that should be reported within time s. For example, 6.83 in the third place of column 25 gives the total number of claims that were reported up to time 25, starting at time 0 from state 3 (2 claims).

TABLE 7. Mean number of claims reported for each year of the horizon time.

	Years						
State	1	5	10	15	20	25	30
1	0.12	0.16	0.20	0.22	0.23	0.22	0.19
2	0.21	0.22	0.25	0.25	0.25	0.19	0.13
3	0.18	0.21	0.26	0.25	0.24	0.14	0.09
4	0.28	0.29	0.25	0.29	0.19	0.08	0.04
5	0.27	0.29	0.21	0.29	0.14	0.04	0.02
6	0.32	0.26	0.15	0.28	0.08	0.02	0.01
7	0.33	0.20	0.10	0.22	0.05	0.01	0.01
8	0.18	0.15	0.18	0.07	0.03	0.016	0.002
9	0.25	0.01	0.12	0.08	0.01	0.002	0.001
10	0	0	0	0	0	0	0

TABLE 8. Total mean number of claims reported for each year of the horizon time.

	Years						
State	1	5	10	15	20	25	30
1	0.12	0.73	1.65	2.94	3.86	5.00	6.02
2	0.21	1.06	3.60	5.12	6.06	7.09	7.86
3	0.18	1.10	3.68	5.20	6.03	6.83	7.36
4	0.28	1.42	3.85	5.18	5.72	6.19	6.45
5	0.26	1.44	4.00	5.04	5.47	5.77	5.91
6	0.32	1.48	6.00	6.76	10.86	11.02	11.07
7	0.33	1.27	4.77	5.21	8.28	8.38	8.412
8	0.18	0.88	3.28	3.56	5.60	5.65	5.66
9	0.25	0.64	1.75	1.78	2.79	2.79	2.79
10	0	0	0	0	0	0	0

6. Conclusions

In this paper, a new way to apply the non-homogeneous Markov approach to classical risk processes was presented. It was demonstrated how it is possible to calculate the aggregate claim amount and the claim number processes. In both cases, the non-homogeneous Markov reward processes were applied. The aggregate claim amount process was determined by means of a discounted non-homogeneous Markov reward process. The claim number process was found by a non-discounted approach.

In the near future, the authors would like to:

- access a large amount of data to apply their model and to compare their results with the results of the classical risk models,
- obtain the relations useful for the calculation of higher order moments that allows the calculation of variance, skewness, and kurtosis,
- demonstrate how to calculate the premium rate.

Acknowledgment

The authors are grateful to the two anonymous referees, whose suggestions and corrections substantially improved the quality of the paper.

Bibliography

[1] E. S. Andersen, *An algebraic treatment of fluctuations of sums of random variables*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), vol. 2, Univ. California Press, Berkeley, 1967. MR0221554

[2] S. Asmussen, *Risk theory in a Markovian environment*, Scand. Actuar. J. (1989), 69–100. MR1035668

[3] S. Asmussen and H. Albrecher, *Ruin Probabilities*, Adv. Ser. Stat. Sci. Appl. Probab., vol. 14, World Scientific Publishing Co., Inc., River Edge, 2012. MR2766220

[4] V. S. Borkar and S. P. Meyn, *Risk-sensitive optimal control for Markov decision processes with monotone cost*, Math. Oper. Res. **27** (2002), 192–209. MR1886226

[5] R. Cavazos-Cadena and D. Hernández-Hernández, *A characterization of the optimal risk-sensitive average cost in finite controlled Markov chains*, Ann. Appl. Probab. **15** (2005), 175–212. MR2115041

[6] W. Ching, X. Huang, M. K. Ng, and T. Siu, *Markov Chains: Models, Algorithms and Applications*, 2nd ed., Springer, New York, 2013. MR3053058

[7] K. L. Chung and F. AitSahlia, *Elementary Probability Theory*, 4th Edition, Springer, 2010. MR1961879

[8] H. Cramér, *On the mathematical theory of risk*, Skandia Jubilee volume, Stockholm, 1930. MR0090177

[9] H. Cramér, *Collective risk theory*, Skandia Jubilee volume, Stockholm, 1955. MR0090177

[10] G. D'Amico, J. Janssen, and R. Manca, *Discrete-time Markov reward processes a motor car insurance example*, Technology and Investment **1** (2010), 135–142.

[11] G. D'Amico, F. Gismondi, J. Janssen, and R. Manca, *Discrete-time homogeneous Markov processes for the study of the basic risk processes*, Methodol. Comput. Appl. Probab. **17** (2015), 983–998. MR3424725

[12] R. De Dominicis and R. Manca, *Some new results on the transient behavior of semi-Markov reward processes*, Methods Oper. Res. **54** (1985), 387–397 MR858862

[13] B. de Finetti, *Il problema dei pieni*, Giornale dell'Istituto italiano degli Attuari **8** (1940), 1–88. MR0019248

[14] R. A. Howard and J. E. Matheson, *Risk-sensitive Markov decision processes*, Management Science **18** (1972), 356–369. MR0292497

[15] J. Janssen, *Some transient results on the M/SM/1 special semi-Markov model in risk and queuing theories*, ASTIN Bulletin **11** (1980), 41–51. MR580670

[16] J. Janssen and R. Manca, *Applied Semi-Markov Processes*, Springer, New York, 2006. MR2175472

[17] J. Janssen and R. Manca, *Semi-Markov Risk Models for Finance, Insurance and Reliability*, Springer, New York, 2007. MR2301626

[18] J. Janssen and J. M. Reinhard, *Probabilité de ruine pour une classe de modeles de risque semi-Markoviens*, ASTIN Bulletin **15** (1985), 123–133.

[19] F. Lundberg, *Approximerad framstallning av sannolikhetsfunktionen. Aterforsakring av kollektivrisker*, Acad. Afhaddling. Almqvist. och Wiksell, Uppsala, 1903.

[20] F. Lundberg, *On the theory of reinsurance* (1909), History of Actuarial Science, vol. VII (S. Habermann and T. A. Sibbett, eds.), 1995, pp. 291–344.

[21] A. J. McNeil, R. Frey, and P. Embrechts, *Quantitative Risk Management: Concepts, Techniques, and Tools*, Princeton, 2005. MR2175089

[22] T. Mikosch, *Non-Life Insurance Mathematics: An Introduction with the Poisson Process*, Springer, Berlin, 2009. MR2503328

[23] M. A. Qureshi and H. W. Sanders, *Reward model solution methods with impulse and rate rewards: an algorithmic and numerical results*, Performance Evaluation **20** (1994), 413–436.

[24] F. Riesz, *Les systémes d'equations lineaires a une infinité d'inconnues*, Gauthier-Villar, Paris, 1913.

[25] T. Rolski, H. Schmidli, V. Schmidt, and J. Teugels, *Stochastic Processes for Insurance and Finance*, Wiley, London, 1999. MR1680267

Department of Pharmacy, University "G. d'Annunzio" of Chieti-Pescara, 66013 Chieti, Italy

DSEA, University "Guglielmo Marconi", 00193, Rome, Italy

Université Libre de Bruxelles, 1050 Bruxelles, Belgium

MEMOTEF Department, University "La Sapienza" of Rome, 00185 Rome RM, Italy
Email address: raimondo.manca@uniroma1.it

Department of Business, University of Cagliari, 09124 Cagliari CA, Italy

MEMOTEF Department, University "La Sapienza" of Rome, 00185 Rome RM, Italy

Received 22/FEB/2017
Originally published in English