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ON EXTREME VALUES OF SOME REGENERATIVE PROCESSES

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Abstract. A general limit theorem is proved for extreme values of regenerative processes. Some applications of this result are given for birth and death processes that determine the length of the queue in a queueing system.

1. Introduction

Consider an m-channel queueing system whose input is a Poisson flow of customers with intensity λ and whose service time ξ has an exponential distribution

$$\mathsf{P}(\xi < x) = 1 - \exp(-\mu x).$$

Using the commonly accepted notation, we consider a queueing system of type $(M/M/m)$ (see [1–3]).

Assume that the system is empty at the moment $S_0 = 0$ and let S_1 denote the moment when the system becomes free of service after the first busy period. Similarly, S_k denotes the moment when the system becomes free of service after the kth busy period.

Denote by $Q(t)$ the length in the queueing system at the moment t and let

$$\bar{Q}_n = \bar{Q}(S_n) = \max_{1\le k\le n} Y_k,$$

where

$$\bar{Q}(t) = \sup_{0\le s<t} Q(s), \qquad Y_k = \sup_{S_{k-1}\le s<S_k} Q(s).$$

Several authors [4–8] deal with the problem of finding constants $a_n > 0$ and $b_n > 0$ such that

$$\lim_{n\to\infty} \mathsf{P}\big(b_n(\bar{Q}_n - a_n) < x\big) = G(x), \tag{1}$$

where $G(x)$ is a nondegenerate distribution function.

For example, the classical theory of extreme values for independent identically distributed random variables implies the following asymptotic relation for a queueing system $M/M/m$:

$$\lim_{n\to\infty} \mathsf{P}(b_n\bar{Q}_n < x) = \exp\left(-x^{-1}\right), \qquad x \ge 0, \tag{2}$$

(see [9, 10]) with $\lambda = m\mu$ and $b_n = m!\,(nm^m)^{-1}$ (also see [4, 7]).

It turns out that equalities (1) and (2) are not valid in many important cases (for example, if $\lambda < m\mu$). In other words, a nondegenerate limit distribution does not exist

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for linear normalizations of $\bar{Q}_n$. A similar situation is observed for birth and death processes [7].

Some authors (see, for example, [5]) make attempts to find lower and upper bounds for the distribution of random variables $b_n(\bar{Q}_n - a_n)$. Other approximations based on relation (2) are proposed in [7].

A survey of research in this topic can be found in [8].

This paper is organized as follows. A general result is proved in Section 2 for extreme values of regenerative processes. In doing so we change the setting of the problem a bit. Namely, we consider extreme values on nonrandom intervals in contrast to the case of $\bar{Q}_n$ where the supremum of the process $Q(t)$ is evaluated over the random interval $(0, S_n)$. But the main difference is that we use nonlinear normalizations.

In Sections 3 and 4 we provide some applications of the general result to the birth and death processes and to the process $Q(t)$ describing the length of the queue in a queueing system.

2. A limit theorem for regenerative processes

We recall the definition of a regenerative process (see, for example, the corresponding definition in [11, Part II, Chapter 2]).

Definition 2.1. A cycle of duration T is understood as an ordered pair $\mathcal{L} = (T, \xi(t))$ where T is a nonnegative random variable and $\xi(t)$ is a stochastic process defined on the interval $[0, T)$,

$$\mathsf{P}(T = 0) < 1, \qquad \mathsf{P}(T < \infty) = 1.$$

The random variable T and stochastic process $\xi(t)$ are dependent in the general case.

Let $\mathcal{L}_i = (T_i, \xi_i(t))$, $i \geq 1$, be an infinite sequence of independent cycles identically distributed with $\mathcal{L}$. We introduce the stochastic process $X(t)$, $t \geq 0$, by

$$X(t) = \xi_i(t - S_{i-1}) \quad \text{for } t \in [S_{i-1}, S_i),$$

where $S_i = T_1 + \cdots + T_i$, $i \geq 1$, $S_0 = 0$.

Then the process $X(t)$ is called *regenerative*, the points S_i are moments of regeneration, and the interval $[S_{i-1}, S_i)$ is the ith period of regeneration.

Put

$$Z(t) = \sup_{0 \leq s < t} X(s), \qquad Z_k = \sup_{S_{k-1} \leq s < S_k} X(s). \tag{3}$$

We suppose that the stochastic processes $\xi_i(t)$ are separable to avoid the problem of measurability of $Z(t)$ and Z_k.

It is clear that Z_k are independent identically distributed random variables. We also assume that

$$q(u) = \mathsf{P}(Z_k \geq u) > 0 \quad \text{for all } u \in \mathbb{R}, \text{ and}$$
$$q(u) \downarrow 0 \quad \text{as } u \uparrow \infty.$$

The latter condition means that Z_k is a finite random variable almost surely.

Theorem 1. *Let $a_T = \mathsf{E}\, T_k < \infty$, $x > 0$, and $t^* = x/q(u)$. Then*

$$\lim_{u \to \infty} \mathsf{P}(Z(t^*) \geq u) = 1 - \exp\left(-\frac{x}{a_T}\right). \tag{4}$$

Proof of Theorem 1. We start with an auxiliary result.

Let ζ and ε be random variables such that

$$\mathsf{P}(\zeta \geq 0) = 1, \qquad \mathsf{P}(\zeta = 0) < 1.$$

$$\mathsf{P}(\varepsilon = 1) = q, \qquad \mathsf{P}(\varepsilon = 0) = 1 - q, \qquad 0 < q < 1.$$

In general, the random variables ζ and ε are dependent.

Consider a sequence (ζ_n, ε_n) of independent copies of the pair (ζ, ε). Define the random variable ν by

$$\nu = \min(n \geq 1 \colon \varepsilon_n = 1).$$

The random variable ν is geometrically distributed [12, p. 61]:

$$\mathsf{P}(\nu = n) = q(1-q)^{n-1}, \qquad n \geq 1, \tag{5}$$

and

$$\mathsf{E}\,\nu = \frac{1}{q}, \qquad \operatorname{Var}\nu = \frac{1-q}{q^2}.$$

Put

$$S_\nu = \sum_{i=1}^{\nu} \zeta_i. \tag{6}$$

Lemma 1. *If* $\mathsf{E}\,\zeta = a < \infty$ *and* $x > 0$ *are fixed,* S_ν *is defined in* (6), *and*

$$q = \mathsf{P}(\varepsilon = 1) \to 0,$$

then

$$\lim_{q\to 0} \mathsf{P}(qS_\nu < x) = 1 - \exp\left(-\frac{x}{a}\right). \tag{7}$$

Proof of Lemma 1. It is well known (and can easily be checked on your own) that the geometrically distributed random variable ν with parameter q is such that

$$\lim_{q\to 0} \mathsf{P}(q\nu < x) = 1 - \exp(-x).$$

It remains to prove that

$$\lim_{q\to 0} \mathsf{P}\left(\left|\frac{1}{a\nu} S_\nu - 1\right| > \delta\right) = 0 \tag{8}$$

for all $\delta > 0$. According to the Kolmogorov strong law of large numbers,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\zeta_i - a) = 0 \quad \text{a.s.},$$

whence

$$\lim_{n\to\infty} \sup_{m\geq n} \frac{1}{m} \sum_{i=1}^{m} (\zeta_i - a) = 0 \quad \text{a.s.}$$

Thus, given an arbitrary $\delta > 0$, there exists a positive integer number $n_0 = n_0(\delta)$ such that

$$\mathsf{P}\left(\sup_{m\geq n_0} \frac{1}{m} \left|\sum_{i=1}^{m} (\zeta_i - a)\right| > \delta\right) \leq \delta. \tag{9}$$

Further

$$\begin{aligned} \mathsf{P}\left(\frac{1}{\nu}\left|\sum_{i=1}^{\nu}(\zeta_i - a)\right| > \delta\right) &\leq \mathsf{P}\left(\frac{1}{\nu}\left|\sum_{i=1}^{\nu}(\zeta_i - a)\right| > \delta, \nu \geq n_0\right) + \mathsf{P}(\nu < n_0) \\ &\leq \mathsf{P}\left(\sup_{m\geq n_0} \frac{1}{m}\left|\sum_{i=1}^{m}(\zeta_i - a)\right| > \delta\right) + \mathsf{P}(\nu < n_0). \end{aligned} \tag{10}$$

The last term in (10) is easy to estimate. In view of equality (5),

$$\mathsf{P}(\nu \geq n_0) = (1-q)^{n_0-1} \geq 1-\delta \quad \text{or} \quad \mathsf{P}(\nu < n_0) \leq \delta \tag{11}$$

for

$$q \leq 1-(1-\delta)^{1/(n_0-1)}.$$

Combining together bounds (9)–(11) we prove equality (8). □

Other approaches to asymptotic equalities of type (7) and their applications in the reliability theory can be found in [13–15].

Now we pass to the proof of equality (4). Let

$$\varepsilon_k(u) = I(Z_k \geq u),$$
$$\nu(u) = \min(k \geq 1\colon \varepsilon_k(u) = 1),$$

where $I(A)$ denotes the indicator of a random event A. Let

$$T_k^* = \begin{cases} \inf(t \geq 0\colon \xi_k(t) \geq u) & \text{for } \varepsilon_k(u) = 1, \\ T_k & \text{otherwise.} \end{cases}$$

It is clear from the definition that the random events

$$(Z(t) \geq u) \quad \text{and} \quad \left(\sum_{k=1}^{\nu(u)} T_k^* \leq t \right)$$

are equivalent. Therefore

$$\mathsf{P}\big(Z(t^*) \geq u\big) = \mathsf{P}\left(\sum_{k=1}^{\nu(u)} T_k^* \leq t^* \right) = \mathsf{P}\left(q(u) \sum_{k=1}^{\nu(u)} T_k^* \leq x \right). \tag{12}$$

It is also obvious that $T_k^* \leq T_k$ and

$$\sum_{k=1}^{\nu(u)} T_k - T_{\nu(u)} \leq \sum_{k=1}^{\nu(u)} T_k^* \leq \sum_{k=1}^{\nu(u)} T_k. \tag{13}$$

Under assumptions of Theorem 1,

$$\lim_{u\to\infty} \mathsf{P}\left(q(u) \sum_{k=1}^{\nu(u)} T_k \leq x \right) = 1 - \exp\left(-\frac{x}{a_T} \right)$$

according to Lemma 1. This together with relations (12) and (13) imply that equality (4) follows if

$$\lim_{u\to\infty} \mathsf{P}\big(q(u) T_{\nu(u)} > \delta\big) = 0 \tag{14}$$

for all $\delta > 0$. We have

$$\mathsf{P}\big(q(u) T_{\nu(u)} > \delta\big) = \sum_{k=1}^{\infty} \mathsf{P}(\nu(u) = k)\, \mathsf{P}\left(T_k > \frac{\delta}{q(u)} \Big/ \nu(u) = k \right). \tag{15}$$

Further

$$
\begin{aligned}
&\mathsf{P}\left(T_k > \frac{\delta}{q(u)} \Big/ \nu(u) = k\right)\\
&\quad = \mathsf{P}\left(T_k > \frac{\delta}{q(u)} \Big/ \varepsilon_1(u) = 0, \dots, \varepsilon_{k-1}(u) = 0, \varepsilon_k(u) = 1\right)\\
&\quad = \mathsf{P}\left(T_k > \frac{\delta}{q(u)} \Big/ \varepsilon_k(u) = 1\right) = \frac{1}{q(u)}\,\mathsf{P}\left(T_k > \frac{\delta}{q(u)}, \varepsilon_k(u) = 1\right)\\
&\quad \le \frac{1}{q(u)}\,\mathsf{P}\left(T_1 > \frac{\delta}{q(u)}\right) \stackrel{\text{def}}{=} b(u).
\end{aligned}
\tag{16}
$$

Since $\mathsf{E}\,T_1 = a_T < \infty$,

$$x\,\mathsf{P}(T_1 > x) \to 0 \quad \text{as } x \to \infty,$$

whence $b(u) \to 0$ as $u \to \infty$. The latter asymptotic relation together with bounds (15) and (16) yield equality (14). □

Considering the result of Theorem 1, the probability of attaining a high level u by a process $X(s)$ in the interval $[0,t)$ can numerically be approximated in many cases as follows:

$$\mathsf{P}(Z(t) \ge u) \approx 1 - \exp\left(-\frac{tq(u)}{a_T}\right).$$

This is the case, for example, if u is sufficiently large and $tq(u)$ is not very large.

Consider a class of regenerative processes $X(t)$ that often appear in reliability theory and queueing theory. This class is defined as follows: each period of regeneration consists of two parts whose lengths as well as the trajectories of the process on these parts are independent. The length of the first part denoted by τ_k is exponentially distributed, $\mathsf{P}(\tau_k < x) = 1 - \exp(-\lambda x)$. The distribution of the length of the second part denoted by η_k is arbitrary with $\mathsf{E}\,\eta_k < \infty$.

We assume that $X(t) \in (0,1,2,\dots)$ almost surely and $X(t) = 0$ during the first part (the process stays in the state 0) and $X(t) \in (1,2,\dots)$ during the second part.

Using the language of [14], $X(t)$ is a regenerative process of a special type. The following result holds for such a type of process.

Corollary 1. *If $X(t)$ is a regenerative process of a special type and all assumptions of Theorem 1 hold for this process, then*

$$\lim_{u\to\infty} \mathsf{P}\big(Z(t^*) \ge u\big) = 1 - \exp(-\lambda p_0 x), \tag{17}$$

where

$$p_0 = \lim_{t\to\infty} p_0(t), \qquad p_0(t) = \mathsf{P}(X(t) = 0). \tag{18}$$

Proof. Corollary 1 follows from Theorem 1. Indeed, the process $X(t)$ can be viewed as an alternating process with two states, 0 and $(1,2,\dots)$. Then the limit in (18) exists and $p_0 = K_\Gamma$ is the stationary availability coefficient [15, p. 110]. Moreover

$$K_\Gamma = \frac{\mathsf{E}\,\tau_k}{\mathsf{E}\,\tau_k + \mathsf{E}\,\eta_k} = \frac{1/\lambda}{\mathsf{E}\,T_k},$$

that is, $\mathsf{E}\,T_k = \frac{1}{\lambda p_0}$. It remains to apply Theorem 1. □

Remark 1. In the general case, where $X(t) = m > 0$ in the part τ_k (the process stays in state m) and $X(t) \ne m$ in the part η_k, the above argument allows one to rewrite equality (17) as follows:

$$\lim_{u\to\infty} \mathsf{P}\big(Z(t^*) \ge u\big) = 1 - \exp(-\lambda p_m x).$$

We denote by $\gamma_k(t)$ the total time spent in a state k by the process $X(t)$ during the interval $(0,t)$.

The following result improves Corollary 1 to some extent.

Lemma 2. *Let $X(t)$ be a stochastic process that assumes either a finite or countable number of values, $X(t) \in \{0,1,2,\dots\}$. Assume that there exist the moments of regeneration for the process $X(t)$: $S_0 = 0, S_1, S_2, \dots,$ where $T_i = S_i - S_{i-1}$, $i = 1,2,\dots,$ are independent identically distributed random variables.*

(i) *If*

$$\mathsf{E}\,T_1 = a_T < \infty,$$

then the limit

$$\lim_{t\to\infty} \frac{\gamma_k(t)}{t} = \frac{\mathsf{E}\,\gamma_k(T_1)}{a_T} = \frac{1}{a_T}\int_0^\infty \mathsf{P}\big(X(t)=k, T_1 > t\big)\,dt \tag{19}$$

exists almost surely for all k.

(ii) *If the limit*

$$\lim_{t\to\infty} \mathsf{P}(X(t)=k) = \lim_{t\to\infty} p_k(t) = p_k > 0 \tag{20}$$

exists for some k and

$$\mathsf{E}\,\gamma_k(T_1) = c_k < \infty, \tag{21}$$

then equality (19) *holds and*

$$\mathsf{E}\,T_1 = a_T = \frac{c_k}{p_k}. \tag{22}$$

Proof of Lemma 2. The first part is a well-known result in reliability theory (see [11, pp. 184–185], [16, pp. 429–430], or [17]). Thus we turn to the proof of part (ii). Assume that $\mathsf{E}\,T_1 = \infty$.

Then

$$\frac{\gamma_k(S_n)}{S_n} = \frac{\frac{1}{n}\sum_{i=1}^n \big(\gamma_k(S_i) - \gamma_k(S_{i-1})\big)}{\frac{1}{n}\sum_{i=1}^n T_i} \to 0 \quad \text{a.s.} \tag{23}$$

as $n\to\infty$ by the strong law of large numbers, since the nominator tends to $c_k < \infty$, while the denominator tends to $+\infty$.

Thus

$$\frac{\gamma_k(t)}{t} \le \frac{\gamma_k(S_{n-1})}{S_{n-1}} + \frac{\gamma_k(S_n) - \gamma_k(S_{n-1})}{S_{n-1}} \quad \text{a.s.} \tag{24}$$

for $S_{n-1} < t \le S_n$. The second term on the right hand side of (24) tends to 0. Indeed,

$$\frac{\gamma_k(S_n)}{n} \to c_k, \qquad \frac{\gamma_k(S_{n-1})}{n} \to c_k, \qquad \frac{S_n}{n} \to \infty \quad \text{a.s.} \tag{25}$$

Relations (23)–(25) imply

$$\frac{\gamma_k(t)}{t} \to 0, \qquad t\to\infty \quad \text{a.s.}$$

Since $0 \le \gamma_k(t)/t \le 1$,

$$\lim_{t\to\infty} \frac{\mathsf{E}\,\gamma_k(t)}{t} = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathsf{E}\,I(X(s)=k)\,ds = \lim_{t\to\infty} \frac{1}{t}\int_0^t p_k(s)\,ds = 0,$$

which contradicts condition (20).

Therefore $\mathsf{E}\, T_1 = a_T < \infty$. According to equality (19) we have

$$\frac{c_k}{a_T} = \lim_{t\to\infty} \frac{\gamma_k(t)}{t} = \lim_{t\to\infty} \frac{\mathsf{E}\,\gamma_k(t)}{t} = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathsf{E}\, I(X(s)=k)\, ds$$
$$= \lim_{t\to\infty} \frac{1}{t}\int_0^t p_k(s)\, ds = p_k$$

which proves equality (22). □

Corollary 2. *Let $X(t)$ be a regenerative process, $X(t) \in (0,1,2,\dots)$ almost surely, and conditions (20) and (21) hold for some k. Let $x > 0$ and $t^* = x/q(u)$. Then*

$$\lim_{u\to\infty} \mathsf{P}\big(Z(t^*) \ge u\big) = 1 - \exp\left(-\frac{p_k}{c_k}x\right). \tag{26}$$

Remark 2. Let all assumptions of Lemma 2 hold and let the parameter t belong to a countable set $\Im = \{t_0 = 0 < t_1 < t_2 < \dots\}$ such that $t_i \to \infty$ as $n \to \infty$. The numbers t_i can be random and $X(t)$ may depend on the sequence (t_i). Assume further that moments of regeneration S_i belong to $\Im$, $\widehat{\gamma}_k(t)$ denotes the number of visits to the state k by the sequence $X(t_i)$ for $t_i \in [0,t)$, and

$$N(t) = \max(i \ge 0\colon t_i < t).$$

The proof of Lemma 2 can be used to prove the following equality:

$$\hat{a}_T = \mathsf{E}\, N(T_1) = \frac{\mathsf{E}\,\widehat{\gamma}_k(T_1)}{p_k} = \frac{c_k}{p_k}$$

if relations (20) and (21) hold. Here $\hat{a}_T$ denotes the mean number of points t_i belonging to a single regeneration interval. In this proof, the only change needed is to use the sums instead of the corresponding integrals.

Remark 3. An anonymous reviewer pointed out that Theorem 1 follows easily from Lemma 1.1 of the paper [8]. However, the method of the proof of Theorem 1 is of its own interest. It can be used for the estimation of the rate of convergence in limit theorems of the type studied above. For example, the proof of Theorem 1 and some known results for "geometric" sums yield the following proposition.

Proposition 1. *Let all conditions of Theorem 1 hold. Assume that random variables Z_k and T_k are independent for all k. Also let $\mathsf{E}\, T_1^s < \infty$ for some s, $1 < s < 2$. Then*

$$\sup_{x>0} \left| \mathsf{P}\big(Z(t^*) \ge u\big) - 1 + \exp\left(-\frac{x}{a_T}\right)\right| \le Cq(u)^{s-1}\frac{\mathsf{E}\, T_1^s}{a_T^s}, \tag{27}$$

where C is an absolute constant.

We omit the proof of Proposition 1, since random variables Z_k and T_k are dependent in all applications known to the authors.

3. Birth and death processes

Assume that $X(t)$ is a Markov process whose state space is $0,1,2,\dots$ and whose transition probabilities $p_{i,j}(t)$ are stationary and such that

$$\begin{aligned} &1.\quad p_{i,i+1}(h) = \lambda_i h + o(h), \quad i \ge 0,\\ &2.\quad p_{i,i-1}(h) = \mu_i h + o(h), \quad i \ge 1,\\ &3.\quad p_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h), \quad i \ge 0,\\ &4.\quad \mu_0 = 0,\ \lambda_0 > 0,\ \mu_i > 0,\ \lambda_i > 0, \quad i = 1,2,\dots, \end{aligned} \tag{28}$$

as $h \to 0$. Then $X(t)$ is called a birth and death process. This model is widely used in biology, queueing theory, reliability theory, etc. (See [1, § 1.4], [2, § 7.4], [15, § 6.3].)

Put

$$\theta_0 = 1, \qquad \theta_k = \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}, \qquad k \geq 1.$$

Throughout this section we assume that the birth and death process satisfies condition (28) and

$$\sum_{k\geq 1} \theta_k < \infty, \tag{29}$$

$$\sum_{k\geq 1} \frac{1}{\lambda_k \theta_k} = \infty. \tag{30}$$

It is known that [2, 18] the stationary probabilities of the states exist, namely

$$\lim_{t\to\infty} \mathsf{P}\big(X(t) = k\big) = \lim_{t\to\infty} p_k(t) = p_k \tag{31}$$

and moreover

$$p_k = \theta_k p_0, \quad p_0 = \left(\sum_{k=0}^{\infty} \theta_k\right)^{-1}. \tag{32}$$

In addition, the embedded Markov chain is recurrent [2, pp. 85–86].

In what follows we use the following notation:

$$\alpha_0(m) = 1, \qquad \alpha_k(m) = \prod_{i=1+m}^{k+m} \frac{\mu_i}{\lambda_i}, \qquad k \geq 1.$$

The expression $(u \in \mathbb{N}, u \to \infty)$ means that u is integer and tends to ∞. The process $Z(t)$, as above, is defined by equality (3).

Theorem 2. *Let $X(t)$ be a birth and death process that satisfies conditions* (28)–(30)*. If*

$$X(0) = m \quad a.s., \qquad x > 0, \qquad t^* = x \sum_{k=0}^{u-m-1} \alpha_k(m) \quad \text{for } u \geq m+1,$$

then

$$\lim_{u\in\mathbb{N},\, u\to\infty} \mathsf{P}\big(Z(t^*) \geq u\big) = 1 - \exp(-\lambda_m p_m x), \tag{33}$$

where p_m is defined by equalities (32)*.*

Remark 4. Equality (33) can be rewritten for an important case of $m = 0$ (meaning that the process starts from the state 0) as follows:

$$\lim_{u\in\mathbb{N},\, u\to\infty} \mathsf{P}\big(Z(t^*) \geq u\big) = 1 - \exp(-\lambda_0 p_0 x),$$

where

$$t^* = x \sum_{k=0}^{u-1} \alpha_k, \qquad \alpha_0 = \alpha_0(0) = 1, \qquad \alpha_k = \alpha_k(0) = \prod_{i=1}^{k} \frac{\mu_i}{\lambda_i}.$$

Proof of Theorem 2. The assumptions of the theorem imply that $X(t)$ is a regenerative process of a special kind whose regeneration moments are $S_0 = 0, S_1, S_2, \dots,$ where S_k is the moment of the first visit to a state m after kth exits from it. In this case

$$\mathsf{P}(\tau_k < x) = 1 - \exp\bigl(-(\lambda_m + \mu_m)x\bigr), \qquad x \ge 0,$$
$$\mathsf{E}\, T_k = \frac{1}{(\lambda_m + \mu_m)p_m}.$$

The latter relation follows from equality (22) in Lemma 2.

Now we are going to show that

$$q(u) = \mathsf{P}(Z_1 \ge u) = \frac{\lambda_m}{\lambda_m + \mu_m}\left(\sum_{k=0}^{u-m-1} \alpha_k(m)\right)^{-1} \tag{34}$$

for $u \ge m+1$. Choosing

$$t^* = x\left(\frac{\lambda_m + \mu_m}{\lambda_m}\right)\sum_{k=0}^{u-m-1} \alpha_k(m), \tag{35}$$

we derive from Corollary 1 that

$$\lim_{u\in\mathbb{N},\, u\to\infty} \mathsf{P}\bigl(Z(t^*) \ge u\bigr) = 1 - \exp\bigl(-(\lambda_m + \mu_m)p_m x\bigr) \tag{36}$$

(see Remark 1). Changing the variables in (35) and (36),

$$y = x\left(\frac{\lambda_m + \mu_m}{\lambda_m}\right),$$

we prove equality (33).

It remains to prove equality (34). For this, we need the result of Lemma 3.

Lemma 3. *Consider a Markov chain with the states* $0, 1, 2, \dots, d$ *and transition probabilities*

$$p_{i,j} = \frac{\lambda_i}{\lambda_i + \mu_i} \quad \textit{for } j = i+1,$$
$$p_{i,j} = \frac{\mu_i}{\lambda_i + \mu_i} \quad \textit{for } j = i-1$$

for all $i = 1, 2, \dots, d-1$, *where* $\lambda_i > 0$, $\mu_i > 0$, $i = 1, 2, \dots, d-1$. *Assume that* $p_{0,0} = p_{d,d} = 1$, *that is, the states* 0 *and* d *are absorbing. If the Markov chain starts from the state* 1, *then*

$$\mathsf{P}(\textit{absorption in the state } d) = 1 - \mathsf{P}(\textit{absorption in the state } 0) = \frac{1}{\sum_{k=0}^{d-1} \alpha_k}.$$

Lemma 3 is a particular case of a known result for Markov chains [2, pp. 89, 201–202].

Equality (34) easily follows from Lemma 3. Indeed, let $m = 0$ and consider the first regeneration cycle $[0, S_1)$ of the birth and death process $X(t)$, $Z_1 = \sup_{0\le s<S_1} X(s)$. The process $X(t)$ on the interval $[0, S_1)$ passes from the state 0 to the state 1 at the moment τ_1. Then the random event $(Z_1 \ge u)$ is equivalent to the event that the embedded Markov chain for the process $X(t)$ attains the level u during the first regeneration cycle. The latter event is equivalent to the event that the Markov chain in Lemma 3 is absorbed by the state $d = u$. This together with Lemma 3 implies that

$$q(u) = \mathsf{P}(Z_1 \ge u) = \frac{1}{\sum_{k=0}^{u-1} \alpha_k},$$

that is, equality (34) is proved for $m = 0$.

Let $m \geq 1$. It is well known that the embedded Markov chain for the process $X(t)$ has the same transition probabilities as the Markov chain in Lemma 3 (without absorption). Hence

$$\mathsf{P}(Z_1 = m) = \frac{\mu_m}{\lambda_m + \mu_m} \tag{37}$$

and

$$\mathsf{P}(Z_1 \geq u) = \frac{\lambda_m}{(\lambda_m + \mu_m)\sum_{k=0}^{u-m-1} \alpha_k(m)} \tag{38}$$

for $u \geq m+1$. Indeed, equality (37) corresponds to the case where the process $X(t)$ passes from the state m to the state $m-1$ at the moment τ_1 with probability $\mu_m/(\lambda_m + \mu_m)$. Equality (38) means that the process passes from the state m to the state $m+1$ at the moment τ_1 with probability $\lambda_m/(\lambda_m + \mu_m)$ and thus we deal, in fact, with the case $m = 0$ considered above and treated with the help of Lemma 3. Equality (34) as well as Theorem 2 are proved. □

4. Length of the queue in a queueing system

Below we present some applications of the result proved above to extreme values of the length of the queue in some queueing systems.

Example 1 (Queueing system (M/M/m))**.** The definition of this system is given in the beginning of this paper (also see [1,2]). The length of a queue is understood as the total number of customers that either are serviced in the system or are waiting for the service. Denote by $Q(t)$ the length of a queue at the moment t and put $\bar{Q}(t) = \sup_{0\leq s<t} Q(s)$.

It is known [2, pp. 195–197] that $Q(t)$ is a birth and death process with parameters

$$\lambda_k = \lambda, \qquad k = 0, 1, 2, \dots,$$

$$\mu_0 = 0, \qquad \mu_k = \begin{cases} k\mu & \text{for } 1 \leq k \leq m, \\ m\mu & \text{for } \quad k > m. \end{cases}$$

Then

$$\alpha_k = \prod_{i=1}^{k} \frac{\mu_i}{\lambda_i} = \begin{cases} k!\, \rho^{-k} & \text{for } 1 \leq k \leq m, \\ m!\, m^{k-m} \rho^{-k} & \text{for } k > m, \end{cases}$$

where $\rho = \lambda/\mu$.

For sufficiently large u,

$$q(u) = \left(\sum_{k=0}^{u-1} \alpha_k \right)^{-1} = \left(\sum_{k=0}^{m} \frac{k!}{\rho^k} + \frac{m!\big((m/\rho)^u - (m/\rho)^{m+1}\big)}{m^m (m/\rho - 1)} \right)^{-1}. \tag{39}$$

We impose the following assumption on the parameters λ and μ:

$$\rho = \frac{\lambda}{\mu} < m. \tag{40}$$

Then it is known that conditions (29) and (30) hold and the stationary probabilities exist, namely

$$p_k = \frac{\rho^k}{k!} p_0 \quad \text{for } 1 \le k \le m,$$

$$p_k = \frac{\rho^k}{m!\, m^{k-m}} p_0 \quad \text{for } k > m,$$

$$p_0 = \left(\sum_{k=0}^{m} \frac{\rho^k}{k!} + \frac{\rho^{m+1}}{m!\,(m-\rho)} \right)^{-1} \tag{41}$$

(see [1,2]). This together with Theorem 2 and Remark 4 yield the following result.

Proposition 2. *Let condition* (40) *hold for a queueing system* $(M/M/m)$. *Assume that*

$$Q(0) = 0 \quad a.s., \qquad x > 0, \qquad t^* = x \left(\sum_{k=0}^{m} \frac{k!}{\rho^k} + \frac{m!\,\big((m/\rho)^u - (m/\rho)^{m+1}\big)}{m^m (m/\rho - 1)} \right).$$

Then

$$\lim_{u \in \mathbb{N}, u \to \infty} \mathsf{P}(\bar{Q}(t^*) \ge u) = 1 - \exp(-\lambda p_0 x), \tag{42}$$

where p_0 *is defined by* (41).

Remark 5. (i) It is clear from the proof of Theorem 1 that $q(u)$ in the definition of t^* can be changed by $q_1(u)$ if $q(u) \sim q_1(u)$. Thus $q(u)$ in Proposition 2 can be changed by a simpler variable

$$q_1(u) = \frac{m!\,(m/\rho)^u}{m^m (m/\rho - 1)}.$$

However one can expect that the rate of convergence becomes slower after this change.

(ii) If $m = \infty$ in Example 1 (that is, a queueing system contains infinitely many service channels), then

$$\lambda_k = \lambda, \quad k = 0, 1, 2, \ldots; \qquad \mu_k = k\mu, \quad \alpha_k = \frac{k!}{\rho^k}, \quad k = 1, 2, \ldots, \quad \mu_0 = 0.$$

Thus, for all $\lambda > 0$ and $\mu > 0$, equality (42) holds with

$$t^* = x \sum_{k=0}^{u-1} \frac{k!}{\rho^k}, \qquad p_0 = \exp(-\rho).$$

Example 2 (Queueing system (M/G/1))**.** This is a single channel queueing system with a Poissonian flow of customers arriving with intensity λ and with an arbitrary distribution of the service time ξ, that is, $\mathsf{P}(\xi < x) = G(x)$ and G is arbitrary. Assume that $\mathsf{E}\,\xi = b < \infty$ and

$$\rho = \lambda b < 1. \tag{43}$$

Condition (43) implies that the stationary probabilities of states exist,

$$\lim_{t \to \infty} \mathsf{P}(Q(t) = k) = p_k, \tag{44}$$

and $\varphi(s) = \sum_{k=0}^{\infty} p_k s^k$ is given by the Pollachek–Khinchine identity. This means that

$$p_0 = 1 - \rho \tag{45}$$

(see [2, pp. 441–442]).

If the system is empty at the moment $S_0 = 0$ and if $S_1, S_2, \ldots$ denote the sequential moments when the system becomes free of service, then $Q(t)$ is a regenerative process of a special kind with regeneration moments $S_0 = 0, S_1, S_2, \ldots$. The first part τ_k of the

kth regeneration period (S_{k-1}, S_k) (where the system is empty) is distributed according to $\mathsf{P}(\tau_k < x) = 1 - \exp(-\lambda x)$. The second part η_k (busy period) has a finite mean, $\mathsf{E}\,\eta_k = b/(1-\lambda b) < \infty$ [2, p. 466].

Therefore one can apply Theorem 1 and Corollary 1 if $q(u)$ is known.

Let

$$d_k = \int_0^\infty \frac{(\lambda x)^k}{k!} \exp(-\lambda x)\, dG(x)$$

be the probability of the event that k new customers arrive to the queueing system during the service time of a customer, and let

$$D_k = \sum_{i=k+1}^{\infty} d_i$$

be the probability that more than k customers arrive to the queueing system during the service time of a single customer.

We fix a large positive number u and denote by q_k, $1 \le k < u$, the probability of the event that the process $Q(t)$ visits the state u during the busy period given there are k customers in the queueing system at a certain moment in the second part η and the system starts the service of one of the customers (this moment coincides with the beginning of the busy period if $k = 1$). In fact, q_k is equal to the probability that the embedded Markov chain starts from the state k and reaches u earlier than 0 (here the values of the process $Q(t)$ at the ends of service periods form the embedded Markov chain).

Using the full probability formula for the vector $(q_k), k = 1, 2, \dots, u-1$, one obtains the following system of linear equations:

$$\begin{aligned} q_{u-1} &= D_0 + d_0 q_{u-2}, \\ q_{u-2} &= D_1 + d_0 q_{u-3} + d_1 q_{u-2}, \\ &\dots\dots\dots\dots\dots\dots\dots \\ q_1 &= D_{u-2} + \sum_{i=1}^{u-2} d_i q_i. \end{aligned} \tag{46}$$

It is easy to understand that the process reaches the state 1 at a moment τ_1 during the first regeneration cycle. Hence the events

$$\{\text{the embedded Markov chain starts from 1 and reaches } u \text{ earlier than } 0\}$$

and

$$\{Z_1 \ge u\} = \left\{ \sup_{0 \le s < S_1} Q(s) \ge u \right\}$$

are equivalent, that is, $q(u) = q_1$.

This together with Corollary 1 and equalities (45) and (46) prove the following result.

Proposition 3. *Let condition* (43) *hold for a queueing system* $(M/G/1)$. *If*

$$Q(0) = 0 \quad a.s., \qquad x > 0, \qquad t^* = \frac{x}{q(u)},$$

then

$$\lim_{u \in \mathbb{N},\, u \to \infty} \mathsf{P}\big(\bar{Q}(t^*) \ge u\big) = 1 - \exp\big(-\lambda(1-\rho)x\big), \tag{47}$$

where $q(u) = q_1$ *and* (q_k) *is a solution of the system of equations* (46).

Remark 6. A system of linear equations similar to (46) has been used in some problems of reliability theory (see [14, p. 99]). Moreover a closed form for q_1 is found in [14] (q_1 in [14] is expressed in terms of the moment generating functions of the vector D_k). This result, unfortunately, is not easy to apply in concrete problems.

A numerical solution of the system of equations (46) can be obtained with the help of the sequential substitutions even if u is sufficiently large.

Example 3 (Queueing system (G/M/1))**.** This is a single channel queueing system with a recurrent input flow of customers, and let $t_0 = 0,\ t_1, t_2, \dots$ be the moments when customers arrive to the system. We suppose that $(\tau_k = t_k - t_{k-1})$ is a sequence of independent identically distributed random variables with a distribution function $G(x)$ and that the service time ξ_k is exponentially distributed with parameter μ.

Consider the regeneration moments $S_0 = 0, S_1, S_2, \dots$ of the process $Q(t)$. Here S_k denotes the arrival time of a new customer after kth busy period. For the sake of a simpler notation we assume that a customer arrives to the system at the moment $t_0 = 0$. Let T_k be the duration of the kth regeneration cycle.

The embedded Markov chain (Q_n) is determined by the moments (t_n) and the values of the length of the queue prior to the arrival of a customer constitute the set of its states, that is,

$$Q_n = Q(t_n - 0), \qquad n = 0, 1, 2, \dots .$$

Denote by

$$d_k^* = \int_0^\infty \frac{(\mu x)^k}{k!} \exp(-\mu x)\, dG(x), \qquad k = 0, 1, 2, \dots,$$

the probability of the event that k customers are served during a time interval of length τ, and let

$$f(v) = \sum_{k=0}^\infty d_k^* v^k.$$

Assume further that

$$\mathsf{E}\,\tau_k = a_\tau < \infty \quad 3 \quad \rho = \frac{1}{a_\tau \mu} < 1. \tag{48}$$

It is known [2, pp. 446–447], [19] that the stationary probabilities of the states of the sequence (Q_n) exist in this case,

$$\lim_{n\to\infty} \mathsf{P}(Q_n = k) = p_k = (1 - v_0)v_0^k, \qquad k = 0, 1, 2, \dots, \tag{49}$$

where v_0 is a unique solution of the equation

$$f(v_0) = v_0, \qquad 0 < v_0 < 1. \tag{50}$$

Moreover, if the distribution $G(x)$ is nonlattice, then condition (48) implies that the stationary probabilities exist for the process $Q(t)$, as well:

$$\lim_{t\to\infty} \mathsf{P}(Q(t) = k) = p_k^* = \frac{p_{k-1}}{a_\tau \mu}, \qquad k = 1, 2, \dots, \tag{51}$$

(see [19]).

To apply Theorem 1 or its corollaries in a way similar to the previous examples, one can find the quantity $q(u)$, the probability that the process $Q(t)$ attains the state u over a regeneration cycle.

For the sake of definiteness, let us analyze the first regeneration cycle $[0, S_1)$. Denote by q_k, $k = 1, 2, \dots, u-1$, the probability of the event that the process $Q(t)$ attains the state u during the interval $[0, S_1)$ given its state is k at the moment when a customer arrives, $t_i \in [0, S_1)$.

Consider the values of the process $Q(t)$ at the moments t_i when customers arrive and apply the full probability formula. Like Example 2, we obtain the system of linear equations for the vector (q_k):

$$
\begin{aligned}
q_1 &= d_0^* q_2, \\
q_2 &= d_0^* q_3 + d_1^* q_2, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
q_k &= d_0^* q_{k+1} + d_1^* q_k + \cdots + d_{k-1}^* q_2, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
q_{u-1} &= d_0^* + d_1^* q_{u-1} + \cdots + d_{u-2}^* q_2.
\end{aligned} \tag{52}
$$

This system of linear equations can be solved similarly to the system (46) by sequential substitutions. Then $q(u) = q_1$.

Proposition 4. *Let conditions* (48) *hold for a queueing system* $(G/M/1)$. *If*

$$Q(0) = 1 \quad a.s., \qquad x > 0, \qquad t^* = \frac{x}{q(u)},$$

then

$$\lim_{u\in\mathbb{N},\, u\to\infty} \mathsf{P}\big(\bar{Q}(t^*) \ge u\big) = 1 - \exp\left(-\frac{(1-v_0)x}{a_\tau}\right), \tag{53}$$

where a_τ *and* v_0 *are defined by* (48) *and* (50), *respectively,* $q(u) = q_1$, *and* (q_k) *is a solution of the system of equations* (52).

Proposition 4 follows from Theorem 1 and the following auxiliary result.

Lemma 4. *If conditions* (48) *hold for a queueing system* $(G/M/1)$, *then the mean duration of the regeneration cycle is given by*

$$\mathsf{E}\, T_k = \frac{a_\tau}{1-v_0}. \tag{54}$$

Proof of Lemma 4. Consider the first regeneration cycle $[0, S_1)$ of the process $Q(t)$. Let $T_1 = S_1$ be its duration, $\kappa = \min\,(n \ge 1\colon \sum_{i=1}^n \tau_i \ge \sum_{i=1}^n \xi_i)$. According to the definition,

$$T_1 = \sum_{i=1}^{\kappa} \tau_i.$$

In fact, κ is the first overjump moment over 0 for the sequence $\sum_{i=1}^n (\tau_i - \xi_i)$. Therefore κ is a Markov time with respect to the σ-algebra $\mathfrak{G} = \sigma(\tau_i, \xi_i, i = 1, 2, \dots)$. Then

$$\mathsf{E}\, T_1 = \mathsf{E}\, \tau_1\, \mathsf{E}\, \kappa \tag{55}$$

by the Wald identity [11, p. 229]. The variable κ can be viewed as the number of steps needed for the embedded Markov chain (Q_n) to visit the state 0 after the first visit to 0. Using Lemma 2 (see Remark 2) and equality (49) we obtain

$$\mathsf{E}\, \kappa = \frac{c_0}{p_0} = \frac{c_0}{1-v_0}. \tag{56}$$

Since the chain (Q_n) visits the state 0 only once during the interval $[0, S_1)$, we conclude that

$$c_0 = \mathsf{E}\, \gamma_0(T_1) = 1. \tag{57}$$

Combining together equalities (55)–(57) we prove (54). □

Remark 7. The authors are not aware whether or not uniform bound (27) is valid under the assumptions of Theorem 2 or those of Propositions 2–4.

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Bibliography

[1] B. V. Gnedenko and I. N. Kovalenko, *Introduction to Queueing Theory*, 2nd ed., Mathematical Modeling, vol. 5, Birkhäuser, Boston, 1989. MR1035707

[2] S. Karlin, *A First Course in Stochastic Processes*, Academic Press, New York–London, 1966. MR0208657

[3] V. V. Anisimov, O. K. Zakusylo, and V. S. Donchenko, *Elements of the Queueing Theory and Asymptotic Analysis of Systems*, "Vyshcha shkola", Kiev, 1987. (Russian)

[4] J. W. Cohen, *Extreme values distribution for the M/G/1 and G/M/1 queueing systems*, Ann. Inst. H. Poincare., Sect.B **4** (1968), 83–98. MR0232466

[5] C. W. Anderson, *Extreme value theory for a class of discrete distribution with application to some stochastic processes*, J. Appl. Prob. **7** (1970), 99–113. MR0256441

[6] D. L. Iglehart, *Extreme values in the GI/G/1 gueue*, Ann. Math. Statist. **43** (1972), 627–635. MR0305498

[7] R. F. Serfozo, *Extreme values of birth and death processes and queues*, Stoch. Process. Appl. **27** (1988), 291–306. MR931033

[8] S. Asmussen, *Extreme values theory for queues via cycle maxima*, Extremes **1** (1998), 137–168. MR1814621

[9] J. Galambos, *The Asymptotic Theory of Extreme Order Statistics*, John Wiley & Sons, New York–Chichester–Brisbane, 1978. MR489334

[10] M. R. Leadbetter, G. Lindgren, and H. Rootzén, *Extremes and Related Properties of Random Sequences and Processes*, Springer Series in Statistics, Springer-Verlag, New York–Berlin, 1983. MR691492

[11] D. Cox and W. Smith, *Renewal Theory*, "Sovetskoye Radio", Moscow, 1967. (Russian) MR0224173

[12] I. I. Gikhman, A. V. Skorokhod, and M. I. Yadrenko, *Theory of Probability and Mathematical Statistics*, "Vyshcha shkola", Kiev, 1988. (Russian)

[13] I. K. Matsak, *A lemma for random sums and its applications*, Ukrain. Mat. Zh. (2017). (Ukrainian) (to appear)

[14] E. Yu. Barzilovich, Yu. K. Belyaev, V. A. Kashtanov, I. N. Kovalenko, A. D. Solovyev, and I. A. Ushakov, *Problems of the Mathematical Theory of Reliability* (B. V. Gnedenko, ed.), "Radio i Svyaz", Moscow, 1983. (Russian) MR758792

[15] B. V. Gnedenko, Yu. K. Belyayev, and A. D. Solovyev, *Mathematical Methods of Reliability Theory*, Academic Press, New York–London, 1969. MR0345234

[16] W. Feller, *An Introduction to Probability Theory and its Applications*, vol. 2, 2nd ed., John Wiley & Sons, New York–London–Sydney, 1971.

[17] B. V. Dovgaĭ and I. K. Matsak, *On a redundant system with renewals*, Theory Probab. Math. Statist. **94** (2017), 63–76. MR3553454

[18] S. Karlin and J. McGregor, *The classification of birth and death processes*, Trans. Amer. Math. Soc. **86** (1957), 366–400. MR0094854

[19] L. Takács, *Some probability questions in the theory of telephone traffic*, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **8** (1958), 151–210. (Hungarian) MR0096319

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