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ASYMPTOTIC EXPANSION OF A FUNCTIONAL CONSTRUCTED FROM A SEMI-MARKOV RANDOM EVOLUTION IN THE SCHEME OF DIFFUSION APPROXIMATION

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Abstract. The regular and singular components of an expansion of a functional of a semi-Markov decomposition of a random evolution are found in the paper. A procedure is proposed for finding the explicit form of initial conditions for $t = 0$ by using the boundary conditions for the singular component of the expansion,

1. Introduction

Various stochastic systems can be described with the help of an abstract mathematical model in the Banach space $\mathcal{B}(\mathbb{R}^d)$ of functions $\varphi(u), u \in \mathbb{R}^d$, called a random evolution (a detailed survey can be found in [18]). Griego and Hersh [2–4] were the first to consider such a model.

Asymptotic methods in the theory of random evolutions have been used by many authors (see, for example, [5, 11, 17]). Some applications of these methods for different stochastic systems can be found in [14]. Pinsky [18] studies the models of a kinetic theory of gases, isotropic transport on manifolds, stability of random oscillators, etc., by means of analogous methods.

We mention here the results by Hillen and Othmer [6, 15] among the papers related to random evolutions in the mathematical biology (also see [16]). Namely, the transport equations are used in the mathematical biology as a model of movement and evolution of a population. The bacterial movement can be described as follows: the cycles when bacteria move along a straight line are changed with the random rotation cycles that result in the change of direction of the further movement. One can model such a movement picture with the help of a jump process that determines the speed and this, in turn, leads to the transport equation.

The linear transport equation is written as

$$\frac{\partial}{\partial t}p(x, v, t) + v\nabla p(x, v, t) = -\lambda p(x, v, t) + \int_V \lambda T(v, v')p(x, v', t)\, dv',$$

where $p(x, v, t)$ is the density of particles at a point $x \in \mathbb{R}^d$ that move with speed $v \in V \subset \mathbb{R}^d$ at time $t \geq 0$. The intensity of rotation λ may depend on both the current position and speed. The kernel determining the rotation or the distribution of the angle of rotation $T(v, v')$ determines the probability of the jump from v' to v.

The evolution equation studied in the current paper generalizes the above transport equation. An analogous generalization of the telegraph equation is described in [20].

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Another application of asymptotic methods is described by Yin and Zhang [26]. They study a model for production planning of a failure-prone manufacturing system consisting of several machines whose production capacity is modeled by a Markov or semi-Markov chain. In large-scale controlled systems different components may be renewed with different intensities. Thus the system can be split into separate components for which the corresponding state of the chain are aggregated. Introducing a small parameter $\varepsilon > 0$, the system is transformed to a system scaled by two time parameters. The equation for such a model is given by

$$\frac{dp^\varepsilon(t)}{dt} = \frac{1}{\varepsilon} p^\varepsilon(t) Q(t).$$

Here $p^\varepsilon(t)$ is the probability distribution of the corresponding Markov or semi-Markov chain and $Q(t)$ is its generator. As a rule, the analysis of quickly changing processes $p^\varepsilon(t)$ in a physical or manufacturing system is complicated. Instead of $p^\varepsilon(t)$, one can use "averaged" version constructed with the help of limit properties obtained from asymptotic decompositions.

In particular, the problems of minimization of the discounted cost function and optimal control are solved in [26] with the use of asymptotic approximation. Similar methods are also used when solving Markov problems of decision making, problems of stochastic control in dynamic systems, numerical methods of control, and optimization.

More details and a survey of recent results concerning the methods of the asymptotic analysis of random evolutions and applications of these results in various fields can be found in [22, 27].

Some of the results obtained in [26] are generalized in this paper to the case of semi-Markov processes. The current paper deals with the scheme of diffusion approximation and is a continuation of the paper [1], where a similar study is presented for the scheme of averaging. An analogous problem is investigated in [19], where the model includes some additional parameters in order to transform a semi-Markov random evolution into a Markov one. This essentially simplifies the technical part of the investigation; however, this method requires introducing an additional variable, being an argument of a functional under consideration, and thus the questions concerning the inverse transform remain open. Moreover, the author of [19] does not formulate any result about the form of the regular component of the expansion. An algorithm for finding the regular component of the expansion is proposed in [19], but the regularization of the limit conditions is not done. Such a regularization allows one to propose an algorithm for finding initial conditions in an explicit form at $t = 0$ by using the limit conditions for the singular component of the decomposition.

All problems mentioned above are successfully solved in the current paper. The convergence of the asymptotic series is proved by using an estimate for the remainder term. The authors leave this result for a separate publication elsewhere.

A semi-Markov random evolution in the scheme of series (its properties are studied in [10] in more detail) is defined in terms of the solution of the following evolution equation in the Euclidean space $\mathbb{R}^d$, $d \geq 1$:

$$\frac{du^\varepsilon(t)}{dt} = \frac{1}{\varepsilon} v\left(u^\varepsilon(t); æ\left(t/\varepsilon^2\right)\right). \tag{1}$$

The process that switches the speeds $æ(t), t \geq 0$, is a semi-Markov process [9] with the state space $(E, \mathcal{E})$, where E is a complete separable metric space and $\mathcal{E}$ is the corresponding σ-algebra of its subsets. The process is defined in terms of its semi-Markov kernel (see [10])

$$Q(x, B, t) = P(x, B) F_x(t), \qquad x \in E,\ B \in \mathcal{E},\ t \geq 0,$$

that determines the transition probabilities of the Markov renewal process $æ_n, \tau_n, n \ge 0$,

$$\begin{aligned} Q(x_n, B, t) &= \mathsf{P}\{æ_{n+1} \in B, \tau_{n+1} - \tau_n \le t \mid æ_n = x\} \\ &= \mathsf{P}\{æ_{n+1} \in B \mid æ_n = x\}\, \mathsf{P}\{\tau_{n+1} - \tau_n \le t \mid æ_n = x\}. \end{aligned}$$

The stochastic kernel

$$P(x, B) = \mathsf{P}\{æ_{n+1} \in B \mid æ_n = x\}$$

generates the transition probabilities of the embedded Markov chain $æ_n = æ(\tau_n)$, $n \ge 0$, while the distribution functions

$$F_x(t) = \mathsf{P}\{\tau_{n+1} - \tau_n \le t \mid æ_n = x\} =: \mathsf{P}\{\tau_{n+1} - \tau_n \le t\}, \qquad x \in E,$$

generate the distributions of sojourn times θ_x in the states $x \in E$.

Let $\mathfrak{B}(E)$ denote the Banach space of real-valued test functions $\varphi(x)$ being bounded together with all their derivatives. Let this space be equipped with the sup-norm. The generator of the associated Markov process acts on $\mathfrak{B}(E)$ and is of the form

$$Q = q(x)(P - I),$$

where

$$P\varphi(x) = \int_E P(x, dy)\varphi(y), \qquad x \in E,$$

is the operator of transition probabilities $q(x) = 1/m_1(x)$, $m_k(x) = \int_0^\infty s^k\, F_x(ds)$.

Let the switching semi-Markov process $æ(t)$, $t \ge 0$, be uniformly ergodic (see [10] for details). Denote by $\pi(B)$, $B \in \mathcal{E}$, the stationary distribution of the switching semi-Markov process $æ(t)$, $t \ge 0$, and assume that

$$\begin{aligned} \pi(dx) &= \rho(dx)\, m_1(x)/\widehat{m}, \\ \widehat{m} &= \int_E \rho(dx)\, m_1(x). \end{aligned}$$

Here $\rho(B), B \in \mathcal{E}$, is the stationary distribution of the embedded Markov chain $æ_n, n \ge 0$, defined by

$$\rho(B) = \int_E \rho(dx) P(x, B), \qquad \rho(E) = 1.$$

In this case, the Banach space $\mathfrak{B}(E)$ is the direct sum of the null-subspace $N_Q := \{\varphi(x) \colon Q\varphi(x) = 0\}$ of the operator Q and the subspace $R_Q := \{\psi(x) \colon Q\varphi(x) = \psi(x)\}$ of values of the operator Q (see [10, 12]).

Denote by Π the projector to the null-space of the operator Q: $\Pi\varphi(x) := \widehat{\varphi}\, \mathbf{1}(x)$, where $\mathbf{1}(x) = 1$ for all $x \in E$,

$$\widehat{\varphi} := \int_E \varphi(x)\, \pi(dx).$$

In [10, Section 3.4.3], the conditions for the weak convergence

$$u^\varepsilon(t) \Rightarrow \hat{u}(t), \qquad \varepsilon \to 0,$$

are found, and an equation is derived that determines the limit process.

In general, there are two lines of attack when studying the rate of the weak convergence; namely,

(i) either the asymptotic analysis of the fluctuations

$$\zeta^\varepsilon(t) = u^\varepsilon(t) - \hat{u}(t), \qquad \varepsilon \to 0,$$

(ii) or the asymptotic analysis of the functional defining the expectation of the semi-Markov random evolution

$$\Phi_t^\varepsilon(u,x) = \mathsf{E}\big[\varphi\big(u^\varepsilon(t)\big) \mid u^\varepsilon(0)=u, æ(0)=x\big],$$

where $\varphi(u)$ belongs to the Banach space $\mathfrak{B}(\mathbb{R}^d)$ of real-valued test functions being bound together with all their derivatives. The norm in the space $\mathfrak{B}(\mathbb{R}^d)$ is defined by

$$\|\varphi\| = \sup_{u\in\mathbb{R}^d} |\varphi(u)| < C_\varphi$$

for some $C_\varphi > 0$.

The aim of the current paper is to study the rate of convergence following the second approach. More precisely, we construct an asymptotic expansion of the functional of the semi-Markov random evolution in the form

$$\Phi_t^\varepsilon(u,x) = U^\varepsilon(t) + W^\varepsilon(\tau) = U_0(t) + \sum_{k=1}^\infty \varepsilon^k \big(U_k(t)+W_k(\tau)\big), \tag{2}$$

where $\tau = t/\varepsilon^2$.

Remark 1.1. The initial conditions are given by

$$\Phi_0^\varepsilon(u,x) = U^\varepsilon(0) + W^\varepsilon(0) = \varphi(u),$$

whence

$$U_0(0) = \varphi(u),$$
$$U_k(0) + W_k(0) = 0, \qquad k\ge 1.$$

The singular component of the decomposition satisfies the boundary conditions

$$W^\varepsilon(\infty) = 0.$$

Asymptotic expansions with a boundary layer are studied by many authors; see, for example, [8, 25]. In particular, functionals of Markov and semi-Markov processes are studied in [11, 20, 24] by following the second approach mentioned above.

Asymptotic expansion (2) for a functional of the semi-Markov random evolution in the scheme of diffusion approximation is constructed in the current paper by using the integral equation of the Markov renewal. The procedure for the construction of an explicit form of regular and singular components of the asymptotic expression and boundary conditions are given in Theorem 1.1. The proof of the main result consists of several steps.

Now we introduce some notation. The deterministic evolution

$$\Phi_x(t,u) = \varphi\big(u_x^\varepsilon(t)\big), \qquad u_x^\varepsilon(0) = u,$$

generates the corresponding semigroup

$$V_t(x)\varphi(u) := \varphi(u_x^\varepsilon(t)), \qquad u_x^\varepsilon(0)=u,$$

whose generator is of the form

$$\mathbb{V}^\varepsilon(x)\varphi(u) = \frac{1}{\varepsilon} v(u,x)\varphi'(u).$$

For convenience, the auxiliary generator is denoted by

$$\mathbb{V}(x)\varphi(u) := v(u,x)\varphi'(u).$$

Below is some more notation:

$$\mu_k(x) = \frac{m_k(x)}{k!\, m_1(x)}, \qquad \mu_1(x) := 1,$$
$$L^i_{k,n} U_n(t) := (-1)^i C^{k-n-i}_{k-n-2i} \mathbb{V}^{k-n-2i}(x) P U_n^{(i)}(t),$$
$$\Pi\mathfrak{L}_k := \sum_{n=1}^{k-1} \sum_{i=0}^{\left[\frac{k-n}{2}\right]} \Pi\mu_{k-n-i}(x) L^i_{n,k} \mathbb{R}_0 \mathfrak{L}_n + \sum_{i=0}^{\left[\frac{k}{2}\right]} \Pi\mu_{k-i}(x) L^i_{0,k},$$
$$\nu_k(x) = (-1)^k [m_k(x) - \mu_{k+1}(x)],$$
$$\widehat{L}_{k-1}(x) := \sum_{n=0}^{k-1} (-1)^n C^n_k \mathbb{V}^{k-n}(x) P U^{(n)}(t),$$
$$\widehat{L}_k(x) := \mathbb{V}^k(x) P.$$

Finally,

$$\mathbf{Q} W(\tau) = \int_0^\infty F_x(ds)\, P W(\tau - s),$$

$$\psi^k(\tau) = \bar{F}_x^{(k)}(\tau) \mathbb{V}^k(x) P \varphi(u), \qquad \psi^k_0(\tau) = \sum_{r=1}^{k-1} \mathbf{Q}^r W_{k-r}(\tau), \tag{3}$$

$$\bar{F}_x^{(k)}(\tau) = \int_\tau^\infty \frac{s^{k-1}}{(k-1)!} \bar{F}_x(s)\, ds, \qquad \mathbf{Q}^r W(\tau) = \int_0^\infty \frac{s^r}{r!} F_x(ds) \mathbb{V}^r(x) P W(\tau - s).$$

In what follows, we assume that the balance condition holds; that is, we assume that the mean value of $v(u,x)$ with respect to the stationary measure of the switching semi-Markov process is equal to zero:

$$\Pi\mathbb{V}(x)\Pi = \int_E v(u,x)\, \pi(dx) = 0. \tag{4}$$

Theorem 1.1. *Assume that the uniform ergodicity condition holds for the switching semi-Markov process. Then balance condition* (4) *implies that the asymptotic expansion of the random evolution*

$$\Phi^\varepsilon_t(u,x) = \mathsf{E}\left[\varphi\left(u^\varepsilon(t)\right) \mid u^\varepsilon(0) = u, æ(0) = x\right]$$

is given by

$$\Phi^\varepsilon_t(u,x) = U^\varepsilon(t) + W^\varepsilon(\tau) = U_0(t) + \sum_{k=1}^\infty \varepsilon^k \left(U_k(t) + W_k(\tau)\right), \qquad \tau = t/\varepsilon^2,$$

where

$$U_0(t) = c_0(t,u)\mathbf{1},$$

and the function $c_0(t,u)$ *satisfies the equation*

$$\frac{\partial c_0(t,u)}{\partial t} = \tilde{v}(u) \frac{\partial c_0(t,u)}{\partial u} + \frac{1}{2} \tilde{\sigma}(u) \frac{\partial^2 c_0(t,u)}{\partial u^2} \tag{5}$$

with initial condition

$$c_0(0,u) = \varphi(u).$$

Here

$$\tilde{v}(u) := \Pi\mu_2(x) \frac{\partial v(u,x)}{\partial u} - \Pi v(x,u) \mathbb{R}_0 \frac{\partial v(x,u)}{\partial u},$$
$$\tilde{\sigma}(u) := 2\left(\Pi\mu_2(x) v(u,x) - \Pi v(x,u) \mathbb{R}_0 v(x,u)\right).$$

The rest of the regular terms are of the form

$$U_k(t) = \mathbb{R}_0 \left(\sum_{n=0}^{k-1} \sum_{i=0}^{\left[\frac{k-n}{2}\right]} \mu_{k-n-i}(x) L_{k,n}^i U_n(t) \right) + c_k(t,u),$$

where $\mathbb{R}_0 = \Pi - [Q + \Pi]^{-1}$ *according to* [8].

The functions $c_k(t,u)$ *are such that*

$$\frac{\partial c_k(t,u)}{\partial t} - \Pi \mu_2(x) \mathbb{V}^2(x) c_k(t,u) + \Pi \mathbb{V}(x) \mathbb{R}_0 \mathbb{V}(x) c_k(t,u) = -\Pi \mathfrak{L}_k c_0(t,u) - \cdots - \Pi \mathfrak{L}_1 c_{k-1}(t,u).$$

The singular terms of the expansion ("boundary layer") are of the form

$$W_1(\tau) = \mathbf{R_0} \left[\psi^1(\tau) + \bar{F}_x(\tau) P U_1(0) + \int_\tau^\infty (\tau - s) F_x(ds) P U_0'(0) \right],$$

$$W_k(\tau) = \mathbf{R_0} \left[\psi^k(\tau) - \psi_0^k(\tau) + \bar{F}_x(\tau) P U_k(0) + \sum_{n=1}^{\left[\frac{k}{2}\right]} \int_\tau^\infty \frac{(\tau - s)^n}{n!} F_x(ds) P U_{k-2n}^{(n)}(0) \right],$$

where $\mathbf{R_0}$ *is the matrix of the Markov renewal* [21].

The initial conditions are

$$(I - \Pi)[U_k(0) + W_k(0)] = 0,$$
$$c_k(0,u) = -\Pi W_k(0),$$

$$c_k(0,u) = \left[\sum_{r=0}^{k-1} \int \pi(dx) \nu_{k-r}(x) \widehat{L}_{k-r}(x) U_r(0) - \sum_{r=1}^{k-1} \int \rho(dx) \int_0^\infty \int_0^\tau \frac{s^r}{r!} F_x(ds) \mathbb{V}^r(x) P W_{k-r}(\tau - s)\, d\tau \right] \Big/ \widehat{m}.$$

Remark 1.2. Equation (5) for the function $c_0(t,u)$ is completely compatible with results obtained in Section 3.4.3 of the monograph [10] and means that the stochastic process to which the prelimit random evolution converges weakly as $\varepsilon \to 0$ is defined by the diffusion type equation

$$du^0(t) = \tilde{v}(u)\, dt + \tilde{\sigma}^{1/2}(u)\, dw(t).$$

2. Equation of the Markov renewal

Lemma 2.1. *The functional* $\Phi_t^\varepsilon(u,x)$ *of the semi-Markov evolution satisfies the equation*

$$\int_0^\infty F_x(ds) \mathbb{V}_{\varepsilon^2 s}(x) P \Phi_{t-\varepsilon^2 s}^\varepsilon(u,x) - \Phi_t^\varepsilon(u,x) = \varepsilon^2 \mathbb{V}^\varepsilon(x) \int_\tau^\infty \bar{F}_x(s) V_{\varepsilon^2 s}(x) \varphi(u)\, ds, \tag{6}$$

where $\tau = t/\varepsilon^2$.

Proof. Considering the first jump moment of the switching process, the functional is rewritten as

$$\Phi_t^\varepsilon(u,x) = E_{u,x}\left[\varphi\big(u^\varepsilon(t)\big); \theta_x > t/\varepsilon^2\right] + E_{u,x}\left[\varphi\big(u^\varepsilon(t)\big); \theta_x \le t/\varepsilon^2\right]$$
$$= \bar{F}_x\left(t/\varepsilon^2\right) V_t(x) P \varphi(u) + \int_0^{t/\varepsilon^2} F_x(ds) V_{\varepsilon^2 s}(x) P \Phi_{t-\varepsilon^2 s}^\varepsilon(u,x).$$

Thus

$$\Phi_t^\varepsilon(u,x) - \int_0^{t/\varepsilon^2} F_x(ds) V_{\varepsilon^2 s}(x) P \Phi^\varepsilon_{t-\varepsilon^2 s}(u,x) = \bar{F}_x(\tau) V_t(x) P\varphi(u).$$

Extending $\Phi^\varepsilon_{t-\varepsilon^2 s}(u,x) = \varphi(u)$ by continuity for $t-\varepsilon^2 s \le 0$, the latter equation becomes of the form

$$\begin{aligned}
\Phi_t^\varepsilon(u,x) &- \int_0^{\infty} F_x(ds) V_{\varepsilon^2 s}(x) P \Phi^\varepsilon_{t-\varepsilon^2 s}(u,x) \\
&= \bar{F}_x(\tau) V_t(x) P\varphi(u) - \int_\tau^\infty F_x(ds) V_{\varepsilon^2 s}(x) P \Phi^\varepsilon_{t-\varepsilon^2 s}(u,x) \\
&= \bar{F}_x(\tau) V_t(x) P\varphi(u) - \int_\tau^\infty F_x(ds) V_{\varepsilon^2 s}(x) P \varphi(u).
\end{aligned}$$

Hence

$$\begin{aligned}
\Phi_t^\varepsilon(u,x) &- \int_0^{\infty} F_x(ds) V_{\varepsilon^2 s}(x) P \Phi^\varepsilon_{t-\varepsilon^2 s}(u,x) \\
&= \bar{F}_x(\tau) V_t(x) P\varphi(u) - \bar{F}_x(s) V_{\varepsilon^2 s}(x) P\varphi(u)\Big|_\tau^\infty - \varepsilon^2 \mathbb{V}^\varepsilon(x) \int_\tau^\infty \bar{F}_x(s) V_{\varepsilon^2 s}(x) \varphi(u) ds.
\end{aligned}$$

Deleting the terms we obtain equality (6). The lemma is proved. □

3. Equation for regular terms

Let

$$L_k = \sum_{n=0}^{k} (-1)^n C_k^n (\mathbb{V}^\varepsilon(x))^{k-n} P\left(U^{\varepsilon(n)}(t)\right).$$

Lemma 3.1. *The equations for the regular terms of the expansion can be written as*

$$QU(t) = -\left[\sum_{k=1}^{\infty} \varepsilon^{2k} \mu_k(x) L_k\right] U^\varepsilon(t). \tag{7}$$

Proof. We will use the equality

$$aPb - 1 = (P-1) + (a-1)P + P(b-1) + (a-1)P(b-1),$$

where

$$a = V_{\varepsilon^2 s}(x) = I + \sum_{k=1}^\infty \varepsilon^{2k} \frac{s^k}{k!} \left(\mathbb{V}^\varepsilon(x)\right)^k, \qquad b = \Phi^\varepsilon_{t-\varepsilon^2 s} = \sum_{k=0}^\infty (-1)^k \varepsilon^{2k} \frac{s^k}{k!} \Phi_t^{(k)}(u,x).$$

Then we rewrite (6) as

$$\begin{aligned}
(P-I)\Phi_t^\varepsilon(u,x) &+ \int_0^\infty F_x(ds) \left(\sum_{k=1}^\infty \varepsilon^{2k} \frac{s^k}{k!} (\mathbb{V}^\varepsilon(x))^k\right) P \Phi_t^\varepsilon(u,x) \\
&+ \int_0^\infty F_x(ds) P \left(\sum_{k=1}^\infty (-1)^k \varepsilon^{2k} \frac{s^k}{k!} \Phi_t^{(k)}(u,x)\right) \\
&+ \int_0^\infty F_x(ds) \left(\sum_{k=1}^\infty \varepsilon^{2k} \frac{s^k}{k!} (\mathbb{V}^\varepsilon(x))^k\right) P \left(\sum_{k=0}^\infty (-1)^k \varepsilon^{2k} \frac{s^k}{k!} \Phi_t^{(k)}(u,x)\right) \\
&= \varepsilon^2 \mathbb{V}^\varepsilon(x) \int_\tau^\infty \bar{F}_x(s) V_{\varepsilon^2 s}(x) P \varphi(u)\, ds.
\end{aligned}$$

Substituting expression (2) for regular terms we conclude that

$$
\begin{aligned}
(P-I)U^\varepsilon(t) = &-\int_0^\infty F_x(ds)\left(\sum_{k=1}^\infty \varepsilon^{2k}\frac{s^k}{k!}(\mathbb{V}^\varepsilon(x))^k\right)PU^\varepsilon(t)\\
&-\int_0^\infty F_x(ds)P\left(\sum_{k=1}^\infty (-1)^k\varepsilon^{2k}\frac{s^k}{k!}\left(U^{\varepsilon(k)}(t)\right)\right)\\
&-\int_0^\infty F_x(ds)\left(\sum_{k=1}^\infty \varepsilon^{2k}\frac{s^k}{k!}(\mathbb{V}^\varepsilon(x))^k\right)P\left(\sum_{k=0}^\infty (-1)^k\varepsilon^{2k}\frac{s^k}{k!}\left(U^{\varepsilon(k)}(t)\right)\right).
\end{aligned}
$$

Collecting the terms with the same powers of ε we get

$$
\begin{aligned}
&(P-I)U^\varepsilon(t)\\
&\quad=\sum_{k=1}^\infty \varepsilon^{2k}\Bigg[-\int_0^\infty F_x(ds)\frac{s^k}{k!}(\mathbb{V}^\varepsilon(x))^k PU^\varepsilon(t)\\
&\qquad\qquad-\int_0^\infty F_x(ds)\left(\sum_{n=1}^{k-1}(-1)^n\frac{s^k}{n!(k-n)!}(\mathbb{V}^\varepsilon(x))^n P\left(U^{\varepsilon(k-n)}(t)\right)\right)\\
&\qquad\qquad\qquad-\int_0^\infty (-1)^k F_x(ds)P\frac{s^k}{k!}\left(U^{\varepsilon(k)}(t)\right)\Bigg]\\
&\quad=-\sum_{k=1}^\infty \varepsilon^{2k}\frac{m_k(x)}{k!}L_kU^\varepsilon(t).
\end{aligned}
$$

Dividing the latter equality by $m_1(x)$ we obtain (7). The lemma is proved. □

Substituting the expansion

$$
U^\varepsilon(t)=\sum_{k=0}^\infty \varepsilon^k U_k(t)
$$

into (7), taking into account the relation $\mathbb{V}^\varepsilon(x)=\frac{1}{\varepsilon}\mathbb{V}(x)$, and collecting the terms with the same powers of ε we prove the following result.

Corollary 3.1. *The regular terms of the asymptotic expansion satisfy the following system of equations:*

$$
\begin{cases}
QU_0(t)=0,\\
QU_1(t)=-\mathbb{V}(x)PU_0(t),\\
QU_2(t)=\dfrac{\partial PU_0(t)}{\partial t}-\mu_2(x)\mathbb{V}^2(x)PU_0(t)-\mathbb{V}(x)PU_1(t),\\
QU_3(t)=\dfrac{\partial PU_1(t)}{\partial t}-\mu_2(x)\mathbb{V}^2(x)PU_1(t)-\mathbb{V}(x)PU_2(t)\\
\qquad\qquad+\mu_2(x)C_1^2\mathbb{V}(x)P\dfrac{\partial U_0(t)}{\partial t}-\mu_3(x)\mathbb{V}^3(x)PU_0(t),\\
\qquad\dots\\
QU_k(t)=-\displaystyle\sum_{n=0}^{k-1}\sum_{i=0}^{\left[\frac{k-n}{2}\right]}\mu_{k-n-i}(x)L^i_{k,n}U_n(t),\quad k\ge 3,\\
\qquad\dots
\end{cases}
\tag{8}
$$

where

$$L^i_{k,n}U_n(t) := (-1)^i C^{k-n-i}_{k-n-2i}\mathbb{V}^{k-n-2i}(x)PU^{(i)}_n(t).$$

The first equation of system (8) implies that $U_0(t) \in N_Q$. Therefore one can set

$$U_0(t) = c_0(t,u)\mathbf{1},$$

where $c_0(t,u)$ is a scalar function that does not depend on x but does depend on u.

The right hand side of the second equation belongs to R_Q as seen from balance condition (4), whence

$$U_1(t) = \mathbb{R}_0\mathbb{V}(x)PU_0(t) + c_1(t,u) = \mathbb{R}_0\mathbb{V}(x)c_0(t,u) + c_1(t,u)$$

(see [9]).

As a result, the third equation is rewritten as

$$QU_2(t) = \frac{\partial c_0(t,u)}{\partial t} - \mu_2(x)\mathbb{V}^2(x)c_0(t,u) - \mathbb{V}(x)\mathbb{R}_0\mathbb{V}(x)c_0(t,u) - \mathbb{V}(x)c_1(t,u),$$

and thus balance condition (4) and the condition of solvability of the latter equation imply the equation for the function $c_0(t,u)$:

$$\Pi Q\Pi U_2(t) = 0 = \frac{\partial c_0(t,u)}{\partial t} - \Pi\mu_2(x)\mathbb{V}^2(x)\Pi c_0(t,u) - \Pi\mathbb{V}(x)\mathbb{R}_0\mathbb{V}(x)\Pi c_0(t,u).$$

Corollary 3.2. *The function $c_0(t,u)$ satisfies the diffusion type equation*

$$\frac{\partial c_0(t,u)}{\partial t} = \tilde{v}(u)\frac{\partial c_0(t,u)}{\partial u} + \frac{1}{2}\tilde{\sigma}(u)\frac{\partial^2 c_0(t,u)}{\partial u^2}$$

with the initial condition

$$c_0(0,u) = \varphi(u).$$

Here

$$\tilde{v}(u) := \Pi\mu_2(x)\frac{\partial v(u,x)}{\partial u} - \Pi v(x,u)\mathbb{R}_0\frac{\partial v(x,u)}{\partial u},$$
$$\tilde{\sigma}(u) := 2\big(\Pi\mu_2(x)v(u,x) - \Pi v(x,u)\mathbb{R}_0 v(x,u)\big).$$

Then we obtain for $U_2(t)$ that

$$U_2(t) = \mathbb{R}_0\left(\frac{\partial c_0(t,u)}{\partial t} - \mu_2(x)\mathbb{V}^2(x)c_0(t,u) - \mathbb{V}(x)PU_1(t)\right) + c_2(t,u).$$

Using balance condition (4) and the condition of solvability of the fourth equation of system (8), we get

$$\begin{aligned}0 = \Pi\bigg(&R_0\mathbb{V}(x)\frac{\partial c_0(t,u)}{\partial t} + \frac{\partial c_1(t,u)}{\partial t} - \mu_2(x)\mathbb{V}^2(x)\mathbb{R}_0\mathbb{V}(x)c_0(t,u)\\ &- \mu_2(x)\mathbb{V}^2(x)c_1(t,u) - \mathbb{V}(x)\mathbb{R}_0\frac{\partial c_0(t,u)}{\partial t} + \mathbb{V}(x)\mathbb{R}_0\mu_2(x)\mathbb{V}^2(x)c_0(t,u)\\ &+ \mathbb{V}(x)\mathbb{R}_0\mathbb{V}(x)\mathbb{R}_0\mathbb{V}(x)c_0(t,u) + \mathbb{V}(x)\mathbb{R}_0\mathbb{V}(x)c_1(t,u) - \mathbb{V}(x)c_2(t,u)\\ &+ \mu_2(x)C^2_1\mathbb{V}(x)\frac{\partial c_0(t,u)}{\partial t} - \mu_3(x)\mathbb{V}^3(x)c_0(t,u)\bigg)\Pi,\end{aligned}$$

which yields the equation for $c_1(t,u)$:

$$\begin{aligned}
&\frac{\partial c_1(t,u)}{\partial t} - \Pi\mu_2(x)\mathbb{V}^2(x)\Pi c_1(t,u) + \Pi\mathbb{V}(x)\mathbb{R}_0\mathbb{V}(x)\Pi c_1(t,u)\\
&\quad = -\Pi\mathbb{R}_0\mathbb{V}(x)\frac{\partial c_0(t,u)}{\partial t} + \Pi\mu_2(x)\mathbb{V}^2(x)\mathbb{R}_0\mathbb{V}(x)\Pi c_0(t,u) + \Pi\mathbb{V}(x)\mathbb{R}_0\frac{\partial c_0(t,u)}{\partial t}\\
&\qquad - \Pi\mathbb{V}(x)\mathbb{R}_0\mu_2(x)\mathbb{V}^2(x)\Pi c_0(t,u) - \Pi\mathbb{V}(x)\mathbb{R}_0\mathbb{V}(x)\mathbb{R}_0\mathbb{V}(x)\Pi c_0(t,u)\\
&\qquad - \Pi\mu_2(x)C_1^2\mathbb{V}(x)\frac{\partial c_0(t,u)}{\partial t} + \Pi\mu_3(x)\mathbb{V}^3(x)\Pi c_0(t,u).
\end{aligned}$$

For $U_k(t)$, we proceed analogously:

$$U_k(t) = \mathbb{R}_0\left(\sum_{n=0}^{k-1}\sum_{i=0}^{\left[\frac{k-n}{2}\right]}\mu_{k-n-i}(x)L_{k,n}^iU_n(t)\right) + c_k(t,u),$$

$$\begin{aligned}
&\frac{\partial c_k(t,u)}{\partial t} - \Pi\mu_2(x)\mathbb{V}^2(x)\Pi c_k(t,u) + \Pi\mathbb{V}(x)\mathbb{R}_0\mathbb{V}(x)\Pi c_k(t,u)\\
&\quad = -\Pi\mathfrak{L}_kc_0(t,u) - \cdots - \Pi\mathfrak{L}_1c_{k-1}(t,u),
\end{aligned}$$

where

$$\Pi\mathfrak{L}_k := \sum_{n=1}^{k-1}\sum_{i=0}^{\left[\frac{k-n}{2}\right]}\Pi\mu_{k-n-i}(x)L_{n,k}^i\mathbb{R}_0\mathfrak{L}_n + \sum_{i=0}^{\left[\frac{k}{2}\right]}\Pi\mu_{k-i}(x)L_{0,k}^i.$$

4. Equations for singular terms (“boundary layer”)

We will use notation (3).

Lemma 4.1. *The equations for singular terms are given by*

$$\begin{aligned}
&(\mathbf{Q}-I)W_1(\tau) = \psi^1(\tau),\\
&(\mathbf{Q}-I)W_k(\tau) = \psi^k(\tau) - \psi_0^k(\tau).
\end{aligned} \tag{9}$$

Proof. Substituting the expansion

$$W^\varepsilon(\tau) = \sum_{k=1}^\infty \varepsilon^kW_k(\tau)$$

of the singular component into (6) and taking into account $\mathbb{V}^\varepsilon(x) = \frac{1}{\varepsilon}\mathbb{V}(x)$, we obtain

$$\begin{aligned}
&\int_0^\infty F_x(ds)\left[I + \sum_{k=1}^\infty \varepsilon^k\frac{s^k}{k!}\mathbb{V}^k(x)\right]P\left[\sum_{k=1}^\infty\varepsilon^kW_k(\tau-s)\right] - \sum_{k=1}^\infty\varepsilon^kW_k(\tau)\\
&\quad = \int_\tau^\infty \bar{F}_x(s)\left[\sum_{k=1}^\infty\varepsilon^k\frac{s^{k-1}}{(k-1)!}\mathbb{V}^k(x)\right]P\varphi(u)ds.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\varepsilon[\mathbf{Q}-I]W_1(\tau) + \sum_{k=2}^\infty\varepsilon^k[\mathbf{Q}-I]W_k(\tau) + \sum_{k=2}^\infty\varepsilon^k\sum_{r=1}^{k-1}\mathbf{Q}^rW_{k-r+1}(\tau)\\
&\quad = \sum_{k=1}^\infty\varepsilon^k\bar{F}_x^k(\tau)\mathbb{V}^k(x)P\varphi(u).
\end{aligned}$$

Collecting the terms with the same powers of ε we get (9). Lemma 4.1 is proved. ☐

Corollary 4.1. *The singular terms of the asymptotic expansion are given by*

$$W_1(\tau) = \mathbf{R_0}\left[\psi^1(\tau) - \int_\tau^\infty F_x(ds)PW_1(\tau - s)\right],$$
$$W_k(\tau) = \mathbf{R_0}\left[\psi^k(\tau) - \psi_0^k(\tau) - \int_\tau^\infty F_x(ds)PW_k(\tau - s)\right], \qquad \tau \geq 0,\ k \geq 2.$$

5. Initial conditions. The regularity of initial conditions

Consider the Taylor series for the function $\Phi_{\varepsilon^2\tau}(u,x)$ with respect to ε and for negative numbers τ:

$$\varphi(u) = \Phi_{\varepsilon^2\tau}(u,x)\Big|_{\tau<0}$$
$$= U_0(0) + \sum_{k=1}^\infty \varepsilon^{2k}\frac{\tau^k}{k!}U_0^{(k)}(0) + \varepsilon U_1(0) + \ + \varepsilon\sum_{k=1}^\infty \varepsilon^{2k}\frac{\tau^k}{k!}U_1^{(k)}(0) + \cdots + \sum_{k=1}^\infty \varepsilon^k W_k(\tau).$$

Hence

$$W^\varepsilon(\tau) = W^\varepsilon(0) - \sum_{k=1}^\infty \varepsilon^{2k}\frac{\tau^k}{k!}U^{\varepsilon(k)}(0) \tag{10}$$

and

$$W_k(\tau) = W_k(0) - \sum_{n=1}^{\left[\frac{k}{2}\right]} \frac{\tau^n}{n!}U_{k-2n}^{(n)}(0). \tag{11}$$

Lemma 5.1. *For $\tau = 0$, we have*

$$Q[U^\varepsilon(0) + W^\varepsilon(0)] = 0.$$

Proof. We need to prove the following equalities:

$$QU^\varepsilon(0) = -\left(\sum_{k=1}^\infty \varepsilon^{2k}\mu_k(x)L_k\right)U^\varepsilon(0)$$

and

$$QW^\varepsilon(0) = \left(\sum_{k=1}^\infty \varepsilon^{2k}\mu_k(x)L_k\right)U^\varepsilon(0).$$

The first of the latter equalities follows from equality (7).

Let

$$\mathbf{Q}(\tau)W(\tau) := \int_0^\tau F_x(ds)PW(\tau - s).$$

Now we substitute (10)–(11) into equation (9):

$$
\begin{aligned}
&(\mathbf{Q}(\tau) - I)W^\varepsilon(\tau) \\
&\quad = -\int_\tau^\infty F_x(ds)P\left(W^\varepsilon(0) - \sum_{k=1}^\infty \varepsilon^{2k}\frac{(\tau-s)^k}{k!}U^{\varepsilon(k)}(0)\right) \\
&\qquad + \int_\tau^\infty \bar{F}_x(ds)\left(\sum_{k=1}^\infty \varepsilon^k \frac{s^{k-1}}{(k-1)!}\mathbb{V}^k(x)\right)PU_0(0)\,ds \\
&\qquad - \int_0^\infty F_x(ds)\left(\sum_{k=1}^\infty \varepsilon^k \frac{s^k}{k!}\mathbb{V}^k(x)\right)P\left(W^\varepsilon(0) - \sum_{k=1}^\infty \varepsilon^{2k}\frac{(\tau-s)^k}{k!}U^{\varepsilon(k)}(0)\right) \\
&\quad = -PW^\varepsilon(0) + \int_\tau^\infty F_x(ds)P\left(\sum_{k=1}^\infty \varepsilon^{2k}\frac{(\tau-s)^k}{k!}U^{\varepsilon(k)}(0)\right) \\
&\qquad + \int_\tau^\infty \bar{F}_x(ds)\left(\sum_{k=1}^\infty \varepsilon^{2k}\frac{s^{k-1}}{(k-1)!}(\mathbb{V}^\varepsilon(x))^k\right)PU_0(0)\,ds \\
&\qquad + \int_0^\infty F_x(ds)\left(\sum_{k=1}^\infty \varepsilon^{2k}\frac{s^k}{k!}(\mathbb{V}^\varepsilon(x))^k\right)P(U^\varepsilon(0) - U_0(0)) \\
&\qquad + \int_0^\infty F_x(ds)\left(\sum_{k=1}^\infty \varepsilon^{2k}\frac{s^k}{k!}(\mathbb{V}^\varepsilon(x))^k\right)P\left(\sum_{k=1}^\infty \varepsilon^{2k}\frac{(\tau-s)^k}{k!}U^{\varepsilon(k)}(0)\right).
\end{aligned}
$$

If $\tau = 0$, we consider the initial condition described in Remark 1.1, namely,

$$W^\varepsilon(0) = -U^\varepsilon(0) + \varphi(u) = -U^\varepsilon(0) + U_0(0),$$

and use the equalities

$$\mathbf{Q}(0)W(0) = 0, \qquad \int_0^\infty F_x(ds)PW(0) = PW(0).$$

Then we conclude that

$$(P - I)W^\varepsilon(0) = \left(\sum_{k=1}^\infty \varepsilon^{2k}\frac{m_k(x)}{k!}L_k\right)U^\varepsilon(0),$$

which implies the equality for $QW^\varepsilon(0)$. Lemma 5.1 is proved. □

Corollary 5.1. *We have*

$$(I - P)[U^\varepsilon(0) + W^\varepsilon(0)] = 0$$

or, which is the same,

$$(I - \Pi)[U_k(0) + W_k(0)] = 0.$$

We see that the regular and singular parts of the solution imply that the initial condition described in Remark 1.1 holds in the space of values of the operator Q.

At the same time, the initial conditions in the null-space of the operator Q are determined by those for the "boundary layer". Summarizing what has been said above we derive the following result.

Corollary 5.2. *Let* $c_k(0,u) = -\Pi W_k(0)$, $k \geq 1$.

Proof. It is obvious that $\Pi[W_k(0) + U_k(0)] = \Pi W_k(0) + c_k(0,u) = 0$. □

Corollary 5.3. *The singular terms of the asymptotic expansion can be written in the following explicit form:*

$$W_1(\tau) = \mathbf{R_0}\left[\psi^1(\tau) + \bar{F}_x(\tau)PU_1(0) + \int_\tau^\infty (\tau - s)F_x(ds)PU_0'(0)\right],$$

$$W_k(\tau) = \mathbf{R_0}\left[\psi^k(\tau) - \psi_0^k(\tau) + \bar{F}_x(\tau)PU_k(0) + \sum_{n=1}^{[k/2]}\int_\tau^\infty \frac{(\tau - s)^n}{n!}F_x(ds)PU_{k-2n}^{(n)}(0)\right].$$

Proof. Using relations (10)–(11) and Corollary 5.1, one easily obtains

$$\begin{aligned}\int_\tau^\infty F_x(ds)PW_k(\tau - s) &= \int_\tau^\infty F_x(ds)P\left[-U_k(0) - \sum_{n=1}^{[k/2]}\frac{(\tau - s)^n}{n!}U_{k-2n}^{(n)}(0)\right]\\ &= -\bar{F}_x(\tau)PU_k(0) - \sum_{n=1}^{[k/2]}\int_\tau^\infty \frac{(\tau - s)^n}{n!}F_x(ds)PU_{k-2n}^{(n)}(0). \quad \square\end{aligned}$$

6. Initial conditions for regular terms

Using the limit conditions as $\tau \to \infty$ (see Remark 1.1), we can describe the procedure for finding the initial conditions for the regular component of the expansion at $\tau = 0$. The first singular term $W_1(\tau)$ satisfies the equation

$$\int_0^\infty Q(ds)W_1(\tau - s) - W_1(\tau) = \bar{F}_x^{(1)}(\tau)\mathbb{V}(x)P\varphi(u) \tag{12}$$

(see (9)), where

$$\bar{F}_x^{(1)}(\tau) = \int_\tau^\infty \bar{F}_x(s)\,ds.$$

Splitting the first integral into two parts yields the equation

$$\int_0^\tau Q(ds)W_1(\tau - s) - W_1(\tau) = \bar{F}^{(1)}(\tau)\mathbb{V}(x)P\varphi(u) - \int_\tau^\infty Q(ds)W_1(\tau - s).$$

According to the renewal theorem (see [21]),

$$\begin{aligned}0 &= W_1(\infty)\\ &= \Big(\int \rho(dx)\int_0^\infty\int_\tau^\infty \bar{F}_x(s)dsd\tau\mathbb{V}(x)P\varphi(u)\\ &\qquad - \int \rho(dx)\int_0^\infty\int_\tau^\infty Q(ds)W_1(\tau - s)\,d\tau\Big)\Big/\widehat{m},\end{aligned} \tag{13}$$

where

$$\widehat{m} = \int \rho(dx)m_1(x).$$

We derive from (10)–(11) that

$$W_1(\tau) = W_1(0) \tag{14}$$

for $\tau < 0$. Substituting the latter expression into equation (13) we conclude that

$$\begin{aligned}
&\left(\int \rho(dx)\int_0^\infty\int_\tau^\infty \bar F_x(s)\,ds\,d\tau\mathbb{V}(x)P\varphi(u) - \int \rho(dx)\int_0^\infty\int_\tau^\infty Q(ds)W_1(0)\,d\tau\right)/\widehat{m}\\
&\quad= \left(\int \rho(dx)\int_0^\infty \bar F_x^{(1)}(s)\,d\tau\mathbb{V}(x)P\varphi(u)\right.\\
&\qquad\left. - \int \rho(dx)\int_0^\infty\int_\tau^\infty F_x(s)PW_1(0)\,ds\,d\tau\right)/\widehat{m}\\
&\quad= \left(\int \rho(dx)\frac{m_2(x)}{2}\mathbb{V}(x)P\varphi(u) - \int \rho(dx)m_1(x)PW_1(0)\right)/\widehat{m}\\
&\quad= \left(-\int \rho(dx)m_1(x)\mu_2(x)\mathbb{V}(x)P\varphi(u)\right.\\
&\qquad\left. - \int \rho(dx)m_1(x)(P-I)W_1(0) - \int \rho(dx)m_1(x)W_1(0)\right)/\widehat{m}\\
&\quad= \left(-\int \rho(dx)m_1(x)\mu_2(x)\mathbb{V}(x)P\varphi(u) - \int \rho(dx)m_1(x)(P-I)W_1(0)\right)/\widehat{m}\\
&\qquad - c_1(0,u)\\
&\quad= 0,
\end{aligned} \tag{15}$$

where $\mu_2(x) = m_2(x)/(2m_1(x))$.

Setting $\tau = 0$ in (12), we get

$$\int_0^\infty Q(ds)W_1(-s) - W_1(0) = \int_0^\infty \bar F_x(s)\,ds\mathbb{V}(x)P\varphi(u).$$

Now (14) implies that $W_1(-s) = W_1(0)$. Substituting this expression into the latter relation results in

$$\int_0^\infty Q(ds)W_1(0) - W_1(0) = m_1(x)\mathbb{V}(x)P\varphi(u).$$

Therefore

$$[P-I]W_1(0) = m_1(x)\mathbb{V}(x)P\varphi(u).$$

Finally, we substitute this expression into (15) and conclude that

$$0 = \left(-\int \rho(dx)m_1(x)\mu_2(x)\mathbb{V}(x)P\varphi(u) + \int \rho(dx)m_1^2(x)\mathbb{V}(x)P\varphi(u)\right)/\widehat{m} - c_1(0,u)$$

or

$$c_1(0,u) = \int \pi(dx)\nu_1(x)\mathbb{V}(x)P\varphi(u)/\widehat{m},$$

where

$$\pi(dx) = \rho(dx)m_1(x), \qquad \nu_1(x) = m_1(x) - \mu_2(x) = \frac{2m_1^2(x) - m_2(x)}{2m_1(x)}.$$

Remark 6.1. It is known that $F_x(t)$ has the exponential distribution if $\nu_1(x) = 0$. In this case,

$$c_1(0,u) = 0.$$

The procedure for finding the other terms of the asymptotic expansion is explained for a particular case of the second term $W_2(\tau)$. We start this procedure by writing the equality

$$
\begin{aligned}
&\int_0^\infty Q(ds)W_2(\tau-s)-W_2(\tau)\\
&\qquad=\bar F_x^{(2)}(\tau)\mathbb{V}^2(x)P\varphi(u)-\int_0^\infty\frac{s}{1!}F_x(ds)\mathbb{V}(x)PW_1(\tau-s),
\end{aligned}\tag{16}
$$

where $\bar F^{(2)}(\tau)=\int_\tau^\infty s\bar F_x(s)\,ds$.

Splitting the first integral in equality (16) into two parts we arrive at the equation

$$
\begin{aligned}
&\int_0^\tau Q(ds)W_2(\tau-s)-W_2(\tau)\\
&\quad=\bar F^{(2)}(\tau)\mathbb{V}^2(x)P\varphi(u)-\int_0^\infty\frac{s}{1!}F_x(ds)\mathbb{V}(x)PW_1(\tau-s)-\int_\tau^\infty Q(ds)W_2(\tau-s).
\end{aligned}
$$

Now we pass to the limit as $\tau\to\infty$ and derive from the renewal theorem (see, for example, [21]) that

$$
\begin{aligned}
0&=W_2(\infty)\\
&=\Big(\int\rho(dx)\int_0^\infty\int_\tau^\infty s\bar F_x(s)\,ds\,d\tau\mathbb{V}^2(x)P\varphi(u)\\
&\qquad-\int\rho(dx)\int_0^\infty\Big[\int_0^\tau\frac{s}{1!}F_x(ds)\mathbb{V}(x)PW_1(\tau-s)\,d\tau\\
&\qquad\qquad+\int_\tau^\infty\frac{s}{1!}F_x(ds)\mathbb{V}(x)PW_1(\tau-s)\,d\tau\Big]\\
&\qquad\qquad-\int\rho(dx)\int_0^\infty\int_\tau^\infty Q(ds)W_2(\tau-s)\,d\tau\Big)\Big/\widehat m.
\end{aligned}\tag{17}
$$

For $\tau<0$, we use (10)–(11):

$$
W_2(\tau)=W_2(0)-\tau U_0'(0).\tag{18}
$$

Substituting the latter expression into equation (17) gives

$$
\begin{aligned}
&\Big(\int\rho(dx)\int_0^\infty\int_\tau^\infty s\bar F_x(s)\,ds\,d\tau\mathbb{V}^2(x)P\varphi(u)\\
&\qquad-\int\rho(dx)\int_0^\infty\Big[\int_0^\tau\frac{s}{1!}F_x(ds)\mathbb{V}(x)PW_1(\tau-s)\,d\tau\\
&\qquad\qquad+\int_\tau^\infty\frac{s}{1!}\,F_x(ds)\mathbb{V}(x)PW_1(0)\Big]\\
&\qquad\times\int\rho(dx)\int_0^\infty\int_\tau^\infty Q(ds)[W_2(0)-(\tau-s)U_1'(0)]\,d\tau\Big)\Big/\widehat m\\
&\quad=\Big(\int\rho(dx)\frac{m_3(x)}{3!}\mathbb{V}^2(x)P\varphi(u)\\
&\qquad-\int\rho(dx)\int_0^\infty\int_0^\tau\frac{s}{1!}F_x(ds)\mathbb{V}(x)PW_1(\tau-s)\,d\tau\\
&\qquad-\int\rho(dx)\frac{m_2(x)}{2!}\mathbb{V}(x)PW_1(0)-\int\rho(dx)\int_0^\infty\int_\tau^\infty Q(ds)W_2(0)\,d\tau\\
&\qquad\qquad+\int\rho(dx)\int_0^\infty\int_\tau^\infty Q(ds)(\tau-s)U_1'(0)\,d\tau\Big)\Big/\widehat m
\end{aligned}\tag{19}
$$

$$
\begin{aligned}
&= \Big(\int \rho(dx) \frac{m_3(x)}{3!} \mathbb{V}^2(x) P\varphi(u) \\
&\quad - \int \rho(dx) \int_0^\infty \int_0^\tau \frac{s}{1!} F_x(ds) \mathbb{V}(x) P W_1(\tau - s)\, d\tau \\
&\quad + \int \rho(dx) \frac{m_2(x)}{2!} \mathbb{V}(x) P W_1(0) - \int \rho(dx) m(x) [P - I] W_2(0) \\
&\qquad - \int \rho(dx) m_1(x) U_2(0) - \int \rho(dx) \frac{m_2(x)}{2!} P U_1'(0) \Big) \Big/ \widehat{m} \\
&= \Big(- \int \rho(dx) m_1(x) \mu_2(x) \left(\mathbb{V}(x) P U_1(0) - P U_1'(0) \right) \\
&\quad - \int \rho(dx) m_1(x) \mu_3(x) \mathbb{V}^2(x) P U_0(0) - \int \rho(dx) m_1(x) (P - I) W_2(0) \\
&\quad - \int \rho(dx) \int_0^\infty \int_0^\tau \frac{s}{1!} F_x(ds) \mathbb{V}(x) P W_1(\tau - s) d\tau \Big) / \widehat{m} - c_2(0, u) \\
&= 0.
\end{aligned}
$$

Setting $\tau = 0$ in (16) yields

$$
\int_0^\infty Q(ds) W_2(-s) - W_2(0) = m_2(x) \mathbb{V}^2(x) P\varphi(u) - \int_0^\infty s F_x(ds) \mathbb{V}(x) P W_1(-s),
$$

whence

$$
\int_0^\infty Q(ds)[W_2(0) + s U_1'(0)] - W_2(0) = m_2(x) \mathbb{V}^2(x) P\varphi(u) - \int_0^\infty s F_x(ds) \mathbb{V}(x) P W_1(0).
$$

Therefore

$$
\begin{aligned}
[P - I] W_2(0) &= m_2(x) \mathbb{V}^2(x) P\varphi(u) - m_1(x) P U_1'(0) + m_1(x) \mathbb{V}(x) P U_1(0) \\
&= m_2(x) \mathbb{V}^2(x) P U_0(0) - m_1(x) (\mathbb{V}(x) P U_1(0) - P U_1'(0)).
\end{aligned}
$$

Finally we substitute this expression into (17):

$$
\begin{aligned}
c_2(0, u) = \Big[&\int \pi(dx) \nu_2(x) \mathbb{V}^2(x) P U_0(0) + \int \pi(dx) \nu_1(x) \left(\mathbb{V}(x) P U_1(0) - P U_1'(0) \right) \\
&- \int \rho(dx) \int_0^\infty \int_0^\tau \frac{s}{1!} F_x(ds) \mathbb{V}(x) P W_1(\tau - s)\, d\tau \Big] \Big/ \widehat{m},
\end{aligned}
$$

where

$$
\nu_2(x) = \mu_3(x) - m_2(x) = \frac{m_3(x) - 2 m_1(x) m_2(x)}{3!\, m_1(x)}.
$$

We proceed analogously for other terms:

$$
\begin{aligned}
c_k(0, u) = \Big[&\sum_{r=0}^{k-1} \int \pi(dx) \nu_{k-r}(x) \widehat{L}_{k-r}(x) U_r(0) \\
&- \sum_{r=1}^{k-1} \int \rho(dx) \int_0^\infty \int_0^\tau \frac{s^r}{r!} F_x(ds) \mathbb{V}^r(x) P W_{k-r}(\tau - s)\, d\tau \Big] \Big/ \widehat{m},
\end{aligned}
$$

$$
\nu_k(x) = (-1)^k [m_k(x) - \mu_{k+1}(x)], \qquad \widehat{L}_{k-1}(x) := \sum_{n=0}^{k-1} (-1)^n C_k^n \mathbb{V}^{k-n}(x) P U^{(n)}(t),
$$

$$
\widehat{L}_k(x) := \mathbb{V}^k(x) P.
$$

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