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CONSISTENCY OF THE LEAST SQUARES ESTIMATORS OF PARAMETERS IN THE TEXTURE SURFACE SINUSOIDAL MODEL

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Abstract. We consider the texture surface sinusoidal model of observations. In other words, we consider a model where the regression function is the sum of two-parameter harmonic oscillations while the noise is an isotropic and homogeneous Gaussian random field on the plane. Conditions for the joint consistency of the least squares estimator of unknown amplitudes and angular frequencies are obtained for this trigonometric regression model.

1. Introduction

In the paper, we consider a two-dimensional texture surface sinusoidal model of observations. Various discrete modifications of this model have attracted considerable interest in the literature on signal processing, since those models are used when analyzing textures [1–4]. In particular, some applications are known in the theory of processing of the so-called symmetric gray-scale texture images under the assumption that the intensity of the gray color at every pixel of an image is proportional to the value of a process observed at this pixel. Special interest in this problem appears in spectral analysis [5, 6] (also see [4] and references therein).

The consistency of the least squares estimator of unknown parameters of the sinusoidal model is studied in the case where the random noise is an isotropic and homogeneous Gaussian field on the plane [7, 8]. From the point of view of mathematics, such a setting of the problem is a natural generalization of the well-known problem on detecting hidden periodicities (see, for example, [9, 10]).

Asymptotic properties of the least squares estimator are considered in the papers [11, 12] for the discrete setting where the errors of observations are independent identically distributed (Gaussian, for example) random variables. These results are generalized in [13] for the case of errors of observations represented by a discrete linear homogeneous field. Note that multiparameter harmonic oscillations are studied in the paper [14] under the assumption that the errors of observations constitute a homogeneous random field for which spectral densities of all orders exist. Some results on the asymptotic behavior of peridogram estimators as well as those of the least squares estimators of unknown amplitudes and angular frequencies of these harmonic oscillations are also obtained in [14].

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2. Setting of the problem

Consider the observation model

$$X(t_1,t_2) = g\left(t_1,t_2;\theta^0\right) + \varepsilon(t_1,t_2), \qquad t=(t_1,t_2)\in\mathbb{R}^2_+, \tag{1}$$

where

$$g\left(t_1,t_2;\theta^0\right) = \sum_{k=1}^{N}\left(A_k^0\cos\left(\lambda_k^0 t_1+\mu_k^0 t_2\right)+B_k^0\sin\left(\lambda_k^0 t_1+\mu_k^0 t_2\right)\right), \tag{2}$$

$$\begin{aligned}\theta^0 &= \left(\theta_1^0,\theta_2^0,\theta_3^0,\theta_4^0,\dots,\theta_{4N-3}^0,\theta_{4N-2}^0,\theta_{4N-1}^0,\theta_{4N}^0\right)\\ &= \left(A_1^0,B_1^0,\lambda_1^0,\mu_1^0,\dots,A_N^0,B_N^0,\lambda_N^0,\mu_N^0\right);\end{aligned}$$

here the number $N\ge 1$ is known and $(A_k^0)^2+(B_k^0)^2>0$, $k=1,\dots,N$, is a vector of true values of unknown parameters. The random field $\varepsilon=\{\varepsilon(t_1,t_2),(t_1,t_2)\in\mathbb{R}^2\}$ is defined on a complete probability space $(\Omega,\mathcal{F},\mathsf{P})$ such that

N. ε is a mean square and almost surely continuous homogeneous Gaussian random field with zero mean and covariance function

$$B(t_1,t_2)=\mathsf{E}\varepsilon(t_1,t_2)\varepsilon(0,0), \qquad (t_1,t_2)\in\mathbb{R}^2,$$

such that either

(i) the field ε is isotropic and $B(t_1,t_2)=B(\|t\|)=L(\|t\|)\|t\|^{-\alpha}$, $\alpha\in(0,1)$, where L is a nondecreasing slowly varying at infinity function, $t=(t_1,t_2)$, and $\|t\|=\left(t_1^2+t_2^2\right)^{1/2}$; or

(ii) $\int_{\mathbb{R}^2}|B(t_1,t_2)|\,dt_1\,dt_2<\infty$.

The regression functions (2) like the classical trigonometric regression functions with $\mu_k^0=0$, $k=1,\dots,N$, do not distinguish the parameters in an optimal way if $N\ge 2$ in the sense that the functions (2) do not satisfy any condition of a general theorem on the consistency of the least squares estimator of parameters in a model of nonlinear regression (see, for example, [8,15]). Therefore one needs to impose an additional condition allowing the trigonometric regression function to distinguish the parameters and to be able to prove the consistency of the least squares estimator of parameters (2). This can be achieved by choosing a parametric set for determining the least squares estimator such that the parameters are well distinguished.

We write $(a,b)<(c,d)$ for two points (a,b) and (c,d) in the plane if $a<c$ and $b<d$. The model (1)–(2) is considered in this paper under the following assumption.

R1. The numbers λ_j^0 and μ_j^0, $i,j=1,\dots,N$, are positive and all different; moreover, $\left(\lambda_k^0,\mu_k^0\right)<\left(\lambda_{k+1}^0,\mu_{k+1}^0\right)$, $k=1,\dots,N-1$.

This assumption means that the parametric sets containing the values of parameters $\lambda^0=(\lambda_1^0,\dots,\lambda_N^0)$ and $\mu^0=(\mu_1^0,\dots,\mu_N^0)$ are such that

$$\Lambda\left(\underline{\lambda},\overline{\lambda}\right)=\left\{\lambda=(\lambda_1,\dots,\lambda_N)\in\mathbb{R}^N\colon 0\le\underline{\lambda}<\lambda_1<\dots<\lambda_N<\overline{\lambda}<\infty\right\}, \tag{3}$$

$$M\left(\underline{\mu},\overline{\mu}\right)=\left\{\mu=(\mu_1,\dots,\mu_N)\in\mathbb{R}^N\colon 0\le\underline{\mu}<\mu_1<\dots<\mu_N<\overline{\mu}<\infty\right\}. \tag{4}$$

Put

$$Q_T(\theta)=T^{-2}\int_0^T\int_0^T[X(t_1,t_2)-g(t_1,t_2;\theta)]^2\,dt_1\,dt_2. \tag{5}$$

According to the standard definition, any random vector

$$\theta_T=(A_{1T},B_{1T},\lambda_{1T},\mu_{1T},\dots,A_{NT},B_{NT},\lambda_{NT},\mu_{NT}) \tag{6}$$

is called the least squares estimator of the parameter θ^0 constructed from observations after the field $X(t_1,t_2)$, $(t_1,t_2)=[0,T]\times[0,T]$, if (6) minimizes the functional (5) in the parametric set $\Theta \subset \mathbb{R}^{4N}$, where A_k and B_k, $k = 1, \dots, N$, may assume arbitrary values, while λ and μ assume values in the closed sets $\Lambda^c(\underline{\lambda}, \overline{\lambda})$ and $M^c(\underline{\mu}, \overline{\mu})$, respectively.

To prove relations (27) and (28) and to perform further calculations, one needs to guarantee the almost sure convergence to zero as $T \to \infty$ of the variables

$$\frac{\sin T(\lambda_{kT} - \lambda_{jT})}{T(\lambda_{kT} - \lambda_{jT})}, \qquad \frac{\sin T(\mu_{kT} - \mu_{jT})}{T(\mu_{kT} - \mu_{jT})}, \qquad \frac{\sin T(\lambda_{kT} - \lambda_j^0)}{T(\lambda_{kT} - \lambda_j^0)},$$
$$\frac{\sin T(\mu_{kT} - \mu_j^0)}{T(\mu_{kT} - \mu_j^0)}, \quad k \neq j; \qquad \frac{\sin T\lambda_{kT}}{T\lambda_{kT}}, \quad \frac{\sin T\mu_{kT}}{T\mu_{kT}}, \qquad k = 1, \dots, N. \tag{7}$$

On the other hand, one cannot derive the behavior of the denominators of ratios (7) as $T \to \infty$ from the above definition of the estimators

$$\lambda_T = (\lambda_{1T}, \dots, \lambda_{NT}) \qquad \text{and} \qquad \mu_T = (\mu_{1T}, \dots, \mu_{NT}).$$

Walker [16] proposed a modification of the definition of the least squares estimator of angular frequencies for the classical problem of determining hidden periodicities. This definition in our case guarantees the almost sure convergence to zero of the variables (7) and to prove the consistency of the above estimators. Walker [16] defines the estimator (6) as a point of minimum of the functional (5) in a parametric set that depends on T and asymptotically, as $T \to \infty$, distinguishes the set of frequencies λ and μ.

Consider the two families of nondecreasing open sets

$$\Lambda_T \subset \Lambda(\underline{\lambda}, \overline{\lambda}), \qquad M_T \subset M(\underline{\mu}, \overline{\mu}), \qquad T \geq T_0 > 0, \tag{8}$$

that contain true values of parameters λ^0 and μ^0, respectively, and that satisfy the following conditions:

R2.
$$\lim_{T\to\infty} \inf_{\substack{1 \leq j \leq N-1 \\ \lambda \in \Lambda_T}} T(\lambda_{j+1} - \lambda_j) = \lim_{T\to\infty} \inf_{\substack{1 \leq j \leq N-1 \\ \mu \in M_T}} T(\mu_{j+1} - \mu_j) = \infty, \tag{9}$$

$$\lim_{T\to\infty} \inf_{\lambda \in \Lambda_T} T\lambda_1 = \lim_{T\to\infty} \inf_{\mu \in M_T} T\mu_1 = \infty. \tag{10}$$

Condition (10) holds if $\underline{\lambda} > 0$ and $\underline{\mu} > 0$. If $\Lambda_T \subset \Lambda(0, \overline{\lambda})$ and $M_T \subset M(0, \overline{\mu})$, then conditions (9) and (10) are satisfied for sets Λ_T and M_T such that

$$\begin{aligned} \inf_{\substack{1 \leq j \leq N-1 \\ \lambda \in \Lambda_T}} (\lambda_{j+1} - \lambda_j) &= \inf_{\substack{1 \leq j \leq N-1 \\ \mu \in M_T}} (\mu_{j+1} - \mu_j) \\ &= \inf_{\lambda \in \Lambda_T} \lambda_1 = \inf_{\mu \in M_T} \mu_1 = T^{-1/2}. \end{aligned} \tag{11}$$

Conditions (9) and (10) allow one to treat the case of close frequencies in the families λ^0 and μ^0 and the case where the frequencies λ_1^0 and μ_1^0 are close to zero.

Definition 2.1. Any random vector θ_T of the form (6) that minimizes the functional (5) in the set of parameters $\Theta \subset \mathbb{R}^{4N}$, where the amplitudes A_k and B_k, $k = 1, \dots, N$, assume arbitrary values, while the frequencies λ and μ assume values in the closed sets Λ_T^c and M_T^c, respectively, is called the least squares estimator (Walker least squares estimator) of the vector parameter θ^0 of the form (2) in the model (1), (2).

In the rest of the paper, we study the Walker least squares estimator θ_T of the parameter θ^0 in the sense of Definition 2.1.

3. Auxiliary results

Lemma 3.1 below generalizes the corresponding result of [9]. Let $\varphi = (\varphi_1, \varphi_2) \in \mathbb{R}^2$.

Lemma 3.1. *If condition* $\mathbf{N}$(i) *holds and* $\rho < \alpha/6$, *then*

$$\xi(T) = \sup_{\varphi \in \mathbb{R}^2} T^{-2+\rho} \left| \int_0^T \int_0^T e^{-i(\varphi_1 t_1 + \varphi_2 t_2)} \varepsilon(t_1, t_2)\, dt_1\, dt_2 \right| \to 0 \quad \text{almost surely} \tag{12}$$

as $T \to \infty$.

Proof. Changing the variables we obtain

$$\begin{aligned}
&\left| \int_0^T \int_0^T e^{-i(\varphi_1 t_1 + \varphi_2 t_2)} \varepsilon(t_1, t_2) dt_1 dt_2 \right|^2 \\
&\quad = \int_0^T \int_0^T e^{-i\varphi_1 (t_1 - s_1)} \int_0^T \int_0^T e^{-i\varphi_2 (t_2 - s_2)} \varepsilon(t_1, t_2) \varepsilon(s_1, s_2)\, dt_1\, dt_2\, ds_1\, ds_2 \\
&\quad = 2 \int_0^T \int_0^T \cos(\varphi_1 u_1 + \varphi_2 u_2) \\
&\qquad\qquad \times \int_0^{T-u_1} \int_0^{T-u_2} \varepsilon(v_1, v_2) \varepsilon(v_1 + u_1, v_2 + u_2)\, dv_1\, dv_2\, du_1\, du_2 \\
&\qquad + 2 \int_0^T \int_0^T \cos(\varphi_1 u_1 - \varphi_2 u_2) \\
&\qquad\qquad \times \int_0^{T-u_1} \int_0^{T-u_2} \varepsilon(v_1 + u_1, v_2) \varepsilon(v_1, v_2 + u_2)\, dv_1\, dv_2\, du_1\, du_2.
\end{aligned}$$

Further

$$\begin{aligned}
\mathsf{E}\, \xi^2(T) &\le 2T^{-4+2\rho} \int_0^T \int_0^T \mathsf{E} \left| \int_0^{T-u_1} \int_0^{T-u_2} \varepsilon(v_1, v_2) \varepsilon(v_1 + u_1, v_2 + u_2)\, dv_1\, dv_2 \right| du_1\, du_2 \\
&\quad + 2T^{-4+2\rho} \int_0^T \int_0^T \mathsf{E} \left| \int_0^{T-u_1} \int_0^{T-u_2} \varepsilon(v_1 + u_1, v_2) \varepsilon(v_1, v_2 + u_2)\, dv_1\, dv_2 \right| du_1\, du_2 \\
&\le 2T^{-4+2\rho} \int_0^T \int_0^T \Psi_1^{1/2}(u_1, u_2)\, du_1\, du_2 + 2T^{-4} \int_0^T \int_0^T \Psi_2^{1/2}(u_1, u_2)\, du_1\, du_2,
\end{aligned}$$

where

$$\begin{aligned}
\Psi_1(u_1, u_2) &= \int_0^{T-u_1} \int_0^{T-u_2} \int_0^{T-u_1} \int_0^{T-u_2} \mathsf{E}\, \varepsilon(v_1 + u_1, v_2 + u_2) \varepsilon(v_1, v_2) \\
&\qquad \times \varepsilon(w_1 + u_1, w_2 + u_2) \varepsilon(w_1, w_2)\, dv_1\, dv_2\, dw_1\, dw_2 \\
&= (T - u_1)^2 (T - u_2)^2 B^2(u_1, u_2) \\
&\quad + \int_0^{T-u_1} \int_0^{T-u_2} \int_0^{T-u_1} \int_0^{T-u_2} B^2(v_1 - w_1, v_2 - w_2)\, dv_1\, dv_2\, dw_1\, dw_2 \\
&\quad + \int_0^{T-u_1} \int_0^{T-u_2} \int_0^{T-u_1} \int_0^{T-u_2} B(v_1 - w_1 + u_1, v_2 - w_2 + u_2) \\
&\qquad \times B(v_1 - w_1 - u_1, v_2 - w_2 - u_2)\, dv_1\, dv_2\, dw_1\, dw_2 \\
&= \sum_{j=1}^{3} \Psi_{1j}(u_1, u_2)
\end{aligned}$$

and

$$\begin{aligned}\Psi_2(u_1,u_2) &= \int_0^{T-u_1}\int_0^{T-u_2}\int_0^{T-u_1}\int_0^{T-u_2} \mathsf{E}\,\varepsilon(v_1+u_1,v_2)\varepsilon(v_1,v_2+u_2) \\ &\qquad \times \varepsilon(w_1+u_1,w_2)\varepsilon(w_1,w_2+u_2)\,dv_1\,dv_2\,dw_1\,dw_2 \\ &= (T-u_1)^2(T-u_2)^2B^2(u_1,-u_2) \\ &\quad + \int_0^{T-u_1}\int_0^{T-u_2}\int_0^{T-u_1}\int_0^{T-u_2} B^2(v_1-w_1,v_2-w_2)\,dv_1\,dv_2\,dw_1\,dw_2 \\ &\quad + \int_0^{T-u_1}\int_0^{T-u_2}\int_0^{T-u_1}\int_0^{T-u_2} B(v_1-w_1+u_1,v_2-w_2-u_2) \\ &\qquad \times B(v_1-w_1-u_1,v_2-w_2+u_2)\,dv_1\,dv_2\,dw_1\,dw_2 \\ &= \sum_{j=1}^{3}\Psi_{2j}(u_1,u_2)\end{aligned}$$

by the Isserlis theorem.

Since $\sqrt{a+b+c}\le\sqrt{a}+\sqrt{b}+\sqrt{c}$ for all nonnegative real numbers a, b, and c, we obtain

$$\Psi_i^{1/2}(u_1,u_2)\le\sum_{j=1}^{3}\Psi_{ij}^{1/2}(u_1,u_2)$$

for $i=1,2$. Now we conclude from the above consideration that the second moment of the random variable ξ is estimated as follows:

$$\mathsf{E}\,\xi^2(T)\le\sum_{i=1}^{2}\sum_{j=1}^{3}I_{ij}(T), \tag{13}$$

where the terms $I_{ij}(T)$, $i=1,2$, $j=1,2,3$, in the sum on the right-hand side of (13) are given by

$$I_{ij}(T)=2T^{-4+2\rho}\int_0^T\int_0^T\Psi_{ij}^{1/2}(u_1,u_2)\,du_1\,du_2.$$

Now we estimate each term $I_{ij}(T)$, $i=1,2$, $j=1,2,3$, separately. For the sake of brevity, let

$$b_u(v_1-w_1,v_2-w_2)=B(v_1-w_1+u_1,v_2-w_2+u_2)B(v_1-w_1-u_1,v_2-w_2-u_2).$$

We have

$$
\begin{aligned}
\Psi_{13}(u_1,u_2) &= \int_0^{T-u_1}\int_0^{T-u_1}\int_0^{T-u_2}\int_0^{T-u_2} b(v_1-w_1,v_2-w_2)\,dv_1\,dw_1\,dv_2\,dw_2\\
&= (T-u_1)(T-u_2)\\
&\quad\times\int_{-(T-u_1)}^{T-u_1}\int_{-(T-u_2)}^{T-u_2}\left(1-\frac{|t_1|}{T-u_1}\right)\left(1-\frac{|t_2|}{T-u_2}\right) b_u(t_1,t_2)\,dt_1\,dt_2\\
&= T^2(T-u_1)(T-u_2)\\
&\quad\times\int_{-(1-u_1T^{-1})}^{1-u_1T^{-1}}\int_{-(1-u_2T^{-1})}^{1-u_2T^{-1}}\left(1-\frac{|t_1|}{1-u_1T^{-1}}\right)\left(1-\frac{|t_2|}{1-u_2T^{-1}}\right) b_u(Tt_1,Tt_2)\,dt_1\,dt_2\\
&\le T^2(T-u_1)(T-u_2)\int_{-1}^{1}\int_{-1}^{1} b_u(Tt_1,Tt_2)\,dt_1\,dt_2\\
&\le T^2(T-u_1)(T-u_2)\Bigg[B(0)\int_0^1\int_0^1 B(Tt_1+u_1,Tt_2+u_2)\,dt_1\,dt_2\\
&\qquad + B(0)\int_{-1}^{0}\int_{-1}^{0} B(Tt_1-u_1,Tt_2-u_2)\,dt_1\,dt_2\\
&\qquad + \left(\int_0^1\int_{-1}^0 + \int_{-1}^0\int_0^1\right) b_u(Tt_1,Tt_2)\Bigg]\\
&= T^2(t-u_1)(T-u_2)\sum_{k=1}^{4}\Psi_{13}^{(k)}(u_1,u_2).
\end{aligned}
$$

By the assumption of Lemma 3.1, $\Psi_{13}^{(1)}=\Psi_{13}^{(2)}$ and $\Psi_{13}^{(3)}=\Psi_{13}^{(4)}$. Thus we need to estimate $\Psi_{13}^{(1)}$ and $\Psi_{13}^{(3)}$. Since $\|Tt\pm u\|\le 2\sqrt{2}\,T$, we obtain $L\left(\|Tt\pm u\|\right)\le(1+\varepsilon)L(T)$ for an arbitrary $\varepsilon>0$ and for sufficiently large T (for $T>T_0$, say) in view of the monotonicity of L. On the other hand,

$$\|Tt+u\|^\alpha \ge T^\alpha t_1^\alpha, \tag{14}$$

$$\Psi_{13}^{(1)} \le (1+\varepsilon)(1-\alpha)^{-1}B(0)B(T), \qquad T>T_0. \tag{15}$$

Passing to the term $\Psi_{13}^{(3)}$, note that the bound (14) holds for the first factor $b_u(Tt_1,Tt_2)$, while the second one is estimated by

$$\|Tt-u\|^\alpha \ge T^\alpha t_2^\alpha, \tag{16}$$

that is

$$\Psi_{13}^{3} \le (1+\varepsilon)^2(1-\alpha)^{-2}B^2(T), \qquad T>T_0, \tag{17}$$

and

(18)

$$I_{13}(T)\le \frac{8}{9}\sqrt{2}\left((1+\varepsilon)^{1/2}(1-\alpha)^{-1/2}B^{1/2}(0)B^{1/2}(T)+(1+\varepsilon)(1-\alpha)^{-1}B(T)\right)T^{2\rho}$$

for the same T.

Reasoning similarly, we get the bounds

$$\Psi_{12}(u_1,u_2)\le 4B(0)T^2(T-u_1)(T-u_2)\int_0^1\int_0^1 B(Tt_1,Tt_2)\,dt_1\,dt_2,$$

$$I_{12}(T)\le\frac{16}{9}(1+\varepsilon)^{1/2}(1-\alpha)^{-1/2}B^{1/2}(0)B^{1/2}(T)T^{2\rho}. \tag{19}$$

In addition,

$$
\begin{aligned}
I_{11}(T) &\le 2T^{-4+2\rho}\int_0^T\int_0^T (T-u_1)(T-u_2)B(u_1,u_2)\,du_1\,du_2 \\
&\le T^{2\rho}\int_0^1\int_0^1 B(Tu_1,Tu_2)\,du_1\,du_2 \le 2(1+\varepsilon)(1-\alpha)^{-1}B(T)T^{2\rho}
\end{aligned}
\tag{20}
$$

for $T > T_0$. Thus the bounds (18)–(20) imply that

$$
\sum_{j=1}^{3} I_{1j} = O\left(B^{1/2}(T)T^{2\rho}\right) \tag{21}
$$

as $T \to \infty$. Put

$$
c_u(v_1-w_1, v_2-w_2) = B(v_1-w_1+u_1, v_2-w_2-u_2)B(v_1-w_1-u_1, v_2-w_2+u_2).
$$

As in the case of the term $\Psi_{13}(u_1,u_2)$, we conclude that

$$
\begin{aligned}
\Psi_{23}(u_1,u_2) &\le T^2(T-u_1)(T-u_2) \\
&\quad\times\left(\int_0^1\int_0^1 + \int_{-1}^0\int_{-1}^0 + \int_0^1\int_{-1}^0 + \int_{-1}^0\int_0^1\right) c_u(Tt_1,Tt_2)\,dt_1\,dt_2 \\
&= T^2(T-u_1)(T-u_2)\sum_{k=1}^{4}\Psi_{23}^{(k)}(u_1,u_2).
\end{aligned}
$$

By the assumptions of Lemma 3.1,

$$
\Psi_{23}^{(1)} = \Psi_{23}^{(2)} = \Psi_{13}^{(3)} = \Psi_{13}^{(4)}, \qquad \Psi_{23}^{(3)} = \Psi_{23}^{(4)} = \Psi_{13}^{(1)} = \Psi_{13}^{(2)}.
$$

Moreover, $\Psi_{21} = \Psi_{11}$ and $\Psi_{22} = \Psi_{12}$. This means that

$$
\sum_{j=1}^{3} I_{2j} = O\left(B^{1/2}(T)T^{2\rho}\right) \tag{22}
$$

as $T \to \infty$. Relations (21) and (22) together with (14) show that

$$
\mathsf{E}\,\xi^2(T) = O\left(L^{1/2}(T)T^{-\alpha/2+2\rho}\right) \tag{23}
$$

as $T \to \infty$.

Let $T_n = n^\beta$, where $\beta > 0$ is such that $\left(\frac{\alpha}{2} - 2\rho\right)\beta = 1+\delta$ for some $\delta > 0$. Then

$$
\sum_{n=1}^{\infty} \mathsf{E}\,\xi^2(T_n) < \infty,
$$

that is, $\xi(T_n) \to 0$ almost surely as $n \to \infty$.

Consider the following sequence of random variables:

$$\begin{aligned}\zeta_n &= \sup_{T_n \le T \le T_{n+1}} |\xi(T) - \xi(T_n)| \\ &\le \sup_{T_n \le T \le T_{n+1}} \sup_{\varphi \in \mathbb{R}^2} \Bigg| T^{-2+\rho} \int_0^T \int_0^T e^{-i(\varphi_1 t_1 + \varphi_2 t_2)} \varepsilon(t_1, t_2)\, dt \\ &\qquad\qquad - T_n^{-2+\rho} \int_0^{T_n} \int_0^{T_n} e^{-i(\varphi_1 t_1 + \varphi_2 t_2)} \varepsilon(t_1, t_2)\, dt \Bigg| \\ &\le \left(\frac{T_{n+1}^{2-\rho}}{T_n^{2-\rho}} - 1 \right) \xi(T_n) \\ &\quad + T_n^{-2+\rho} \left(\int_{T_n}^{T_{n+1}} \int_0^{T_n} + \int_0^{T_n} \int_{T_n}^{T_{n+1}} + \int_{T_n}^{T_{n+1}} \int_{T_n}^{T_{n+1}} \right) |\varepsilon(t_1, t_2)|\, dt_1\, dt_2 \\ &= \sum_{i=1}^{4} \zeta_n^{(i)}.\end{aligned}$$

It is obvious that $\zeta_n^{(1)} \to 0$ almost surely as $n \to \infty$. For $k \in \mathbb{N}$, consider

$$\begin{aligned}\mathsf{E}\left(\zeta_n^{(2)}\right)^{2k} &= T_n^{-2k(2-\rho)} \int_{T_n}^{T_{n+1}} \int_0^{T_n} \overset{2k}{\cdots} \int_{T_n}^{T_{n+1}} \int_0^{T_n} \mathsf{E} \prod_{j=1}^{2k} \left| \varepsilon\left(t_1^{(j)}, t_2^{(j)}\right) \right| \prod_{j=1}^{2k} dt_1^{(j)}\, dt_2^{(j)} \\ &\le (2k-1)!!\, B^k(0) T_n^{-2k(2-\rho)} (T_{n+1} - T_n)^{2k} T_n^{2k} \\ &= (2k-1)!!\, B^k(0) \left(\frac{T_{n+1}}{T_n} - 1 \right)^{2k} T_n^{2k\rho} = O\left(n^{-2k(1-\beta\rho)}\right), \qquad n \to \infty.\end{aligned}$$

If $\beta\rho < 1$, then the series $\sum_{n=1}^{\infty} \mathsf{E}\big(\zeta_n^{(2)}\big)^{2k}$ converges for an appropriate k and hence $\zeta_n^{(2)} \to 0$ almost surely as $n \to \infty$, whence $\beta\rho = \frac{\rho(1+\delta)}{\alpha/2 - 2\rho} < 1$ or $\rho < \frac{\alpha}{2(3+\delta)}$. Since $\delta > 0$ can be chosen arbitrarily small, the assumption $\rho < \alpha/6$ implies the convergence of $\zeta_n^{(2)}$ as well as the convergence $\zeta_n^{(3)} \to 0$ almost surely as $n \to \infty$. Since

$$\mathsf{E}\left(\zeta_n^{(4)}\right)^2 \le (2k-1)!!\, B^k(0) \left(\frac{T_{n+1}}{T_n} - 1 \right)^{4k} T_n^{2k\rho} = O\left(n^{-2k(2-\beta\rho)}\right), \qquad n \to \infty,$$

and $\zeta_n^{(4)} \to 0$ almost surely as $n \to \infty$, Lemma 3.1 is proved. □

Lemma 3.2. *Assume that condition* $\mathbf{N}$(ii) *holds. Then* $\xi(T) \to 0$ *almost surely as* $T \to \infty$ *if* $\rho < 1/3$.

Proof. Using the notation introduced in Lemma 3.1 and the assumption of Lemma 3.2, we obtain for $i = 1, 2$ that

$$I_{i1}(T) = O\left(T^{-2+2\rho}\right), \qquad I_{i2}(T) = O\left(T^{-1+2\rho}\right), \qquad I_{i3}(T) = O\left(T^{-1+2\rho}\right), \qquad T \to \infty. \tag{24}$$

Let $T_n = n^{\beta}$, where $(1 - 2\rho)\beta = 1 + \delta$ and $\delta > 0$. Then, similarly to the proof of Lemma 3.1, $\xi(T_n) \to 0$ almost surely as $n \to \infty$. Put $\zeta_n = \sum_{i=1}^{4} \zeta_n^{(i)}$. Then $\zeta_n^{(1)} \to 0$ almost surely as $n \to \infty$. As in the proof of Lemma 3.1, the assumption $\rho < 1/3$ implies the convergence $\zeta_n^{(i)} \to 0$ almost surely as $n \to \infty$ for $i = 2, 3, 4$. □

4. Main result

Theorem 4.1. *Assume that conditions* **N**, **R1**, *and* **R2** *hold. Then the Walker least squares estimator θ_T is a strongly consistent estimator of the parameter θ^0, namely*

$$A_{kT} \to A_k^0, \qquad B_{kT} \to B_k^0, \qquad T(\lambda_{kT} - \lambda_k^0) \to 0, \qquad T(\mu_{kT} - \mu_k^0) \to 0$$

almost surely as $T \to \infty$, $k = 1, \dots, N$.

Proof. Consider the following system of linear equations with respect to the least squares estimators A_{kT} and B_{kT}, $k = 1, \dots, N$:

$$\left.\frac{\partial Q_T(\theta)}{\partial A_p}\right|_{\theta=\theta_T} = \left.\frac{\partial Q_T(\theta)}{\partial B_p}\right|_{\theta=\theta_T} = 0, \qquad p = 1, \dots, N.$$

We rewrite this system in the form

$$\begin{cases} \sum_{k=1}^N a_{kp}^{(1)} A_{kT} + \sum_{k=1}^N b_{kp}^{(1)} B_{kT} = c_p^{(1)}, & p = 1, \dots, N; \\ \sum_{k=1}^N a_{kp}^{(2)} A_{kT} + \sum_{k=1}^N b_{kp}^{(2)} B_{kT} = c_p^{(2)}, & p = 1, \dots, N. \end{cases} \tag{25}$$

Let

$$\begin{gathered} \cos(\lambda_{kT} t_1 + \mu_{kT} t_2) = \cos_k, \qquad \sin(\lambda_{kT} t_1 + \mu_{kT} t_2) = \sin_k, \\ \cos(\lambda_k^0 t_1 + \mu_k^0) = \cos_k^0, \qquad \sin(\lambda_k^0 t_1 + \mu_k^0) = \sin_k^0, \qquad k = 1, \dots, N. \end{gathered} \tag{26}$$

Then the coefficients of system (25) are such that

$$\begin{aligned} a_{kp}^{(1)} &= T^{-2} \int_0^T \int_0^T \cos_k \cos_p \, dt_1 \, dt_2, & a_{kp}^{(2)} &= T^{-2} \int_0^T \int_0^T \cos_k \sin_p \, dt_1 \, dt_2, \\ b_{kp}^{(1)} &= T^{-2} \int_0^T \int_0^T \sin_k \cos_p \, dt_1 \, dt_2, & b_{kp}^{(2)} &= T^{-2} \int_0^T \int_0^T \sin_k \sin_p \, dt_1 \, dt_2, \\ c_p^{(1)} &= T^{-2} \int_0^T \int_0^T X(t_1, t_2) \cos_p \, dt_1 \, dt_2, & c_{kp}^{(2)} &= T^{-2} \int_0^T \int_0^T X(t_1, t_2) \sin_p \, dt_1 \, dt_2. \end{aligned}$$

Below we use the symbol $o(1)$ to denote various stochastic processes (possibly different in different places) that depend on the parameter T and that almost surely approach zero as $T \to \infty$.

Taking into account properties (9) and (10) of parametric sets Λ_T and M_T whose closures contain the values of the estimators λ_T and μ_T, respectively, we find after simple algebra that

$$a_{kp}^{(1)} = o(1), \qquad k \neq p, \qquad a_{pp}^{(1)} = \frac{1}{2} + o(1), \qquad a_{kp}^{(2)} = o(1), \qquad k, p = 1, \dots, N; \tag{27}$$

$$\begin{gathered} b_{kp}^{(1)} = a_{pk}^{(2)} = o(1), \qquad b_{kp}^{(2)} = o(1), \qquad k \neq p, \qquad b_{kp}^{(2)} = \frac{1}{2} + o(1), \\ k, p = 1, \dots, N. \end{gathered} \tag{28}$$

Further let

$$\begin{gathered} x_{\lambda p} = \frac{\sin T \left(\lambda_{pT} - \lambda_p^0\right)}{T \left(\lambda_{pT} - \lambda_p^0\right)}, \qquad x_{\mu p} = \frac{\sin T \left(\mu_{pT} - \mu_p^0\right)}{T \left(\mu_{pT} - \mu_p^0\right)}, \qquad p = 1, \dots, N; \\ y_{\lambda p} = \frac{1 - \cos T \left(\lambda_{pT} - \lambda_p^0\right)}{T \left(\lambda_{pT} - \lambda_p^0\right)}, \qquad y_{\mu p} = \frac{1 - \cos T \left(\mu_{pT} - \mu_p^0\right)}{T \left(\mu_{pT} - \mu_p^0\right)}, \qquad p = 1, \dots, N. \end{gathered} \tag{29}$$

Then

$$c_p^{(1)} = T^{-2}\int_0^T\int_0^T \varepsilon(t_1,t_2)\cos_p\,dt_1\,dt_2 + T^{-2}\int_0^T\int_0^T g\left(t_1,t_2;\theta^0\right)\cos_p\,dt_1\,dt_2 \tag{30}$$
$$= \frac{1}{2}\left[A_p^0(x_{\lambda p}x_{\mu p} - y_{\lambda p}y_{\mu p}) - B_p^0(x_{\mu p}y_{\lambda p} + x_{\lambda p}y_{\mu p})\right] + o(1)$$

in view of Lemmas 3.1 and 3.2. Analogously

$$c_p^{(2)} = \frac{1}{2}\left[A_p^0(x_{\mu p}y_{\lambda p} + x_{\lambda p}y_{\mu p}) + B_p^0(x_{\lambda p}x_{\mu p} - y_{\lambda p}y_{\mu p})\right] + o(1). \tag{31}$$

Since $|x_{\lambda p}| \le 1$, $|x_{\mu p}| \le 1$, $|y_{\lambda p}| \le 1$, and $|y_{\mu p}| \le 1$, $p = 1,\dots,N$, solutions of system (25) can be represented as

$$\begin{aligned} A_{pT} &= A_p^0(x_{\lambda p}x_{\mu p} - y_{\lambda p}y_{\mu p}) - B_p^0(x_{\mu p}y_{\lambda p} + x_{\lambda p}y_{\mu p}) + o(1),\\ B_{pT} &= A_p^0(x_{\mu p}y_{\lambda p} + x_{\lambda p}y_{\mu p}) + B_p^0(x_{\lambda p}x_{\mu p} - y_{\lambda p}y_{\mu p}) + o(1),\\ &\qquad p = 1,\dots,N, \end{aligned} \tag{32}$$

according to relations (27), (28), (30), and (31).

In turn, relation (32) implies the inequalities

$$|A_{pT}|,\ |B_{pT}| \le 2\left(|A_p^0| + |B_p^0|\right) + o(1), \qquad p = 1,\dots,N. \tag{33}$$

Put

$$G_T(\theta_1;\theta_2) = T^{-2}\int_0^T\int_0^T \left[g(t_1,t_2;\theta_1) - g(t_1,t_2;\theta_2)\right]^2\,dt_1\,dt_2.$$

By the definition of the least squares estimator,

$$\begin{aligned} 0 &\ge Q_T(\theta_T) - Q_T(\theta^0)\\ &= G_T(\theta_T;\theta^0) + 2T^{-2}\int_0^T\int_0^T \varepsilon(t_1,t_2)\left(g(t_1,t_2;\theta^0) - g(t_1,t_2;\theta_T)\right)\,dt_1\,dt_2. \end{aligned} \tag{34}$$

The second term on the right-hand side of equality (34) is $o(1)$ in view of Lemmas 3.1 and 3.2 and relation (33). This means that

$$G_T(\theta_T;\theta^0) \to 0 \quad \text{almost surely as } T \to \infty. \tag{35}$$

Now we rewrite the expression for $G_T(\theta_T;\theta^0)$ in such a way that relation (35) implies the consistency of the least squares estimators of parameters λ_k^0 and μ_k^0, $k = 1,\dots,N$. We have

$$\begin{aligned} G_T(\theta_T;\theta^0) &= T^{-2}\int_0^T\int_0^T g^2(t_1,t_2;\theta_T)\,dt_1\,dt_2 + T^{-2}\int_0^T\int_0^T g^2(t_1,t_2;\theta^0)\,dt_1\,dt_2\\ &\quad - 2T^{-2}\int_0^T\int_0^T g(t_1,t_2;\theta_T)g(t_1,t_2;\theta^0)\,dt_1\,dt_2\\ &= J_1 + J_2 + J_3. \end{aligned}$$

Using (33) and (27), (28) we obtain

$$J_1 = \frac{1}{2}\sum_{k=1}^N \left(A_{kT}^2 + B_{kT}^2\right) + o(1), \tag{36}$$

$$J_2 = \frac{1}{2}\sum_{k=1}^N \left((A_k^0)^2 + (B_k^0)^2\right) + o(1), \tag{37}$$

$$
\begin{aligned}
J_3 &= -2\sum_{p=1}^{N}\sum_{k=1}^{N} T^{-2}\int_0^T\int_0^T \left(A_{pT}A_k^0 \cos_p \cos_k^0 + A_{pT}B_k^0 \cos_p \sin_k^0\right) dt_1\, dt_2 \\
&\quad - 2\sum_{p=1}^{N}\sum_{k=1}^{N} T^{-2}\int_0^T\int_0^T \left(B_{pT}A_k^0 \sin_p \cos_k^0 + B_{pT}B_k^0 \sin_p \sin_k^0\right) dt_1\, dt_2 \\
&= \sum_{p=1}^{N} \left(A_{pT}A_p^0(x_{\lambda p}x_{\mu p} - y_{\lambda p}y_{\mu p}) - A_{pT}B_p^0(x_{\mu p}y_{\lambda p} + x_{\lambda p}y_{\mu p})\right) \\
&\quad - \sum_{p=1}^{N} \left(B_{pT}A_p^0(x_{\mu p}y_{\lambda p} + x_{\lambda p}y_{\mu p}) + B_{pT}B_p^0(x_{\lambda p}x_{\mu p} - y_{\lambda p}y_{\mu p})\right) + o(1).
\end{aligned} \tag{38}
$$

Put $z_{1p} = x_{\lambda p}x_{\mu p} - y_{\lambda p}y_{\mu p}$ and $z_{2p} = x_{\mu p}y_{\lambda p} + x_{\lambda p}y_{\mu p}$, $p = 1, \ldots, N$. Substituting expressions (32) into (36) and (38), we get

$$
\begin{aligned}
G_T(\theta_T; \theta^0) &= \frac{1}{2}\sum_{p=1}^{N}\left[\left(A_p^0 z_{1p} - B_p^0 z_{2p}\right)^2 + \left(A_p^0 z_{2p} + B_p^0 z_{1p}\right)^2 + \left(A_p^0\right)^2 + \left(B_p^0\right)^2\right] \\
&\quad - \sum_{p=1}^{N}\left[\left(A_p^0\right)^2 z_{1p}^2 - 2A_p^0B_p^0 z_{1p}z_{2p} + \left(B_p^0\right)^2 z_{2p}^2\right] \\
&\quad - \sum_{p=1}^{N}\left[\left(A_p^0\right)^2 z_{2p}^2 + 2A_p^0B_p^0 z_{1p}z_{2p} + \left(B_p^0\right)^2 z_{1p}^2\right] + o(1) \\
&= \frac{1}{2}\sum_{p=1}^{N}\left(\left(A_p^0\right)^2 + \left(B_p^0\right)^2\right)\left(1 - z_{1p}^2 - z_{2p}^2\right) + o(1) \\
&= \frac{1}{2}\sum_{p=1}^{N}\left(\left(A_p^0\right)^2 + \left(B_p^0\right)^2\right)\left(1 - \left(x_{\lambda p}^2 + y_{\lambda p}^2\right)\left(x_{\mu p}^2 + y_{\mu p}^2\right)\right) + o(1) \\
&= \frac{1}{2}\sum_{p=1}^{N}\left(\left(A_p^0\right)^2 + \left(B_p^0\right)^2\right) \\
&\quad \times \left[1 - \left(\frac{\sin \frac{1}{2}T\left(\lambda_{pT} - \lambda_p^0\right)}{\frac{1}{2}T\left(\lambda_{pT} - \lambda_p^0\right)}\right)^2 \left(\frac{\sin \frac{1}{2}T\left(\mu_{pT} - \mu_p^0\right)}{\frac{1}{2}T\left(\mu_{pT} - \mu_p^0\right)}\right)^2\right] + o(1).
\end{aligned} \tag{39}
$$

Equality (39) together with (35) proves that

$$
T\left(\lambda_{pT} - \lambda_p^0\right) \to 0, \qquad T\left(\mu_{pT} - \mu_p^0\right) \to 0
$$

almost surely as $T \to \infty$, $p = 1, \ldots, N$. Now (29) implies that $x_{\lambda p} \to 1$, $x_{\mu p} \to 1$ and $y_{\lambda p} \to 0$, $y_{\mu p} \to 0$ almost surely as $T \to \infty$, $p = 1, \ldots, N$. We also obtain from (32) that

$$
A_{pT} \to A_p^0, \qquad B_{pT} \to B_p^0.
$$

The theorem is proved. □

5. Concluding remarks

The strong consistency of the least squares estimator of parameters in the texture surface sinusoidal model is proved in the paper under the assumption that the random noise is an isotropic and homogeneous Gaussian random field. It is natural to extend this result in order to find conditions for the consistency of least squares estimators in the case of a non-Gaussian noise and to prove the asymptotic normality of the least squares estimators.

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