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MILD SOLUTION OF THE PARABOLIC EQUATION DRIVEN BY A σ-FINITE STOCHASTIC MEASURE

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Abstract. Stochastic parabolic equation driven by a σ-finite stochastic measure in the interval $[0,T]\times\mathbb{R}$ is studied. The only condition imposed on the stochastic integrator is its σ-additivity in probability on bounded Borel sets. The existence, uniqueness, and Hölder continuity of a mild solution are proved. These results generalize those known earlier for usual stochastic measures.

Introduction

The following stochastic equation:

$$\begin{cases}\mathcal{L}u(t,x)\,dt + f\big(t,x,u(t,x)\big)\,dt + \sigma(t,x)\,d\mu^{\sigma}(x) = 0\,,\\ u(0,x) = u_0(x),\end{cases}\tag{1}$$

is considered in this paper, where $(t,x)\in[0,T]\times\mathbb{R}$, μ^{σ} is a σ-finite stochastic measure defined for bounded Borel subsets of $\mathbb{R}$ and $\mathcal{L}$ is the parabolic differential operator (necessary definitions and conditions are given below in Sections 1 and 2).

The only condition imposed on the stochastic integrator μ^{σ} is its σ-additivity in probability for bounded Borel subsets of $\mathbb{R}$. We prove the existence of a mild solution of equation (1) above (see definition (2) below) and Hölder continuity of its trajectories.

Stochastic parabolic equations have been considered by many authors (see, for example, [1, 2]). Detailed studies are known for equations and systems driven by various types of stochastic processes. The results are presented in [3] for the Wiener process, in [4] for the infinite dimensional Wiener process, in [5] for martingale measures, and in [6] for α-stable processes. Certain assumptions are imposed on the stochastic integrator in all those papers, namely either the existence of certain moments, or martingale property, or independence of increments. We consider a more general integrator under the assumption that the stochastic term does not depend on an unknown function. The stochastic integrator below is defined on the set of values of the spatial variable. Analogous equations driven by $d\mu(t)$ are considered in [7] and [8].

Equations of type (1) driven by a usual stochastic measure μ are considered in [9]. The reasoning below is based on the results and methods presented in [9]. Similar results for the heat equation are obtained in [10]. In [9] and [10], μ is such that $\mu(\mathbb{R})$ is an almost surely finite random variable and moreover every measurable bounded function is integrable on $\mathbb{R}$ with respect to μ. This assumption essentially restricts the set of possible stochastic integrators. Here we generalize the results obtained in [9] and [10] to the equations driven by σ-finite random functions of sets.

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The paper is constructed as follows. Section 1 contains some preliminary definitions and results about usual and σ-finite stochastic measures. The definition of a solution of problem (1) is given in Section 2. Necessary assumptions to be imposed on elements of the equations are also collected in Section 2. The main result of the paper concerning the solutions of equation (1) is stated and proved in Section 3. Some auxiliary propositions used in our proofs are discussed in Section 4.

1. Preliminary definitions and results

Let $\mathsf{L}_0 = \mathsf{L}_0(\Omega, \mathcal{F}, \mathsf{P})$ be the set of all real valued random variables defined on a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$. The convergence in the space L_0 is understood in the sense of the convergence in probability. Also let X be an arbitrary set and let $\mathcal{B}$ be some σ-algebra of subsets in X.

Definition 1.1. An arbitrary σ-additive mapping $\mu\colon \mathcal{B} \to \mathsf{L}_0$ is called a *stochastic measure.*

Below is an example of a stochastic measure, namely

$$\mu(A) = \int_0^T \mathbf{1}_A(s)\, dX(s),$$

where $X(s)$ is either a square integrable martingale or a fractional Brownian motion with Hurst index $H > 1/2$. Other examples of stochastic measures and conditions under which the increments of a stochastic process with independent increments generate a stochastic measure can be found in Sections 7 and 8 of [11].

The theory of integration of real valued functions with respect to a stochastic measure is developed in [11, 12]. In particular, every bounded measurable function is integrable with respect to any μ (see [11, 12]). Also, an analogue of the Lebesgue dominated convergence theorem is valid (see [11, Proposition 7.1.1] or [12, Corollary 1.2]).

Stochastic measures are an analogue of finite measures in the sense that $\mu(A)$ is an almost surely finite random variable. In order to generalize the notion of a σ-finite real measure, we adopt the following definition from [12, Section 2].

Definition 1.2. A random function of sets μ^σ is called a *σ-finite stochastic measure* if there exists a representation

$$\mathsf{X} = \bigcup_{j=1}^{\infty} \mathsf{X}_j, \qquad \mathsf{X}_j \in \mathcal{B}, \quad \mathsf{X}_j \subset \mathsf{X}_{j+1}, \tag{1}$$

such that μ^σ is a random measure on $\mathcal{B} \cap \mathsf{X}_j$ for all $j \geq 1$.

Such a measure μ^σ is not defined on the whole σ-algebra $\mathcal{B}$. Rather, it is defined on the class of sets $\bigcup_{j\geq 1}(\mathcal{B} \cap \mathsf{X}_j)$. Obviously, a usual stochastic measure is a particular case of a σ-finite stochastic measure with $\mathsf{X}_j = \mathsf{X}$.

Definition 1.3. A measurable function $g\colon \mathsf{X} \to \mathbb{R}$ is called *integrable* with respect to a σ-finite stochastic measure μ^σ if g is integrable with respect to μ^σ on every set X_j involved in representation (see equation (1) in this section) and the limit

$$\operatorname*{p\,lim}_{j\to\infty} \int_{A\cap \mathsf{X}_j} g\, d\mu^\sigma \tag{2}$$

exists in probability for every $A \in \mathcal{B}$. In such a case we define $\int_A g\, d\mu^\sigma$ as the limit in (2).

Example. Let $\mathcal{B}$ be the Borel σ-algebra in $[0,+\infty)$, $\mathsf{X}_j = [0,j]$, $X(s)$, $s \ge 0$, be a martingale, and let $\mathsf{E}\, X^2(j) < +\infty$ for all j. Then

$$\mu^\sigma(A \cap \mathsf{X}_j) = \int_{\mathsf{X}_j} \mathbf{1}_A(s)\, dX(s)$$

is a σ-finite stochastic measure. If a measurable function $g\colon [0,+\infty) \to \mathbb{R}$ is such that

$$\mathsf{E} \int_{[0,+\infty)} g^2(s)\, d\langle X, X\rangle(s) < +\infty,$$

then g is integrable on $[0,+\infty)$ with respect to μ^σ. The corresponding limit in (2) exists in the square mean sense.

Integrals of real valued functions with respect to a σ-finite stochastic measure are considered in detail in Section 2 of [12]. In particular, the following results are proved in [12].

Theorem 1.1 ([12, Lemma 2.2, Theorem 2.2])**.** 1. *Let a function g be integrable with respect to μ^σ. Then the function of sets*

$$\eta(A) = \int_A g\, d\mu^\sigma, \qquad A \in \mathcal{B},$$

is a stochastic measure.

2. *A measurable function $h\colon \mathsf{X} \to \mathbb{R}$ is integrable with respect to a stochastic measure η if and only if gh is integrable with respect to μ^σ. Moreover*

$$\forall\, A \in \mathcal{B}: \quad \int_A h\, d\eta = \int_A gh\, d\mu^\sigma.$$

Theorem 1.2 ([12, Theorem 2.1])**.** *Let a function $g\colon \mathsf{X} \to \mathbb{R}$ be integrable with respect to μ^σ, let a function $h\colon \mathsf{X} \to \mathbb{R}$ be measurable, and let $|h(x)| \le |g(x)|$ for all x. Then h is integrable with respect to μ^σ.*

Theorem 1.3 ([12, Theorem 2.4])**.** *Let g be integrable with respect to μ^σ in the sense of Definition* 1.3 *with a given representation (see* (1) *in this section). Then g is integrable with respect to μ^σ with any other representation (see* (1) *in this section) such that μ^σ satisfies Definition* 1.2*. Moreover the integrals $\int_A g\, d\mu^\sigma$ for two representations are almost surely equal for every $A \in \mathcal{B}$.*

2. Setting of the problem

In what follows let $\mathsf{X} = \mathbb{R}$, let $\mathcal{B}$ be the Borel σ-algebra of subsets of $\mathbb{R}$, and let μ^σ be a σ-finite stochastic measure satisfying Definition 1.2 with $\mathsf{X}_j = [-j,j]$. Thus $\mu^\sigma(A)$ is defined for all bounded Borel sets $A \subset \mathbb{R}$.

Consider the differential operator

$$\mathcal{L}u(t,x) = a(t,x)\frac{\partial^2 u(t,x)}{\partial x^2} + b(t,x)\frac{\partial u(t,x)}{\partial x} + c(t,x)u(t,x) - \frac{\partial u(t,x)}{\partial t}, \tag{1}$$

where the functions a, b, and c are defined in the cylinder

$$S = [0,T] \times \mathbb{R} = \{(t,x)\colon t \in [0,T], x \in \mathbb{R}\}.$$

Our aim is to study a mild solution of equation (1) in the Introduction. In other words, we study a function

$$u(t,x) = u(t,x,\omega)\colon [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$$

for which

$$
(2)\qquad \begin{aligned} u(t,x) = &\int_{\mathbb{R}} p(t,x;0,y)u_0(y)\,dy + \int_0^t ds \int_{\mathbb{R}} p(t,x;s,y) f\big(s,y,u(s,y)\big)\,dy \\ &+ \int_{\mathbb{R}} d\mu^\sigma(y) \int_0^t p(t,x;s,y)\sigma(s,y)\,ds \end{aligned}
$$

almost surely for every pair $(t,x) \in (0,T] \times \mathbb{R}$.

Here $p(t,x;s,y)$ denotes the fundamental solution of the operator $\mathcal{L}$.

The following conditions are used throughout the paper.

$\mathbf{A}_{\boldsymbol{u_0}}$ $u_0(y) = u_0(y,\omega)\colon \mathbb{R}\times\Omega \to \mathbb{R}$ is measurable and bounded, $|u_0(y,\omega)| \le C_{u_0}(\omega)$. Also, $u_0(y)$ is Hölder continuous with respect to the argument $y \in \mathbb{R}$, that is,

$$
|u_0(y_1) - u_0(y_2)| \le L_{u_0}(\omega)\,|y_1 - y_2|^{\beta(u_0)}\,, \qquad \beta(u_0) \ge 1/2.
$$

$\mathbf{A}_{\boldsymbol{f}}$ $f(s,y,z)\colon [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is measurable and bounded, $|f(s,y,z)| \le C_f$. Also, $f(s,y,z)$ satisfies the Lipschitz condition with respect to the arguments $y \in \mathbb{R}$ and $z \in \mathbb{R}$, that is,

$$
|f(s,y_1,z_1) - f(s,y_2,z_2)| \le L_f\left(|y_1 - y_2| + |z_1 - z_2|\right)\,.
$$

$\mathbf{A}_{\boldsymbol{\sigma}}$ $\sigma(s,y)\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ is measurable and such that

$$
\sigma_\theta(s,y) = \sigma(s,y)e^{-\theta y^2}
$$

is bounded for all $\theta > 0$, that is, $|\sigma_\theta(s,y)| \le C_{\sigma,\theta}$. Also $\sigma_\theta(s,y)$ is Hölder continuous with respect to the argument $y \in \mathbb{R}$, that is,

$$
|\sigma_\theta(s,y_1) - \sigma_\theta(s,y_2)| \le L_{\sigma,\theta}\,|y_1 - y_2|^{\beta(\sigma,\theta)}\,, \qquad \beta(\sigma,\theta) > 1/2.
$$

$\mathbf{A}_{\boldsymbol{\mathcal{L}}}$ The functions $a(t,x)$, $b(t,x)$, and $c(t,x)$ in equation (1) in Section 2 are continuous and bounded in S and such that, for some $L_{\mathcal{L}} > 0$ and $\alpha > 0$,

$$
\begin{gathered} \left|a(t,x) - a\left(t^0,x^0\right)\right| \le L_{\mathcal{L}}\left(\left|x - x^0\right|^\alpha + \left|t - t^0\right|^\alpha\right), \\ \left|b(t,x) - b\left(t,x^0\right)\right| \le L_{\mathcal{L}}\left|x - x^0\right|^\alpha, \qquad \text{and} \\ \left|c(t,x) - c\left(t,x^0\right)\right| \le L_{\mathcal{L}}\left|x - x^0\right|^\alpha \end{gathered}
$$

everywhere in S. Moreover the operator $\mathcal{L}$ is uniformly parabolic in the cylinder S, that is, there are positive constants λ_0 and λ_1 such that $\lambda_0 \le a(t,x) \le \lambda_1$ for all pairs $(t,x) \in S$.

$\mathbf{A}_{\boldsymbol{p}}$ The fundamental solution of the operator $\mathcal{L}$ is homogeneous with respect to spatial arguments, that is,

$$
p(t,x;s,y) = p(t,x-y;s,0).
$$

Note that condition A_p is equivalent to the following one: the functions a, b, and c on the right hand side of equation (1) in Section 2 do not depend on the spatial variable x.

If condition $\mathrm{A}_{\mathcal{L}}$ holds, then Theorem 1 of [13, Section 4] yields the following bounds:

$$
(3)\qquad |p(t,x;s,y)| \le M(t-s)^{-\frac{1}{2}} \exp\left\{-\frac{\lambda|x-y|^2}{t-s}\right\},
$$

$$
(4)\qquad \left|\frac{\partial p(t,x;s,y)}{\partial x}\right| \le M(t-s)^{-1} \exp\left\{-\frac{\lambda|x-y|^2}{t-s}\right\},
$$

$$
(5)\qquad \left|\frac{\partial p(t,x;s,y)}{\partial t}\right| \le M(t-s)^{-\frac{3}{2}} \exp\left\{-\frac{\lambda|x-y|^2}{t-s}\right\},
$$

where λ and M are some positive constants.

Throughout below the symbols C, C_1, C_2, C_3 denote positive constants whose precise values do not matter for our reasoning.

3. Main result

Theorem 3.1. *Let conditions* A_{u_0}, A_f, A_σ, *and* $\mathrm{A}_{\mathcal{L}}$ *hold and a function* $e^{-\theta y^2}$ *be integrable on* $\mathbb{R}$ *with respect to* μ^σ *for all* $\theta > 0$. *Then*

1. *Equation* (2) *in Section 2 possesses a solution* $u(t,x)$. *If* $v(t,x)$ *is another solution of* (2), *then* $u(t,x) = v(t,x)$ *almost surely for all* $(t,x) \in S$.

Assume, in addition, that condition A_p *holds. Then*

2. *For all fixed* $t \in [0,T]$, $K > 0$, *and* $\gamma_1 < 1/2$, *there exists a version of the stochastic process* $u(t,x)$, $x \in [-K,K]$, *being Hölder continuous of order* γ_1.

3. *For all fixed* $\delta > 0$, $K > 0$, $\gamma_1 < 1/2$, *and* $\gamma_2 < 1/4$, *there exists a version* $\tilde{u}(t,x)$ *of the random function* $u(t,x)$ *such that*

$$|\tilde{u}(t_1,x_1) - \tilde{u}(t_2,x_2)| \le C_{\tilde{u}}(\omega)(|t_1-t_2|^{\gamma_2} + |x_1-x_2|^{\gamma_1}), \qquad t \in [\delta,T], \qquad x \in [-K,K]$$

for some $C_{\tilde{u}}(\omega) > 0$.

Proof. We will apply Lemmas 4.1 and 4.2 whose proof is given in Section 4. In what follows we consider versions (16) for all integrals with respect to stochastic measures.

First we rewrite equation (2) in Section 2 in the following form:

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}} p(t,x;0,y)u_0(y)\,dy + \int_0^t ds \int_{\mathbb{R}} p(t,x;s,y) f\big(s,y,u(s,y)\big)\,dy \\ &\quad + \int_{\mathbb{R}} d\eta_\theta(y) \int_0^t p_\theta(t,x;s,y)\sigma_\theta(s,y)\,ds, \end{aligned} \tag{6}$$

where σ_θ is defined in condition A_σ,

$$p_\theta(t,x;s,y) = e^{2\theta y^2} p(t,x;s,y), \qquad 0 < \theta < \frac{\lambda}{2T},$$

and λ is such that inequalities (3)–(5) hold. The stochastic measure η_θ is defined by the following equality:

$$\eta_\theta(A) = \int_A e^{-\theta y^2}\,d\mu^\sigma(y), \qquad A \in \mathcal{B}.$$

The integrals with respect to η_θ in (6) and with respect to μ^σ in equation (2) in Section 2 coincide in accordance with Theorem 1.1.

Using (3)–(5) for a fixed K and some $M = M_{K,\theta}$ and $\lambda = \lambda_\theta > 0$ we obtain the following bounds:

$$|p_\theta(t,x;s,y)| \le M_{K,\theta}(t-s)^{-\frac{1}{2}} \exp\left\{-\frac{\lambda_\theta|x-y|^2}{t-s}\right\}, \tag{7}$$

$$\left|\frac{\partial p_\theta(t,x;s,y)}{\partial x}\right| \le M_{K,\theta}(t-s)^{-1} \exp\left\{-\frac{\lambda_\theta|x-y|^2}{t-s}\right\}, \tag{8}$$

$$\left|\frac{\partial p_\theta(t,x;s,y)}{\partial t}\right| \le M_{K,\theta}(t-s)^{-\frac{3}{2}} \exp\left\{-\frac{\lambda_\theta|x-y|^2}{t-s}\right\} \tag{9}$$

for all $|x| \le K$. Let us prove, for example, bound (7). For $0 < \varepsilon < (\lambda - 2T\theta)/T$, we derive from (3) that

$$|p_\theta(t,x;s,y)| \le \exp\{2\theta y^2\} M (t-s)^{-\frac{1}{2}} \exp\left\{-\frac{\lambda |x-y|^2}{t-s}\right\}$$
$$= M(t-s)^{-\frac{1}{2}} \exp\left\{-\frac{(\lambda - 2T\theta - T\varepsilon)|x-y|^2}{t-s}\right\}$$
$$\times \exp\left\{2\theta y^2 - \frac{(2T\theta + T\varepsilon)|x-y|^2}{t-s}\right\},$$

$$2\theta y^2 - \frac{(2T\theta + T\varepsilon)|x-y|^2}{t-s} \overset{t-s\le T}{\le} 2\theta\left(y^2 - (x-y)^2\right) - \varepsilon (x-y)^2. \tag{10}$$

Put $\lambda_\theta = \lambda - 2T\theta - T\varepsilon$. Then

$$\max_{y\in\mathbb{R}} \left(2\theta\left(y^2 - (x-y)^2\right) - \varepsilon(x-y)^2\right) = x^2 \left(\frac{4\theta^2}{\varepsilon} + 2\theta\right) \overset{|x|\le K}{\le} K^2 \left(\frac{4\theta^2}{\varepsilon} + 2\theta\right) =: M_1. \tag{11}$$

One can choose $M_{K,\theta} = M \exp\{M_1\}$. It is obvious that bounds (8) and (9) hold with the same constants λ_θ and $M_{K,\theta}$.

Proof of statement 1 *of Theorem* 3.1. We apply the fixed-point iteration similarly to the reasoning in [10]. Put $u^{(0)}(t,x) = 0$ and

$$u^{(n+1)}(t,x) = \int_{\mathbb{R}} p(t,x;0,y) u_0(y)\, dy + \int_0^t ds \int_{\mathbb{R}} p(t,x;s,y) f\left(s,y,u^{(n)}(s,y)\right) dy + \int_{\mathbb{R}} d\eta_\theta(y) \int_0^t p_\theta(t,x;s,y)\sigma_\theta(s,y)\, ds, \qquad n \ge 0. \tag{12}$$

Then

$$\left|u^{(n+1)}(t,x) - u^{(n)}(t,x)\right| \overset{\mathrm{A}_f}{\le} L_f \int_0^t ds \int_{\mathbb{R}} p(t,x;s,y) \left|u^{(n)}(s,y) - u^{(n-1)}(s,y)\right| dy \tag{13}$$

for all $\omega \in \Omega$ and $n \ge 2$.

The assumptions of the theorem and equality $\int_{\mathbb{R}} e^{-z^2/b^2} dz = Cb$ imply that

$$\int_{\mathbb{R}} |p(t,x;s,y)|\, dy \le M \int_{\mathbb{R}} (t-s)^{-\frac{1}{2}} e^{-\frac{\lambda|x-y|^2}{t-s}}\, dy = C. \tag{14}$$

Hence

$$\left|u^{(2)}(t,x) - u^{(1)}(t,x)\right| \le 2C_f \int_0^t ds \int_{\mathbb{R}} |p(t,x;s,y)|\, dy \le 2Ct.$$

Considering

$$g_n(t) = \sup_{x\in\mathbb{R}} \left|u^{(n+1)}(t,x) - u^{(n)}(t,x)\right|, \qquad n \ge 1,$$

we derive from (13) that

$$g_n(t) \le L_f \int_0^t g_{n-1}(s)\, ds.$$

Using the induction,

$$g_n(t) \le 2C_f L_f^n \frac{t^{n+1}}{(n+1)!}, \tag{15}$$

whence we conclude that the sequence $\sum_{n=1}^{\infty} g_n(t)$ converges uniformly in $[0,T]$. Put

$$u(t,x) := \lim_{n\to\infty} u^{(n)}(t,x).$$

Passing to the limit in (12) as $n \to \infty$ we prove (2).

Now we prove that a solution is unique. If $u(t,x)$ and $v(t,x)$ are two different solutions of equation (2) in Section 2, then

$$u(t,x) - v(t,x) = \int_0^t ds \int_{\mathbb{R}} p(t,x;s,y) \left[f\big(s,y,u(s,y)\big) - f\big(s,y,v(s,y)\big) \right] dy.$$

Based on condition A_f, one can repeat the same reasoning for

$$g(t) = \sup_{x\in\mathbb{R}} |u(t,x) - v(t,x)|\,.$$

Then we obtain

$$g(t) \le 2C_f L_f^n \frac{t^{n+1}}{(n+1)!} \le 2C_f L_f^n \frac{T^{n+1}}{(n+1)!}\,.$$

Passing to the limit as $n \to \infty$ we conclude that $g = 0$, whence the uniqueness of a solution follows.

Proof of statement 2 *of Theorem* 3.1. We apply the Hölder property with respect to the variable x on bounded subsets of $\mathbb{R}$. Using the induction, we prove that for any $n \ge 0$ there exists $L_{u^{(n)}}(t) > 0$ such that

$$\left| u^{(n)}(t,x_1) - u^{(n)}(t,x_2) \right| \le L_{u^{(n)}}(t)\, |x_1 - x_2|^{\gamma_1}\,.$$

We have $L_{u^{(0)}} = 0$.

With the help of equality (12), Lemma 4.1, the change of the variable $y \to y + x_2 - x_1$ in the integrals with respect to y containing the variable x_2, and taking into account the initial conditions we get

$$\begin{aligned}
&\left| u^{(n+1)}(t,x_1) - u^{(n+1)}(t,x_2) \right| \\
&\quad \le \int_{\mathbb{R}} p(t,x_1;0,y)\, |u_0(y) - u_0(y + x_2 - x_1)|\, dy \\
&\qquad + \int_0^t ds \int_{\mathbb{R}} p(t,x_1;s,y) \Big| f\left(s,y,u^{(n)}(s,y)\right) \\
&\qquad\qquad - f\left(s, y + x_2 - x_1, u^{(n)}(s, y + x_2 - x_1)\right) \Big|\, dy \\
&\qquad + C\, |x_1 - x_2|^{\gamma_1} \\
&\quad \le L_{u_0}\, |x_1 - x_2|^{\beta(u_0)} \\
&\qquad + \int_0^t ds \int_{\mathbb{R}} p(t-s, x_1 - y) L_f \left(|x_1 - x_2| + L_{u^{(n)}}(s)\, |x_1 - x_2|^{\gamma_1} \right) dy \\
&\qquad + C\, |x_1 - x_2|^{\gamma_1}\,.
\end{aligned}$$

Thus

$$L_{u^{(n+1)}}(t) \le L + L \int_0^t L_{u^{(n)}}(s)\, ds$$

for some constant L. Statement 2 is proved. By induction, we find an upper bound

$$L_{u^{(n)}}(t) \le L e^{Lt} \le L e^{LT}$$

and this proves the Hölder continuity with respect to the variable x. Note that the constant L does not depend on t.

Proof of statement 3 *of Theorem* 3.1. We follow the lines of the proof of Proposition 3 in [9] and use Lemma 4.2. □

Remark 3.1. If μ^σ in Theorem 3.1 is an ordinary stochastic measure, then the function $e^{-\theta y^2}$, as a bounded function, is integrable with respect to μ^σ. This implies the integrability of the function $|y|^\tau e^{-\theta y^2}$. Therefore Theorem of the paper [9] is a particular case of Theorem 3.1. In addition, the result of Theorem of [9] remains valid without the assumption that $|y|^\tau$ is integrable with respect to a stochastic measure.

4. Auxiliary results

Consider the Besov space $B^\alpha_{22}([c,d])$, $0<\alpha<1$. A function g belongs to the space $B^\alpha_{22}([c,d])$ if its norm in the Besov space,

$$\|g\|_{B^\alpha_{22}([c,d])} = \|g\|_{L_2([c,d])} + \left(\int_0^{d-c} \big(w_2(g,r)\big)^2 r^{-2\alpha-1}\,dr\right)^{1/2},$$

is finite, where

$$w_2(g,r) = \sup_{0\le h\le r}\left(\int_c^{d-h} |g(v+h)-g(v)|^2\,dv\right)^{1/2}.$$

For an arbitrary $j\in\mathbb{Z}$, let

$$\Delta^{(j)}_{kn} = \big(j+(k-1)2^{-n}, j+k2^{-n}\big], \qquad n\ge 0,\ 1\le k\le 2^n.$$

Let Z be an arbitrary set and let a function $g(z,v)\colon Z\times[j,j+1]\to\mathbb{R}$ be continuous with respect to the second argument for all $z\in Z$. Put

$$g_n(z,v) = g(z,j)\mathbf{1}_{\{j\}}(v) + \sum_{1\le k\le 2^n} g\big(z, j+(k-1)2^{-n}\big)\mathbf{1}_{\Delta^{(j)}_{kn}}(v).$$

Then

$$\zeta(z) = \int_{[j,j+1]} g(z,v)\,d\mu(v), \qquad z\in Z,$$

has a version

$$\widetilde{\zeta}(z) = \int_{[j,j+1]} g_0(z,v)\,d\mu(v) + \sum_{n\ge1}\left(\int_{[j,j+1]} g_n(z,v)\,d\mu(v) - \int_{[j,j+1]} g_{n-1}(z,v)\,d\mu(v)\right) \tag{16}$$

such that

$$\big|\widetilde{\zeta}(z)\big| \le \big|g(z,j)\mu([j,j+1])\big| + C\|g(z,\cdot)\|_{B^\alpha_{22}([j,j+1])}\left\{\sum_{n\ge1} 2^{-n(1-2\alpha)}\sum_{1\le k\le 2^n}\left|\mu\left(\Delta^{(j)}_{kn}\right)\right|^2\right\}^{\frac12} \tag{17}$$

for all $\omega\in\Omega$ and $z\in Z$. This follows from Lemma 3 in [14] and Theorem 1.2 in [15].

The following result is an analogue of Lemma 1 of [9]. Note, however, that we do not impose the restriction that the function p_θ is homogeneous with respect to the spatial variables.

Lemma 4.1. *Let conditions* A_σ, $\mathrm{A}_{\mathcal{L}}$, *and* A_p *hold. Given arbitrary* $t \in [0,T]$, $K > 0$, *and* $\gamma_1 < 1/2$, *the stochastic process*

$$\vartheta_\theta(x) = \int_{\mathbb{R}} d\eta_\theta(y) \int_0^t p_\theta(t,x;s,y)\sigma_\theta(s,y)\,ds, \qquad |x| \le K,$$

has a Hölder continuous version of order γ_1.

Proof. We consider version (16) for all stochastic integrals. Now we follow the lines of the proof of an analogous result in [9].

For fixed t and $x_1 \le x_2$, consider

$$q(z,y) = \int_0^t p_\theta(t,x_1;s,y)\sigma_\theta(s,y)\,ds - \int_0^t p_\theta(t,x_2;s,y)\sigma_\theta(s,y)\,ds,$$
$$z = (t,x_1,x_2), \qquad y \in \mathbb{R}.$$

Our current aim is to estimate the norm of the function $q(z,\cdot)$ in the Besov space in the interval $[j,j+1]$. We have

$$\begin{aligned} q(z,y+h) - q(z,y) &= \int_0^t \big(p_\theta(t,x_1;s,y) - p_\theta(t,x_2;s,y)\big)\big(\sigma_\theta(s,y+h) - \sigma_\theta(s,y)\big)\,ds \\ &+ \int_0^t \big(p_\theta(t,x_1;s,y+h) - p_\theta(t,x_1;s,y) - p_\theta(t,x_2;s,y+h) \\ &+ p_\theta(t,x_2;s,y)\big)\sigma_\theta(s,y+h)\,ds \\ &=: I_1 + I_2. \end{aligned}$$

First we assume that $|y| \le K+1$. Similarly to the corresponding reasoning in [9] we apply the bound

$$\int_0^t \frac{1}{r} e^{-\frac{b}{r}}\,dr = \left|\frac{b}{r} = z\right| = \int_{b/t}^\infty \frac{1}{z} e^{-z}\,dz \le \mathbf{1}_{\{t>b\}} \int_{b/t}^1 \frac{1}{z}\,dz + \int_1^\infty e^{-z}\,dz \le \left|\ln\frac{T}{b}\right| + 1$$

and conclude that

$$\begin{aligned} \int_0^t \big|p_\theta(t,x_1;s,y) - p_\theta(t,x_2;s,y)\big|\,ds &\le \int_0^t \left(\frac{C}{t-s}\int_{x_1}^{x_2} e^{-\frac{\lambda_\theta|x-y|^2}{t-s}}\,dx\right) ds \\ &= \Big|t-s = r\Big| = \int_{x_1}^{x_2} dx \int_0^t \frac{1}{r} e^{-\frac{\lambda_\theta|x-y|^2}{r}}\,dr \\ &\le \int_{x_1}^{x_2} \left(\left|\ln\frac{T}{\lambda_\theta|x-y|^2}\right| + 1\right) dx \le C_1|x_1-x_2| + C_2\int_{x_1}^{x_2} \big|\ln|x-y|\big|\,dx \\ &\le C_1|x_1-x_2| + C_3\int_0^{|x_1-x_2|/2} |\ln z|\,dz \\ &= C_1|x_1-x_2| + C_3(z - z\ln z)\Big|_0^{|x_1-x_2|/2} \le C|x_1-x_2|^\gamma, \end{aligned} \tag{18}$$

where $0 < \gamma < 1$ is arbitrary and the constant C depends on γ, λ, K, and T. Here we used the fact that if $|x_1 - x_2| < 1$ is fixed and either x_1 or x_2 belongs to the interval $[y-1,y+1]$, then the value of the integral $\int_{x_1}^{x_2}\big|\ln|x-y|\big|\,dx$ is maximal if x_1 and x_2 are symmetric around y. Otherwise $[x_1,x_2] \subset [-K,K]$ and $|y| \le K+1$, whence the integral does not exceed $|\ln(2K+1)| \cdot |x_1-x_2|$. We also used the inequality$|x_1-x_2|^{1-\gamma}\ln|x_1-x_2| \le C$ for $x_1,x_2 \in \{x \in \mathbb{R}\colon |x| \le K\}$ and for all $\gamma < 1$. This inequality follows from $|x_1-x_2|^{1-\gamma}\ln|x_1-x_2| \to 0$ as $|x_1-x_2| \to 0$.

If $|y| > K+1$, then $|x-y| \geq 1$ in (18) and the corresponding integral does not exceed

$$\int_0^t \left(\frac{C}{t-s} \int_{x_1}^{x_2} e^{-\frac{\lambda_\theta}{t-s}}\, dx \right) ds = C|x_1 - x_2|.$$

This implies the bounds

$$(19)\quad \begin{aligned} |I_1| &\leq \int_0^t \big(|p_\theta(t,x_1;s,y) - p_\theta(t,x_2;s,y)|\big)\big(|\sigma_\theta(s,y+h) - \sigma_\theta(s,y)|\big)\, ds \\ &\leq Ch^{\beta(\sigma,\theta)}|x_1 - x_2|^\gamma, \end{aligned}$$

$$(20)\quad |I_2| \leq C|x_1 - x_2|^\gamma.$$

Now we estimate those terms in I_2 that contain x_1. The terms with x_2 are considered similarly. We have

$$\begin{aligned} &\left| \int_0^t \big(p_\theta(t,x_1;s,y+h) - p_\theta(t,x_1;s,y)\big)\sigma_\theta(s,y+h)\, ds \right| \\ &\leq C_{\sigma,\theta} \int_0^t \left| e^{2\theta(y+h)^2} p(t,x_1;s,y+h) - e^{2\theta y^2} p(t,x_1;s,y) \right| ds \\ &\leq C \int_0^t e^{2\theta(y+h)^2} \big|p(t,x_1;s,y+h) - p(t,x_1;s,y)\big|\, ds \\ &\quad + C \int_0^t \left| e^{2\theta(y+h)^2} - e^{2\theta y^2} \right| |p(t,x_1;s,y)|\, ds \\ &=: CJ_1 + CJ_2, \\ &J_1 \overset{\mathrm{A}_p}{=} \int_0^t e^{2\theta(y+h)^2} \big|p(t,x_1-h;s,y) - p(t,x_1;s,y)\big|\, ds \\ &\quad = \int_0^t ds \left| \int_{x_1-h}^{x_1} e^{2\theta(y+h)^2}\, \frac{\partial p(t,x;s,y)}{\partial x}\, dx \right| \\ &\quad \overset{(4)}{\leq} C \int_0^t \frac{ds}{t-s} \int_{x_1-h}^{x_1} e^{2\theta(y+h)^2} e^{-\frac{\lambda|x-y|^2}{t-s}}\, dx. \end{aligned}$$

Similarly to (10)–(11) we obtain for $0 \leq h \leq 1$ that

$$e^{2\theta(y+h)^2} e^{-\frac{\lambda|x-y|^2}{t-s}} \leq \widetilde{M}_{K,\theta} \exp\left\{ -\frac{\lambda_\theta|x-y|^2}{t-s} \right\}.$$

Also if $0 < \beta < \lambda_\theta/T$, then

$$\begin{aligned} J_2 &= \int_0^t \left| e^{2\theta\left((y+h)^2 - y^2\right)} - 1 \right| |p_\theta(t,x_1;s,y)|\, ds \\ &\overset{|e^z-1| \leq |z|e^{|z|},\, h\leq 1}{\leq} 2\theta h \int_0^t e^{2\theta(2|y|+1)}(2|y|+1)|p_\theta(t,x_1;s,y)|\, ds \\ &= 2\theta h \int_0^t e^{2\theta(2|y|+1) - \beta y^2}(2|y|+1) \left| e^{\beta y^2} p_\theta(t,x_1;s,y) \right| ds. \end{aligned}$$

Here $e^{2\theta(2|y|+1)-\beta y^2}(2|y|+1)$ is a bounded function and thus one can obtain an inequality similar to (7) for $p_{\beta,\theta} = e^{\beta y^2} p_\theta$, whence $\int_0^t p_{\beta,\theta}\, ds \leq C\int_0^t (t-s)^{-1/2}\, ds = C$. Hence $|J_2| \leq Ch$.

For $\gamma < 1$,

$$(21)\quad |I_2| \leq Ch^\gamma.$$

Now we estimate the modulus of continuity,

$$\big(w_2(q,r)\big)^2 \le 2 \sup_{0\le h\le r} \int_j^{j+1-h} I_1^2\,dy + 2 \sup_{0\le h\le r} \int_j^{j+1-h} I_2^2\,dy.$$

According to relations (19),

$$\sup_{0\le h\le r} \int_j^{j+1-h} I_1^2\,dy \le C|x_1-x_2|^{2\gamma} \sup_{0\le h\le r} h^{2\beta(\sigma,\theta)}(1-h) \le Cr^{2\beta(\sigma,\theta)}|x_1-x_2|^{2\gamma}.$$

Reasoning similarly to the case of the integral I_2, we obtain from bounds (20) and (21) that

$$\sup_{0\le h\le r} \int_j^{j+1-h} I_2^2\,dy \le C|x_1-x_2|^{2\gamma}$$

and

$$\sup_{0\le h\le r} \int_j^{j+1-h} I_2^2\,dy \le Cr^{2\gamma}.$$

The latter two inequalities for the integral $\int_j^{j+1-h} I_2^2\,dy$ imply that, for some number $0<\delta<1$,

$$\sup_{0\le h\le r} \int_j^{j+1-h} I_2^2\,dy \le Cr^{2\gamma(1-\delta)}|x_1-x_2|^{2\gamma\delta}.$$

Therefore,

$$\begin{aligned}\big(w_2(q,r)\big)^2 &\le Cr^{2\beta(\sigma,\theta)}|x_1-x_2|^{2\gamma} + Cr^{2\gamma(1-\delta)}|x_1-x_2|^{2\gamma\delta}\\ &\le C|x_1-x_2|^{2\gamma\delta}\left(r^{2\beta(\sigma,\theta)} + r^{2\gamma(1-\delta)}\right).\end{aligned}$$

Note that the integral involved in the definition of the norm in the Besov space is finite if and only if

$$2\gamma(1-\delta) > 2\alpha \Leftrightarrow \gamma\delta < \gamma-\alpha.$$

Now the Hölder order is such that $\gamma\delta \to 1/2-$ as $\gamma\to 1-$ and $\alpha \to 1/2+$. This means that, given $0<\gamma_1<1/2$, there exists $\alpha > 1/2$ such that

$$\|q(z,\boldsymbol{\cdot})\|_{B^\alpha_{22}([j,j+1])} \le C|x_1-x_2|^{\gamma_1}.$$

The same reasoning as those that led us to bound (20) proves the following two inequalities with some constant C,

$$|q(z,j)| \le C|x_1-x_2|^{\gamma}, \qquad \|q(z,\boldsymbol{\cdot})\|_{L_2([j,j+1])} \le C|x_1-x_2|^{\gamma}.$$

Now we are ready to prove the Hölder property of ϑ_θ:

$$
\begin{aligned}
|\vartheta_\theta(x_1)-\vartheta_\theta(x_2)| &= \left|\int_{\mathbb{R}} g(y)\,d\eta_\theta(y)\right| \le \sum_{j\in\mathbb{Z}}\left|\int_j^{j+1} g(y)\,d\eta_\theta(y)\right| \\
&\overset{(17)}{\le} \sum_{j\in\mathbb{Z}}\big|q(z,j)\,\eta_\theta([j,j+1])\big| \\
&\quad + C\sum_{j\in\mathbb{Z}}\|q(z,\cdot)\|_{B_{22}^{\alpha}([j,j+1])}\left\{\sum_{n\ge1}2^{n(1-2\alpha)}\sum_{1\le k\le 2^n}\left|\eta_\theta\left(\Delta_{kn}^{(j)}\right)\right|^2\right\}^{1/2} \\
&\le C|x_2-x_1|^{\gamma_1}\left[\sum_{j\in\mathbb{Z}}|\eta_\theta([j,j+1])| + \sum_{j\in\mathbb{Z}}\left\{\sum_{n\ge1}2^{n(1-2\alpha)}\sum_{1\le k\le 2^n}\left|\eta_\theta\left(\Delta_{kn}^{(j)}\right)\right|^2\right\}^{1/2}\right] \\
&\le C|x_2-x_1|^{\gamma_1}\left[\left(\sum_{j\in\mathbb{Z}}(|j|+1)^2\left(\eta_\theta([j,j+1])\right)^2\right)^{1/2}\left(\sum_{j\in\mathbb{Z}}(|j|+1)^{-2}\right)^{1/2}\right. \\
&\qquad + \left(\sum_{n\ge1}2^{n(1-2\alpha)}\sum_{j\in\mathbb{Z}}(|j|+1)^2\sum_{1\le k\le 2^n}\left|\eta_\theta\left(\Delta_{kn}^{(j)}\right)\right|^2\right)^{1/2} \\
&\qquad\left.\times\left(\sum_{j\in\mathbb{Z}}(|j|+1)^{-2}\right)^{1/2}\right],
\end{aligned}
$$

where the sums with stochastic measures are of the form

$$
\sum_{l=1}^{\infty}\left(\int_{\mathsf{X}} f_l\,d\eta_\theta\right)^2,
$$

$$
\{f_l(y),\ l\ge1\} = \left\{(|j|+1)\,\mathbf{1}_{[j,j+1]}(y),\ j\in\mathbb{Z}\right\},
$$

$$
\{f_l(y),\ l\ge1\} = \left\{(|j|+1)2^{n(1-2\alpha)/2}\mathbf{1}_{\Delta_{kn}^{(j)}}(y),\ j\in\mathbb{Z},\ n\ge1,\ 1\le k\le 2^n\right\}.
$$

Theorem 1.1 implies

$$
\int_A(|y|+1)\,d\eta_\theta = \int_A(|y|+1)e^{-\theta y^2}\,d\mu^\sigma(y).
$$

Since $(|y|+1)e^{-\theta y^2}\le Ce^{(-\theta/2)y^2}$ and $e^{(-\theta/2)y^2}$ is integrable with respect to the stochastic measure μ^σ, Theorems 1.1 and 1.2 imply that the function $\sum_{l=1}^{\infty} f_l$ is integrable. Now Lemma 3.1 of [10] yields

$$
\sum_{l=1}^{\infty}\left(\int_{\mathsf{X}} f_l\,d\eta_\theta\right)^2 < +\infty \quad \text{almost surely}
$$

and this completes the proof of the theorem. □

The following result is an analogue of Lemma 2 in [9].

Lemma 4.2. *Let conditions* A_f, A_σ, $\mathrm{A}_\mathcal{L}$, *and* A_p *hold. Let* $x \in \mathbb{R}$ *and* $\gamma_2 < 1/4$ *be fixed. Then the stochastic process*

$$\hat{\vartheta}(t) = \int_\mathbb{R} d\eta_\theta(y) \int_0^t p_\theta(t, x; s, y)\sigma_\theta(s, y)\, ds, \quad t \in [0, T],$$

has a Hölder continuous version of order γ_2.

Proof. Let $x \in \mathbb{R}$ and $0 \le t_1 < t_2 \le T$ be arbitrary fixed numbers. Put

$$\hat{q}(z, y) = \int_0^{t_2} p_\theta(t_2, x; s, y)\sigma_\theta(s, y)\, ds - \int_0^{t_1} p_\theta(t_1, x; s, y)\sigma_\theta(s, y)\, ds, \qquad z = (t_1, t_2, x).$$

Then relation (17) holds for the version (16) of the stochastic integral

$$\hat{\eta}(z) = \int_{[j,j+1]} \hat{q}(z, y)\, d\eta_\theta(y).$$

Now we estimate the norm of the function $\hat{q}(z, \cdot)$ in the Besov space. First we represent the increment $\hat{q}(z, y + h) - \hat{q}(z, y)$ as follows:

$$\begin{aligned}
&\hat{q}(z, y + h) - \hat{q}(z, y)\\
&\quad= \int_0^{t_1} \big(p_\theta(t_2, x; s, y + h) - p_\theta(t_1, x; s, y + h)\big)\sigma_\theta(s, y + h)\, ds\\
&\qquad - \int_0^{t_1} \big(p_\theta(t_2, x; s, y) - p_\theta(t_1, x; s, y)\big)\sigma_\theta(s, y)\, ds\\
&\qquad + \int_{t_1}^{t_2} p_\theta(t_2, x; s, y + h)\sigma_\theta(s, y + h)\, ds - \int_{t_1}^{t_2} p_\theta(t_2, x; s, y)\sigma_\theta(s, y)\, ds\\
&\quad= \int_0^{t_1} \big(p_\theta(t_2, x; s, y + h) - p_\theta(t_1, x; s, y + h)\big)\big(\sigma_\theta(s, y + h) - \sigma_\theta(s, y)\big)\, ds\\
&\qquad + \int_0^{t_1} \big(p_\theta(t_2, x; s, y + h) - p_\theta(t_2, x; s, y)\big)\sigma_\theta(s, y)\, ds\\
&\qquad - \int_0^{t_1} \big(p_\theta(t_1, x; s, y + h) - p_\theta(t_1, x; s, y)\big)\sigma_\theta(s, y)\, ds\\
&\qquad + \int_{t_1}^{t_2} p_\theta(t_2, x; s, y + h)\big(\sigma_\theta(s, y + h) - \sigma_\theta(s, y)\big)\, ds\\
&\qquad + \int_{t_1}^{t_2} \big(p_\theta(t_2, x; s, y + h) - p_\theta(t_2, x; s, y)\big)\sigma_\theta(s, y)\, ds\\
&\quad=: J_{11} + J_{12} - J_{13} + J_{21} + J_{22}\\
&\quad=: J_1 + J_2.
\end{aligned}$$

Then we obtain from (7)

$$\begin{aligned}
|J_{21}| &\le Ch^{\beta(\sigma,\theta)} \int_{t_1}^{t_2} (t_2 - s)^{-1/2} e^{-\frac{\lambda_\theta |x-y-h|^2}{t_2-s}}\, ds \le Ch^{\beta(\sigma,\theta)} \int_{t_1}^{t_2} (t_2 - s)^{-1/2}\, ds\\
&= Ch^{\beta(\sigma,\theta)}(t_2 - t_1)^{1/2}
\end{aligned} \tag{22}$$

by condition A_σ. Further,

$$
\begin{aligned}
|J_{22}| &\le \left|\int_{t_1}^{t_2} p_\theta(t_2,x;s,y+h)\sigma_\theta(s,y)ds\right| + \left|\int_{t_1}^{t_2} p_\theta(t_2,x;s,y)\sigma_\theta(s,y)\,ds\right| \\
&\le C\int_{t_1}^{t_2}(t_2-s)^{-1/2}e^{-\frac{\lambda_\theta|x-y-h|^2}{t_2-s}}\,ds + C\int_{t_1}^{t_2}(t_2-s)^{-1/2}e^{-\frac{\lambda_\theta|x-y|^2}{t_2-s}}\,ds \\
&\le C(t_2-t_1)^{1/2}.
\end{aligned} \tag{23}
$$

On the other hand, applying the same reasoning as that leading to (20) we obtain

$$
\begin{aligned}
|J_{22}| &\le C\int_{t_1}^{t_2}\int_{x-h}^{x}\left|\frac{\partial p_\theta(t_2,v;s,y)}{\partial v}\right|\,dv\,ds \\
&\le C\int_{t_1}^{t_2}\int_{x-h}^{x}(t_2-s)^{-1}e^{-\frac{\lambda_\theta|v-y|^2}{t_2-s}}\,dv\,ds \le Ch^{\gamma_0},
\end{aligned} \tag{24}
$$

where $0<\gamma_0<1$ is an arbitrary number and the constant C depends on γ_0.

Raising both sides of inequality (23) to the power δ_0 and both sides of inequality (24) to the power $1-\delta_0$ with some $\delta_0\in(0,1)$ and taking into account (22) we conclude that

$$
\begin{aligned}
|J_2| &\le Ch^{\beta(\sigma,\theta)}(t_2-t_1)^{1/2} + Ch^{(1-\delta_0)\gamma_0}(t_2-t_1)^{\delta_0/2} \\
&\le C(t_2-t_1)^{\delta_0/2}\Big(h^{\beta(\sigma,\theta)} + h^{(1-\delta_0)\gamma_0}\Big).
\end{aligned}
$$

Passing to the limit as $\gamma_0\to 1-$ and $1-\delta_0\to 1/2+$ we prove that $(1-\delta_0)\gamma_0>1/2$ and $\delta_0\to 1/2-$.

Using condition A_σ and relation (9) we get

$$
\begin{aligned}
|J_{11}| &\le L_{\sigma,\theta}h^{\beta(\sigma,\theta)}\int_0^{t_1}\int_{t_1}^{t_2}\left|\frac{\partial p_\theta(\tau,x;s,y+h)}{\partial\tau}\right|\,d\tau\,ds \\
&\le Ch^{\beta(\sigma,\theta)}\int_0^{t_1}\int_{t_1}^{t_2}(\tau-s)^{-3/2}e^{-\frac{\lambda_\theta|x-y-h|^2}{\tau-s}}\,d\tau\,ds \\
&\le Ch^{\beta(\sigma,\theta)}\int_0^{t_1}\int_{t_1}^{t_2}(\tau-s)^{-3/2}\,d\tau\,ds \le Ch^{\beta(\sigma,\theta)}(t_2-t_1)^{1/2}.
\end{aligned} \tag{25}
$$

Similarly, condition A_σ implies

$$
\begin{aligned}
|J_{12}-J_{13}| &= \Bigg|\int_0^{t_1}\big(p_\theta(t_2,x;s,y+h)-p_\theta(t_1,x;s,y+h)\big)\sigma_\theta(s,y)\,ds \\
&\qquad - \int_0^{t_1}\big(p_\theta(t_2,x;s,y)-p_\theta(t_1,x;s,y)\big)\sigma_\theta(s,y)\,ds\Bigg| \\
&\le C\int_0^{t_1}\int_{t_1}^{t_2}\left|\frac{\partial p_\theta(\tau,x;s,y+h)}{\partial\tau}\right|\,d\tau\,ds \\
&\quad + C\int_0^{t_1}\int_{t_1}^{t_2}\left|\frac{\partial p_\theta(\tau,x;s,y)}{\partial\tau}\right|\,d\tau\,ds \\
&\le C(t_2-t_1)^{1/2}.
\end{aligned} \tag{26}
$$

On the other hand, we repeat the same reasoning as that used to prove inequality (24) and obtain the following bound:

$$
|J_{12}-J_{13}| \le |J_{12}|+|J_{13}| \le Ch^{\gamma_0}. \tag{27}
$$

Raising both sides of inequality (26) to the power δ_0 and both sides of inequality(27) to the power $1-\delta_0$ and taking into account inequality (25) we derive the bound

$$\begin{aligned}|J_1| &\le Ch^{\beta(\sigma,\theta)}(t_2-t_1)^{1/2} + Ch^{(1-\delta_0)\gamma_0}(t_2-t_1)^{\delta_0/2}\\ &\le C(t_2-t_1)^{\delta_0/2}\Big(h^{\beta(\sigma,\theta)} + h^{(1-\delta_0)\gamma_0}\Big).\end{aligned}$$

Therefore

$$|\hat q(z,y+h)-\hat q(z,y)| \le C(t_2-t_1)^{\delta_0/2}\Big(h^{\beta(\sigma,\theta)} + h^{(1-\delta_0)\gamma_0}\Big),$$

whence

$$\begin{aligned}&\left(\int_0^1 \big(w_2(g,r)\big)^2 r^{-2\alpha-1}\,dr\right)^{1/2}\\ &\le C(t_2-t_1)^{\delta_0/2}\left(\int_0^1 r^{2\beta(\sigma,\theta)-2\alpha-1}dr + \int_0^1 r^{2(1-\delta_0)\gamma_0-2\alpha-1}dr\right)^{1/2}\\ &\le C(t_2-t_1)^{\delta_0/2}\end{aligned}$$

for an appropriate $1/2<\alpha<\min\{(1-\delta_0)\gamma_0, \beta(\sigma,\theta)\}$.

Moreover, relations (23) and (26) yield

$$\begin{aligned}|\hat q(z,y)| &= \left|\int_0^{t_1}\big(p_\theta(t_2,x;s,y)-p_\theta(t_1,x;s,y)\big)\sigma_\theta(s,y)\,ds + \int_{t_1}^{t_2} p_\theta(t_2,x;s,y)\sigma_\theta(s,y)\,ds\right|\\ &\le C(t_2-t_1)^{1/2}\end{aligned}$$

for $y\in\mathbb{R}$ and thus

$$\|\hat q(z,\cdot)\|_{L_2([j,j+1])} \le C(t_2-t_1)^{1/2}, \qquad |\hat q(z,j)| \le C(t_2-t_1)^{1/2}.$$

The rest of the proof follows the lines of that of Lemma 4.1. □

5. Concluding remarks

We proved the existence and uniqueness as well as the Hölder continuity of trajectories for a solution of the stochastic parabolic equation driven by a σ-finite stochastic measure. An equation with such an integrator is considered for the first time. A result of the paper [9] is generalized and some of the conditions used in [9] are weakened.

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