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ESTIMATES OF FUNCTIONALS CONSTRUCTED FROM RANDOM SEQUENCES WITH PERIODICALLY STATIONARY INCREMENTS

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Dedicated to the blessed memory of Mykhailo Yosypovych Yadrenko

Abstract. The problem of the optimal estimation of the linear functional

$$A_N\xi = \sum_{k=0}^{N} a(k)\xi(k)$$

is studied. The functional depends on unknown values of a random sequence $\xi(k)$ with periodically stationary increments. The estimate is constructed from observations of this sequence at points $\mathbb{Z} \setminus \{0, 1, \dots, N\}$. Expressions for evaluating the mean square error and spectral characteristics are found for the optimal estimate of the functional in the case where the spectral density of the sequence is known. For a given set of admissible spectral densities, the sets of least favorable spectral densities are found and the spectral characteristics of the optimal estimate of the functional are determined.

1. Introduction

Problems of estimation of unknown values of stochastic processes play an important role among modern topics of the theory of stochastic processes. The classical methods of solving the problems of estimation of unknown values of stationary processes (namely, the problems of extrapolation, interpolation, and filtration) with known spectral densities are developed in the papers by Kolmogorov [5], Wiener [26], and Yaglom [27, 28]. The study of stochastic processes with stationary increments of order n was initiated by Yaglom [29] who obtained the spectral representation of stochastic increment processes and solved the problem of prediction of values of a stochastic increment process by using known observations. Stochastic processes with stationary increments are also studied by Pinsker [23] and Pinsker and Yaglom [22].

If the spectral density is unknown but a set of admissible spectral densities is given, then one uses the minimax method which constitutes finding an estimate that minimizes the error for all densities of the given class. Grenander [2] was the first to apply the minimax approach to the problem of extrapolation. Franke [3] studied the problem of minimax extrapolation of stationary sequences with the help of methods of convex optimization. Kassam and Poor [4] provide a survey of earlier papers published before 1985 concerning robust methods of estimation.

The problems of extrapolation, interpolation, and filtration have been studied by Moklyachuk [13–16] for stationary processes and sequences. Solutions of problems of extrapolation, interpolation, and filtration for vector-valued processes are presented in

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the book by Moklyachuk and Masyutka [21]. The corresponding results for periodically correlated processes are published by Moklyachuk and Golichenko [20]. Luz and Moklyachuk [6–12] deal with estimates of functionals constructed from stochastic processes with stationary increments and those of cointegrated sequences. Moklyachuk and Sidei [17–19] studied the problems of extrapolation, interpolation, and filtration for stationary sequences with missing observations.

In the current paper, we study the problem of the optimal estimation of the linear functional

$$A_N\xi=\sum_{k=0}^{N}a(k)\xi(k)$$

depending on unknown values of a random sequence $\xi(k)$ with periodically stationary increments. The estimate is constructed from observations of this sequence at points $\mathbb{Z}\setminus\{0,1,\dots,N\}$. We find expressions for evaluating the mean square error as well as for the spectral characteristic of the optimal estimate of the functional in the case where the spectral density of the sequence is known. If the spectral density is unknown but a set of admissible spectral densities is given we follow the minimax approach for estimation. For a given set of admissible spectral densities, we determine the set of least favorable spectral densities and minimax characteristics of the optimal estimate of the functional.

2. Processes with periodically stationary increments

Definition 2.1. The function

$$\eta^{(n)}(m,\mu)=(1-B_\mu)^n\,\eta(m)=\sum_{l=0}^{n}(-1)^l\binom{n}{l}\eta(m-l\mu) \tag{1}$$

is called the stochastic increment of order n with shift $\mu\in\mathbb{Z}$ constructed from a random sequence $\{\eta(m),m\in\mathbb{Z}\}$, where B_μ is the shift operator with lag $\mu\in\mathbb{Z}$ acting at the sequence η as follows: $B_\mu\eta(m)=\eta(m-\mu)$, $m\in\mathbb{Z}$.

Definition 2.2. The stochastic increment $\eta^{(n)}(m,\mu)$ of order n constructed from a random sequence $\{\eta(m),m\in\mathbb{Z}\}$ is called stationary (wide sense stationary) if the expectations

$$\mathsf{E}\,\eta^{(n)}(m_0,\mu)=c^{(n)}(\mu),$$

$$\mathsf{E}\,\eta^{(n)}(m_0+m,\mu_1)\,\overline{\eta^{(n)}(m_0,\mu_2)}=D^{(n)}(m,\mu_1,\mu_2)$$

exist for all integers m_0, μ, m, μ_1, μ_2 and do not depend on m_0. The function $c^{(n)}(\mu)$ is called the mean value of the stationary increment of order n. Accordingly, $D^{(n)}(m,\mu_1,\mu_2)$ is called the structural function of the stationary increment of order n constructed from a random sequence $\{\eta(m),m\in\mathbb{Z}\}$.

Definition 2.3. A random sequence $\{\eta(m),m\in\mathbb{Z}\}$ is said to have stationary increments of order n if the increment $\eta^{(n)}(m,\mu)$ of order n defined by equality (1) is stationary.

Definition 2.4. A vector random sequence $\boldsymbol{\eta}(m)=\{\eta_p(m)\}_{p=1,2,\dots,T}$ is said to have stationary increments of order n if the increments of order n of the components $\eta_p^{(n)}(m,\mu)$, $p=1,2,\dots,T$, defined by equality (1) are stationary and stationary related.

Consider a random sequence $\zeta(m)$ and the corresponding sequence of stochastic increments $\zeta^{(n)}(m,\mu)$ constructed from the sequence $\zeta(m)$.

Definition 2.5. The stochastic increment $\zeta^{(n)}(m,\mu T)$ of order n constructed from a random sequence $\{\zeta(m), m\in\mathbb{Z}\}$ is called periodically stationary (periodically correlated with period T) if the expectations

$$\mathsf{E}\,\zeta^{(n)}(m+T,\mu T)=\mathsf{E}\,\zeta^{(n)}(m,\mu T)=c^{(n)}(m,\mu T),$$

$$\begin{aligned}\mathsf{E}\,\zeta^{(n)}(m+T,\mu_1 T)\,\overline{\zeta^{(n)}(k+T,\mu_2 T)}&=D^{(n)}(m+T,k+T;\mu_1 T,\mu_2 T)\\&=D^{(n)}(m,k;\mu_1 T,\mu_2 T)\end{aligned}$$

exist and, moreover, given arbitrary integers m,k,μ,μ_1,μ_2 there is no number less than $T>0$ for which the above equalities hold.

Consider the vector sequence $\boldsymbol{\zeta}^{(n)}(m,\mu)$ formed by blocks of elements of the sequence $\zeta^{(n)}(m,\mu T)$. Every coordinate $\zeta_p^{(n)}(m,\mu)$, $p=1,2,\dots,T$, of the vector $\boldsymbol{\zeta}^{(n)}(m,\mu)$ is defined by

$$\zeta_p^{(n)}(m,\mu)=\zeta^{(n)}(mT+p-1,\mu T),\qquad p=1,2,\dots,T.$$

Theorem 2.1. *A stochastic increment $\zeta^{(n)}(m,\mu T)$ of order n is a stationary increment if and only if T, $T>0$, is the least integer number such that the T-dimensional sequence $\boldsymbol{\zeta}^{(n)}(m,\mu)$ is stationary with respect to parameter m for all integer μ.*

Proof. The proof of the theorem follows from the following equalities:

$$\begin{aligned}&\mathsf{E}\,\zeta_{p_1}^{(n)}(m_0+m,\mu_1)\,\overline{\zeta_{p_2}^{(n)}(m_0,\mu_2)}\\&\quad=\mathsf{E}\,\zeta^{(n)}\big((m_0+m)T+p_1-1,\mu_1 T\big)\,\overline{\zeta^{(n)}(m_0T+p_2-1,\mu_2 T)}\\&\quad=\mathsf{E}\,\zeta^{(n)}(m_0T+mT+p_1-1,\mu_1 T)\,\overline{\zeta^{(n)}(m_0T+p_2-1,\mu_2 T)}\\&\quad=\mathsf{E}\,\zeta^{(n)}(mT+p_1-1,\mu_1 T)\,\overline{\zeta^{(n)}(p_2-1,\mu_2 T)}=\mathsf{E}\,\zeta_{p_1}^{(n)}(m,\mu_1 T)\,\overline{\zeta_{p_2}^{(n)}(0,\mu_2 T)},\\&\mathsf{E}\,\zeta_p^{(n)}(m_0,\mu T)=\mathsf{E}\,\zeta^{(n)}(m_0T+p-1,\mu T)=\mathsf{E}\,\zeta^{(n)}(p-1,\mu T)=\mathsf{E}\,\zeta_p^{(n)}(0,\mu T).\end{aligned}$$

□

Theorem 2.1 implies that the change

$$\xi_p(k)=\zeta(kT+p-1),\qquad p=1,2,\dots,T,\qquad k\in\mathbb{Z},\tag{2}$$

results in the vector sequence $\boldsymbol{\xi}(k)=\{\xi_p(k)\}_{p=1,2,\dots,T}$, $k\in\mathbb{Z}$, with stationary increments of order n. Indeed,

$$\begin{aligned}\xi_p^{(n)}(m,\mu)&=\sum_{l=0}^{n}(-1)^l\binom{n}{l}\xi_p(m-l\mu)=\sum_{l=0}^{n}(-1)^l\binom{n}{l}\zeta\big((m-l\mu)T+p-1\big)\\&=\zeta^{(n)}(mT+p-1,\mu T)\end{aligned}$$

for all $p=1,2,\dots,T$, where $\xi_p^{(n)}(m,\mu)$ is the increment of order n constructed from the pth component of the vector sequence $\boldsymbol{\xi}(m)$.

If the matrix of spectral densities $F(\lambda)$ of a stationary sequence

$$\boldsymbol{\xi}^{(n)}(m,\mu)$$

is known, then the sequence itself admits the spectral decomposition

$$\xi_p^{(n)}(m,\mu)=\int_{-\pi}^{\pi}e^{im\lambda}\left(1-e^{-i\mu\lambda}\right)^n\frac{1}{(i\lambda)^n}\,dZ_p(\lambda),\qquad p=1,2,\dots,T,\tag{3}$$

where $\boldsymbol{Z}(\Delta)=\{Z_p(\Delta)\}_{p=1}^{T}$ is an orthogonal random measure of the sequence $\boldsymbol{\xi}^{(n)}(m,\mu)$ (see [1, 12, 29]).

3. Classical method of estimation

Let the vector stochastic sequence $\boldsymbol{\xi}(m)$ be constructed from a sequence $\zeta(m)$ with the help of transformation (2) and determine the stationary increment

$$\boldsymbol{\xi}^{(n)}(m,\mu)=\{\xi_p^{(n)}(m,\mu)\}_{p=1}^{T}$$

of order n with the matrix of spectral densities $F(\lambda)=\{f_{ij}(\lambda)\}_{i,j=1}^{T}$.

Consider the problem of linear mean square optimal estimation of the functional

$$A_N\boldsymbol{\xi}=\sum_{j=0}^{N}\boldsymbol{a}(j)^{\top}\boldsymbol{\xi}(j)=\sum_{j=0}^{N}\sum_{p=1}^{T}a_p(j)\xi_p(j) \tag{4}$$

that depends on unknown values of the vector sequence $\boldsymbol{\xi}(j)=\{\xi_p(j)\}_{p=1}^{T}$. The estimate is constructed from observations of the sequence $\boldsymbol{\xi}(j)$ at points $j\in\mathbb{Z}\setminus\{0,1,\dots,N\}$.

Simple algebra allows one to transform equality (1) to a representation of the random sequence $\xi_p(j)$ in terms of the increments of order n,

$$\xi_p(j)=\frac{1}{(1-B_\mu)^n}\xi_p^{(n)}(j,\mu)=\sum_{k=-\infty}^{j}d_\mu(j-k)\xi_p^{(n)}(k,\mu),\qquad p=1,2,\dots,T, \tag{5}$$

where $\{d_\mu(j)\colon j\geqslant 0\}$ are the coefficients of the term x^j in the equality

$$\sum_{j=0}^{\infty}d_\mu(j)x^j=\left(\sum_{k=0}^{\infty}x^{\mu k}\right)^n.$$

Using representation (5), we rewrite the functional (4) as follows:

$$\begin{aligned}A_N\boldsymbol{\xi}&=\sum_{j=0}^{N}\boldsymbol{a}(j)^{\top}\boldsymbol{\xi}(j)=\sum_{j=0}^{N}\sum_{p=1}^{T}a_p(j)\xi_p(j)\\&=\sum_{p=1}^{T}\left[-\sum_{j=-\mu n}^{-1}v_p(j)\xi_p(j)+\sum_{j=0}^{N}\left(\sum_{k=j}^{N}a_p(k)d_\mu(k-j)\right)\xi_p^{(n)}(j,\mu)\right]\\&=-\sum_{j=-\mu n}^{-1}\sum_{p=1}^{T}v_p(j)\xi_p(j)+\sum_{j=0}^{N}\sum_{p=1}^{T}b_p(j)\xi_p^{(n)}(j,\mu)\\&=-\sum_{j=-\mu n}^{-1}\boldsymbol{v}(j)^{\top}\boldsymbol{\xi}(j)+\sum_{j=0}^{N}\boldsymbol{b}(i)^{\top}\boldsymbol{\xi}^{(n)}(j,\mu).\end{aligned}$$

The latter relation implies that the functional $A_N\boldsymbol{\xi}$ is a difference of functionals,

$$A_N\boldsymbol{\xi}=B_N\boldsymbol{\xi}-V_N\boldsymbol{\xi}, \tag{6}$$

where

$$B_N\boldsymbol{\xi}=\sum_{j=0}^{N}\boldsymbol{b}(j)^{\top}\boldsymbol{\xi}^{(n)}(j,\mu),\qquad V_N\boldsymbol{\xi}=\sum_{j=-\mu n}^{-1}\boldsymbol{v}(j)^{\top}\boldsymbol{\xi}(j),$$

$$v_p(j)=\sum_{l=[-\frac{j}{m}]'}^{n}(-1)^l\binom{n}{l}b_p(l\mu+j),\qquad p=1,2,\dots,T,\qquad j=-1,-2,\dots,-\mu n, \tag{7}$$

$$b_p(j)=\sum_{m=j}^{N}a_p(m)d_\mu(m-j),\qquad p=1,2,\dots,T,\qquad j=0,1,\dots,N, \tag{8}$$

$$\boldsymbol{v}(j)=\big(v_1(j),v_2(j),\dots,v_T(j)\big)^{\top},\qquad \boldsymbol{b}(j)=\big(b_1(j),b_2(j),\dots,b_T(j)\big)^{\top}.$$

The symbol $[x]'$ in relation (7) stands for the least integer number in the set of numbers that are greater than or equal to x.

Let $\widehat{A}_N\boldsymbol{\xi}$ be the mean square optimal linear estimate of the functional $A_N\boldsymbol{\xi}$ constructed from observations of the vector random sequence $\boldsymbol{\xi}(j)$ at points $j\in\mathbb{Z}\setminus\{0,1,\dots,N\}$. Denote by $\widehat{B}_N\boldsymbol{\xi}$ the mean square optimal linear estimate of the functional $B_N\boldsymbol{\xi}$ constructed from observations of the stochastic increment $\boldsymbol{\xi}^{(n)}(m,\mu)$ of order n at the points $m\in\mathbb{Z}\setminus\{0,1,\dots,N+\mu n\}$. Since the values of the sequence $\boldsymbol{\xi}(m)$ at points $m=-1,-2,\dots,-\mu n$ are known, the estimate $\widehat{A}_N\boldsymbol{\xi}$ can be rewritten as

$$\widehat{A}_N\boldsymbol{\xi}=\widehat{B}_N\boldsymbol{\xi}-V_N\boldsymbol{\xi}. \tag{9}$$

The mean square error of the estimate $\widehat{A}_N\boldsymbol{\xi}$ admits the relations

$$\begin{aligned}\Delta\left(F,\widehat{A}_N\boldsymbol{\xi}\right)&=\mathsf{E}\left|A_N\boldsymbol{\xi}-\widehat{A}_N\boldsymbol{\xi}\right|^2=\mathsf{E}\left|A_N\boldsymbol{\xi}+V_N\boldsymbol{\xi}-\widehat{B}_N\boldsymbol{\xi}\right|^2\\&=\mathsf{E}\left|B_N\boldsymbol{\xi}-\widehat{B}_N\boldsymbol{\xi}\right|^2=\Delta\left(F,\widehat{B}_N\boldsymbol{\xi}\right).\end{aligned} \tag{10}$$

Denote by $H^{(N+\mu n)-}\big(\boldsymbol{\xi}_\mu^{(n)}\big)$ the closed linear subspace in the space $H=L_2(\Omega,\mathcal{F},P)$ of second-order random variables generated by

$$\big\{\xi_p^{(n)}(l,\mu),\ p=1,\dots,T,\ l\in\mathbb{Z}\setminus\{0,1,\dots,N+\mu n\}\big\}.$$

Similarly, by $L_2^{(N+\mu n)-}(F)$ we denote the subspace generated in the space $L_2(F)$ by the functions

$$e^{i\lambda l}(1-e^{-i\lambda\mu})^n\boldsymbol{\delta}_p/(i\lambda)^n,\qquad \boldsymbol{\delta}_p=\{\delta_{pk}\}_{p=1}^{T},$$
$$k=1,2,\dots,T,\qquad l\in\mathbb{Z}\setminus\{0,1,\dots,N+\mu n\}.$$

Here δ_{pk} denotes the Kronecker delta symbol.

There is a one-to-one correspondence between the elements $\xi_p^{(n)}(l,\mu)$ of the space $H^{(N+\mu n)-}\big(\boldsymbol{\xi}_\mu^{(n)}\big)$ and the elements

$$e^{i\lambda l}\left(1-e^{-i\lambda\mu}\right)^n\delta_p/(i\lambda)^n$$

of the space $L_2^{(N+\mu n)-}(F)$. This correspondence is defined by relation (3).

We search for a linear estimate $\widehat{B}_N\boldsymbol{\xi}$ of the form

$$\widehat{B}_N\boldsymbol{\xi}=\int_{-\pi}^{\pi}\boldsymbol{h}(\lambda)^{\top}\,d\boldsymbol{Z}(\lambda), \tag{11}$$

where $\boldsymbol{h}(\lambda)=\{h_p(\lambda)\}_{p=1}^{T}$ is the spectral characteristic of the estimate. The optimal estimate $\widehat{B}_N\boldsymbol{\xi}$ is the projection of the element $B_N\boldsymbol{\xi}$ of the space H onto the subspace

$H^{(N+\mu n)-}\big(\boldsymbol{\xi}_{\mu}^{(n)}\big)$. The optimal estimate is determined by the conditions

$$\widehat{B}_N\boldsymbol{\xi} \in H^{(N+\mu n)-}\big(\boldsymbol{\xi}_{\mu}^{(n)}\big), \tag{12}$$

$$B_N\boldsymbol{\xi} - \widehat{B}_N\boldsymbol{\xi} \perp H^{(N+\mu n)-}\big(\boldsymbol{\xi}_{\mu}^{(n)}\big). \tag{13}$$

Condition (13) implies that

$$\mathsf{E}\left[\left(B_N\boldsymbol{\xi} - \widehat{B}_N\boldsymbol{\xi}\right)\overline{\xi_p^{(n)}(l,\mu)}\right] = 0 \tag{14}$$

for all $p = 1,\dots,T$ and $l \in \mathbb{Z}\setminus\{0,1,\dots,N+\mu n\}$. Since

$$B_N\boldsymbol{\xi} = \sum_{j=0}^{N}\boldsymbol{b}(j)^{\top}\boldsymbol{\xi}^{(n)}(j,\mu) = \sum_{j=0}^{N}\boldsymbol{b}(j)^{\top}\int_{-\pi}^{\pi} e^{ij\lambda}\left(1-e^{-i\mu\lambda}\right)^n\frac{1}{(i\lambda)^n}\,d\boldsymbol{Z}(\lambda)$$
$$= \int_{-\pi}^{\pi}\sum_{j=0}^{N}\boldsymbol{b}(j)^{\top}e^{ij\lambda}\frac{\left(1-e^{-i\mu\lambda}\right)^n}{(i\lambda)^n}\,d\boldsymbol{Z}(\lambda) = \int_{-\pi}^{\pi}\boldsymbol{B}_N^{\mu}\left(e^{i\lambda}\right)^{\top}\frac{\left(1-e^{-i\mu\lambda}\right)^n}{(i\lambda)^n}\,d\boldsymbol{Z}(\lambda),$$

where

$$\boldsymbol{B}_N^{\mu}(e^{i\lambda}) = \sum_{j=0}^{N}\boldsymbol{b}(j)e^{ij\lambda},$$

equality (14) is rewritten in the form

$$\mathsf{E}\left[\int_{-\pi}^{\pi}\left(\boldsymbol{B}_N^{\mu}\left(e^{i\lambda}\right)^{\top}\frac{\left(1-e^{-i\mu\lambda}\right)^n}{(i\lambda)^n} - \boldsymbol{h}(\lambda)^{\top}\right)d\boldsymbol{Z}(\lambda)\int_{-\pi}^{\pi}e^{-il\lambda}\frac{\left(1-e^{i\mu\lambda}\right)^n}{(-i\lambda)^n}\,d\overline{\boldsymbol{Z}(\lambda)}\right] = 0,$$
$$l \in \mathbb{Z}\setminus\{0,1,\dots,N+\mu n\}.$$

Thus

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\boldsymbol{B}_N^{\mu}\left(e^{i\lambda}\right)\frac{\left(1-e^{-i\mu\lambda}\right)^n}{(i\lambda)^n} - \boldsymbol{h}(\lambda)\right)^{\top}F(\lambda)\frac{\left(1-e^{i\mu\lambda}\right)^n}{(-i\lambda)^n}e^{-il\lambda}\,d\lambda = 0,$$
$$l \in \mathbb{Z}\setminus\{0,1,\dots,N+\mu n\}.$$

The latter condition implies

$$\left(\boldsymbol{B}_N^{\mu}\left(e^{i\lambda}\right)\frac{\left(1-e^{-i\mu\lambda}\right)^n}{(i\lambda)^n} - \boldsymbol{h}(\lambda)\right)^{\top}F(\lambda)\frac{\left(1-e^{i\mu\lambda}\right)^n}{(-i\lambda)^n} = \boldsymbol{C}_{N+\mu n}\left(e^{i\lambda}\right)^{\top},$$

$$\boldsymbol{C}_{N+\mu n}\left(e^{i\lambda}\right) = \sum_{j=0}^{N+\mu n}\boldsymbol{c}(j)e^{ij\lambda},$$

where $\boldsymbol{c}(j) = \{c_p(j)\}_{p=1}^{T}$ are unknown coefficients to be determined. The latter equality allows us to find the spectral characteristic of the estimate,

$$\boldsymbol{h}(\lambda)^{\top} = \boldsymbol{B}_N^{\mu}\left(e^{i\lambda}\right)^{\top}\frac{\left(1-e^{-i\lambda\mu}\right)^n}{(i\lambda)^n} - \frac{(-i\lambda)^n\boldsymbol{C}_{N+\mu n}\left(e^{i\lambda}\right)^{\top}}{\left(1-e^{i\lambda\mu}\right)^n}F^{-1}(\lambda).$$

Our aim is to derive a system of equations that determine unknown coefficients $\boldsymbol{c}(j) = \{c_p(j)\}_{p=1}^{T}$. It follows from condition (12) that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\boldsymbol{B}_N^{\mu}\left(e^{i\lambda}\right)^{\top} - \frac{\lambda^{2n}\boldsymbol{C}_{N+\mu n}\left(e^{i\lambda}\right)^{\top}}{\left(1-e^{-i\lambda\mu}\right)^n\left(1-e^{i\lambda\mu}\right)^n}F^{-1}(\lambda)\right]e^{-ij\lambda}\,d\lambda = 0$$

for all $j = 0, \dots, N + \mu n$. Now we transform the latter equality as follows for $j = 0, \dots, N + \mu n$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \boldsymbol{B}_N^{\mu}\left(e^{i\lambda}\right)^{\top} e^{-ij\lambda}\, d\lambda = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda^{2n}\boldsymbol{C}_{N+\mu n}\left(e^{i\lambda}\right)^{\top}}{\left(1-e^{-i\lambda\mu}\right)^{n}\left(1-e^{i\lambda\mu}\right)^{n}} F^{-1}(\lambda)e^{-ij\lambda}\, d\lambda. \tag{15}$$

Assume that the spectral density $F(\lambda)$ satisfies the condition of minimality,

$$\int_{-\pi}^{\pi} \operatorname{Tr}\left[\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}} F^{-1}(\lambda)\right] d\lambda < \infty, \tag{16}$$

where $\operatorname{Tr}[A]$ is the trace of the matrix A. Note that this condition is equivalent to the statement that an error-free estimate of an unknown value of a sequence does not exist in the problem of interpolation (see [25]).

Now we consider the Fourier coefficients of the vector function

$$\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}} F^{-1}(\lambda).$$

Put

$$D(j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}} F^{-1}(\lambda)^{\top} e^{-ij\lambda}\, d\lambda, \qquad j = 0, \dots, N+\mu n.$$

Then we derive a system of equations from equality (15) that determine unknown coefficients $\boldsymbol{c}(j) = \{c_p(j)\}_{p=1}^{T}$, $j = 0, \dots, N+\mu n$, namely

$$\begin{aligned}
\boldsymbol{b}(0) &= D(0)\boldsymbol{c}(0) + D(-1)\boldsymbol{c}(1) + \dots + D(-N-\mu n)\boldsymbol{c}(N+\mu n),\\
\boldsymbol{b}(1) &= D(1)\boldsymbol{c}(0) + D(0)\boldsymbol{c}(1) + \dots + D(1-N-\mu n)\boldsymbol{c}(N+\mu n),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\boldsymbol{b}(N+\mu n) &= D(N+\mu n)\boldsymbol{c}(0) + D(N+\mu n-1)\boldsymbol{c}(1) + \dots + D(0)\boldsymbol{c}(N+\mu n).
\end{aligned}$$

Put $\boldsymbol{b}(j) = \boldsymbol{0}$ for $j = N+1, N+2, \dots, N+\mu n$ and denote by

$$\boldsymbol{b}_{N+\mu n} = \{\boldsymbol{b}(j)\}_{j=0}^{N+\mu n} \quad \text{and} \quad \boldsymbol{c}_{N+\mu n} = \{\boldsymbol{c}(j)\}_{j=0}^{N+\mu n}$$

the vectors of dimension $(N+\mu n+1)T$, and by $D_{N+\mu n}$ the matrix of sizes

$$(N+\mu n+1)T \times (N+\mu n+1)T$$

constituted of the matrix blocks of sizes $T \times T$,

$$D_{N+\mu n} = \begin{pmatrix} D(0) & D(-1) & \dots & D(-N-\mu n)\\ D(1) & D(0) & \dots & D(1-N-\mu n)\\ \dots & \dots & \dots & \dots\\ D(N+\mu n) & D(N+\mu n-1) & \dots & D(0)\end{pmatrix}.$$

Then we transform the preceding system of equations to the form

$$\boldsymbol{b}_{N+\mu n} = D_{N+\mu n}\boldsymbol{c}_{N+\mu n},$$

whence we derive the expression for evaluating unknown coefficients $c_p(j)$,

$$\boldsymbol{c}_{N+\mu n} = D_{N+\mu n}^{-1}\boldsymbol{b}_{N+\mu n}. \tag{17}$$

Using the relations obtained above we derive the formula for evaluating the spectral characteristic $\boldsymbol{h}(\lambda) = \{h_p(\lambda)\}_{p=1}^{T}$ of the optimal estimate $\widehat{B}_N\boldsymbol{\xi}$,

$$\begin{aligned}
\boldsymbol{h}(\lambda)^{\top} &= \boldsymbol{B}_N^{\mu}\left(e^{i\lambda}\right)^{\top} \frac{\left(1-e^{-i\lambda\mu}\right)^{n}}{(i\lambda)^{n}}\\
&\quad - \frac{(-i\lambda)^{n}\left[\sum_{j=0}^{N+\mu n}\left(D_{N+\mu n}^{-1}\boldsymbol{b}_{N+\mu n}\right)_j e^{ij\lambda}\right]^{\top}}{\left(1-e^{i\lambda\mu}\right)^{n}} F^{-1}(\lambda).
\end{aligned} \tag{18}$$

The mean square error of the estimate is given by

$$\begin{aligned}\Delta\left(F,\widehat{B}_N\boldsymbol{\xi}\right) &= \mathsf{E}\left|B_N\boldsymbol{\xi}-\widehat{B}_N\boldsymbol{\xi}\right|^2 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(i\lambda)^n\left[\sum_{j=0}^{N+\mu n}\left(D^{-1}_{N+\mu n}\boldsymbol{b}_{N+\mu n}\right)_j e^{ij\lambda}\right]^\top}{(1-e^{i\lambda\mu})^n} \\ &\qquad\times F^{-1}(\lambda)\frac{(-i\lambda)^n\overline{\left[\sum_{j=0}^{N+\mu n}\left(D^{-1}_{N+\mu n}\boldsymbol{b}_{N+\mu n}\right)_j e^{ij\lambda}\right]}}{(1-e^{-i\lambda\mu})^n}\,d\lambda \\ &= \langle\boldsymbol{c}_{N+\mu n},D_{N+\mu n}\boldsymbol{c}_{N+\mu n}\rangle = \left\langle D^{-1}_{N+\mu n}\boldsymbol{b}_{N+\mu n},\boldsymbol{b}_{N+\mu n}\right\rangle.\end{aligned} \tag{19}$$

The results obtained above are collected in the following theorem.

Theorem 3.1. *Let a vector random sequence* $\{\boldsymbol{\xi}(m), m\in\mathbb{Z}\}$ *define the stationary increment* $\boldsymbol{\xi}^{(n)}(m,\mu)$ *of order* n *with the matrix of spectral densities*

$$F(\lambda)=\{f_{ij}(\lambda)\}_{i,j=1}^{T}$$

that satisfies condition (16).

The optimal linear estimate $\widehat{B}_N\boldsymbol{\xi}$ *of the functional* $B_N\boldsymbol{\xi}$ *that depends on unknown values* $\boldsymbol{\xi}^{(n)}(m,\mu)$, $m\in\{0,1,\dots,N\}$, *constructed from observations of the sequence* $\boldsymbol{\xi}(m)$, $m\in\mathbb{Z}\setminus\{0,1,\dots,N\}$, *is defined by relation* (11).

The spectral characteristic $\boldsymbol{h}(\lambda)=\{h_p(\lambda)\}_{p=1}^{T}$ *of the optimal estimate* $\widehat{B}_N\boldsymbol{\xi}$ *is given by equality* (18).

The mean square error $\Delta\big(F,\widehat{B}_N\boldsymbol{\xi}\big)$ *is obtained from equality* (19).

As a consequence of Theorem 3.1, one can construct an estimate of an unknown value of the increment $\xi_p^{(n)}(m,\mu)$, $m=0,1,\dots,N$, $p=1,2,\dots,T$, from observations of the sequence $\boldsymbol{\xi}(m)$, $m\in\mathbb{Z}\setminus\{0,1,\dots,N\}$. This estimate is constructed from the vector $\boldsymbol{b}(m)=e_p$ whose coordinate p is equal to 1, while all other coordinates are equal to 0. The remaining vectors

$$\boldsymbol{b}(j),\qquad j=0,1,\dots,N,\ j\neq m,$$

are equal to 0. Then $\boldsymbol{b}_{N+\mu n}=e_{mT+p}$ is the vector whose coordinate $(mT+p)$ is equal to 1 and all other coordinates are equal to 0.

Now the optimal linear estimate $\widehat{\xi}_p^{(n)}(m,\mu)$ of an unknown increment $\xi_p^{(n)}(m,\mu)$ is determined by the relation

$$\widehat{\xi}_p^{(n)}(m,\mu)=\int_{-\pi}^{\pi}\boldsymbol{\varphi}_m(\lambda,\mu)^\top\,d\boldsymbol{Z}(\lambda). \tag{20}$$

The spectral characteristic $\boldsymbol{\varphi}_m(\lambda,\mu)$ of the optimal estimate $\widehat{\xi}_p^{(n)}(m,\mu)$ can be found from the relation

$$\begin{aligned}\boldsymbol{\varphi}_m(\lambda,\mu)^\top &= e^\top_{mT+p}\,e^{im\lambda}\,\frac{\left(1-e^{-i\lambda\mu}\right)^n}{(i\lambda)^n} \\ &\quad-\frac{(-i\lambda)^n\left[\sum_{j=0}^{N+\mu n}\left(D^{-1}_{N+\mu n}e_{mT+p}\right)_j e^{ij\lambda}\right]^\top}{\left(1-e^{i\lambda\mu}\right)^n}F^{-1}(\lambda).\end{aligned} \tag{21}$$

To evaluate the mean square error of the estimate one can use the following formula:

$$\begin{aligned}
\Delta\left(F,\xi_p^{(n)}(m,\mu)\right) &= \langle \tilde{\boldsymbol{c}}_{N+\mu n}, D_{N+\mu n}\tilde{\boldsymbol{c}}_{N+\mu n}\rangle \\
&= \left\langle D_{N+\mu n}^{-1}e_{mT+p}, D_{N+\mu n}D_{N+\mu n}^{-1}e_{mT+p}\right\rangle \\
&= \left\langle D_{N+\mu n}^{-1}e_{mT+p}, e_{mT+p}\right\rangle = \left(D_{N+\mu n}^{-1}\right)_{mT+p,mT+p},
\end{aligned} \tag{22}$$

where

$$\tilde{\boldsymbol{c}}_{N+\mu n} = \{\tilde{\boldsymbol{c}}(j)\}_{j=0}^{N+\mu n} = D_{N+\mu n}^{-1}e_{mT+p}.$$

Therefore we obtain the following result for the estimate $\widehat{\xi}_p^{(n)}(m,\mu)$ of an unknown increment $\xi_p^{(n)}(m,\mu)$.

Corollary 3.1. *The optimal linear estimate $\widehat{\xi}_p^{(n)}(m,\mu)$ of the increment $\xi_p^{(n)}(m,\mu)$, $m = 0,1,\dots,N$, $p = 1,2,\dots,T$, constructed from observations of the sequence $\boldsymbol{\xi}(m)$, $m \in \mathbb{Z}\setminus\{0,1,\dots,N\}$, is defined by formula* (20). *The spectral characteristic $\boldsymbol{\varphi}_m(\lambda,\mu)$ of the optimal estimate $\widehat{\xi}_p^{(n)}(m,\mu)$ is found from equality* (21). *The mean square error is evaluated according to the relation*

$$\Delta\left(F,\xi_p^{(n)}(m,\mu)\right) = \left(D_{N+\mu n}^{-1}\right)_{mT+p,mT+p}. \tag{23}$$

Theorem 3.1 and equalities (10) and (19) yield the following result.

Theorem 3.2. *Let a vector random sequence $\{\boldsymbol{\xi}(m), m \in \mathbb{Z}\}$ define the stationary increment $\boldsymbol{\xi}^{(n)}(m,\mu)$ of order n whose matrix of spectral densities is given by*

$$F(\lambda) = \{f_{ij}(\lambda)\}_{i,j=1}^{T}.$$

Assume that F satisfies condition (16). *The optimal linear estimate $\widehat{A}_N\boldsymbol{\xi}$ of the functional $A_N\boldsymbol{\xi}$ that depends on unknown elements $\boldsymbol{\xi}(m)$, $m \in \{0,1,\dots,N\}$, is constructed from observations of the sequence $\boldsymbol{\xi}(m)$, $m \in \mathbb{Z}\setminus\{0,1,\dots,N\}$, as follows:*

$$\widehat{A}_N\boldsymbol{\xi} = \int_{-\pi}^{\pi}\boldsymbol{h}(\lambda)^{\top}\,d\boldsymbol{Z}(\lambda) - \sum_{j=-\mu n}^{-1}\boldsymbol{v}(j)^{\top}\boldsymbol{\xi}(j),$$

where the spectral characteristic $\boldsymbol{h}(\lambda) = \{h_p(\lambda)\}_{p=1}^{T}$ of the optimal estimate is found from equality (18) *and $\boldsymbol{v}(j)$ are defined by* (7). *The mean square error of the estimate $\Delta\big(F,\widehat{A}_N\boldsymbol{\xi}\big)$ is given by*

$$\Delta\left(F,\widehat{A}_N\boldsymbol{\xi}\right) = \langle \boldsymbol{c}_{N+\mu n}, D_{N+\mu n}\boldsymbol{c}_{N+\mu n}\rangle = \left\langle D_{N+\mu n}^{-1}\boldsymbol{b}_{N+\mu n}, \boldsymbol{b}_{N+\mu n}\right\rangle,$$

where $\boldsymbol{c}_{N+\mu n} = \{\boldsymbol{c}(j)\}_{j=0}^{N+\mu n} = D_{N+\mu n}^{-1}\boldsymbol{b}_{N+\mu n}$ and the vector $\boldsymbol{b}_{N+\mu n}$ is constituted by vectors $\boldsymbol{b}(j)$ whose components are defined in relation (8).

Now we turn to the case of a one-dimensional random sequence with periodically stationary increments $\{\zeta(m), m \in \mathbb{Z}\}$. Consider the problem of linear mean square optimal estimation of the functional

$$A_M\zeta = \sum_{k=0}^{M} a^{(\zeta)}(k)\zeta(k) \tag{24}$$

that depends on unknown values of the sequence $\zeta(k)$. The estimate is constructed from observations of the sequence $\zeta(k)$ with $k < 0$ and $k > M$. Assume that $M+1$, the number of unknown values of the sequence $\{\zeta(m), m \in \mathbb{Z}\}$, is divided by the number T, the corresponding period. Let the number N be such that

$$N = \frac{M+1}{T} - 1. \tag{25}$$

Making the change of variables (2) and taking into account the equality $M = (N+1)T-1$ we rewrite the expression defining the functional $A_M\zeta$ as follows:

$$\begin{aligned} A_M\zeta &= \sum_{k=0}^{M} a^{(\zeta)}(k)\zeta(k) = \sum_{j=0}^{N}\sum_{p=1}^{T} a^{(\zeta)}(jT+p-1)\zeta(jT+p-1) \\ &= \sum_{j=0}^{N}\sum_{p=1}^{T} a_p(j)\xi_p(j) = \sum_{j=0}^{N} \boldsymbol{a}(j)^\top \boldsymbol{\xi}(j) = A_N\boldsymbol{\xi}, \end{aligned}$$

where $\boldsymbol{a}(j)^\top = \big(a_1(j), a_2(j), \dots, a_T(j)\big)$ and $a_p(j) = a^{(\zeta)}(jT+p-1)$. Considering the latter equality and Theorem 3.2 one can obtain the optimal linear estimate $\widehat{A}_M\zeta$ of the functional $A_M\zeta$.

Theorem 3.3. *Let $\{\zeta(k), k \in \mathbb{Z}\}$ be a random sequence with periodically stationary increments such that the vector sequence $\{\boldsymbol{\xi}(m), m \in \mathbb{Z}\}$ obtained after the change* (2) *satisfies the assumptions of Theorem* 3.2.

Then the optimal linear estimate $\widehat{A}_M\zeta$ of the functional $A_M\zeta$ that depends on unknown values $\zeta(k)$, $k \in \{0, 1, \dots, M\}$, is constructed from observations of the sequence $\zeta(k)$ with $k < 0$ and $k > M$ according to

$$\widehat{A}_M\zeta = \int_{-\pi}^{\pi} \boldsymbol{h}(\lambda)^\top \, d\boldsymbol{Z}(\lambda) - \sum_{j=-\mu n}^{-1} \boldsymbol{v}(j)^\top \boldsymbol{\xi}(j),$$

where the spectral characteristic $\boldsymbol{h}(\lambda) = \{h_p(\lambda)\}_{p=1}^{T}$ of the optimal estimate $\widehat{A}_{(N+1)T-1}\zeta$ is defined by equality (18) *and $\boldsymbol{v}(j)$ is defined in relation* (7).

The mean square error $\Delta\big(F, \widehat{A}_M\zeta\big)$ is given by

$$\Delta(F, \widehat{A}_M\zeta) = \langle \boldsymbol{c}_{N+\mu n}, D_{N+\mu n}\boldsymbol{c}_{N+\mu n}\rangle = \left\langle D_{N+\mu n}^{-1}\boldsymbol{b}_{N+\mu n}, \boldsymbol{b}_{N+\mu n}\right\rangle,$$

where $\boldsymbol{c}_{N+\mu n} = \{\boldsymbol{c}(j)\}_{j=0}^{N+\mu n} = D_{N+\mu n}^{-1}\boldsymbol{b}_{N+\mu n}$ and the vector $\boldsymbol{b}_{N+\mu n}$ is constituted by vectors $\boldsymbol{b}(j)$ whose elements are defined according to equality (8).

4. Minimax estimates of the functional

The matrix of spectral densities $F(\lambda) = \{f_{ij}(\lambda)\}_{i,j=1}^{T}$ of a stationary increment $\boldsymbol{\xi}^{(n)}(m,\mu)$ of order n is needed to use the theorems and equalities of the preceding section. If the spectral density is unknown but a set $\mathcal{D}$ of admissible densities is specified, then one applies the minimax method for estimating a functional. Following this method, one finds an estimate that minimizes the mean square error for all spectral densities of a given set $\mathcal{D}$.

Definition 4.1. Given a set $\mathcal{D}$, a spectral density $F^0(\lambda) \in \mathcal{D}$ is called the least favorable in the class $\mathcal{D}$ for the optimal estimation of a functional $A_N\boldsymbol{\xi}$ if

$$\Delta\left(F^0\right) = \Delta\left(\boldsymbol{h}\left(F^0\right); F^0\right) = \max_{F\in\mathcal{D}} \Delta\big(\boldsymbol{h}(F); F\big).$$

Definition 4.2. Given a set $\mathcal{D}$, a spectral characteristic $\boldsymbol{h}^0(\lambda)$ of the optimal estimate of a functional $A_N\boldsymbol{\xi}$ is called minimax (robust) if

$$\boldsymbol{h}^0(\lambda) \in H_{\mathcal{D}} = \bigcap_{F\in\mathcal{D}} L_2^{N-}(F),$$

$$\min_{h\in H_{\mathcal{D}}} \max_{F\in\mathcal{D}} \Delta(\boldsymbol{h}; F) = \max_{F\in\mathcal{D}} \Delta(\boldsymbol{h}^0; F).$$

The following result follows from the results obtained in the preceding section and uses our definitions of the least favorable spectral density and minimax spectral characteristic.

Lemma 4.1. *A spectral density $F^0(\lambda) \in \mathcal{D}$ that satisfies the condition of minimality* (16) *is the least favorable in the class $\mathcal{D}$ for the optimal linear estimation of a functional $A_N\xi$ from observations $\xi(m)$ with $m \in \mathbb{Z}\setminus\{0,1,2,\dots,N\}$ if the matrix $D^0_{N+\mu n}$ constituted by the Fourier coefficients of the function*

$$\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}} \left(F^0\right)^{-1}(\lambda)$$

determines a solution of the conditional extremum problem

$$\max_{F\in\mathcal{D}} \left\langle D^{-1}_{N+\mu n}\boldsymbol{b}_{N+\mu n}, \boldsymbol{b}_{N+\mu n}\right\rangle = \left\langle \left(D^0_{N+\mu n}\right)^{-1}\boldsymbol{b}_{N+\mu n}, \boldsymbol{b}_{N+\mu n}\right\rangle. \tag{26}$$

The minimax spectral characteristic $\boldsymbol{h}^0 = \boldsymbol{h}(F^0)$ is given by equality (18) *if $\boldsymbol{h}(F^0) \in H_{\mathcal{D}}$.*

A more careful analysis of properties of the least favorable spectral densities and minimax spectral characteristic of the optimal estimate of a functional is based on an observation that the minimax spectral characteristic $\boldsymbol{h}^0$ and least favorable spectral density F^0 form a saddle point of the function $\Delta(\boldsymbol{h}; F)$ in the set $H_{\mathcal{D}}\times\mathcal{D}$. The saddle point inequalities

$$\Delta\left(\boldsymbol{h}; F^0\right) \geq \Delta\left(\boldsymbol{h}^0; F^0\right) \geq \Delta\left(\boldsymbol{h}^0; F\right) \qquad \forall F\in\mathcal{D},\ \forall \boldsymbol{h}\in H_{\mathcal{D}}$$

hold if $\boldsymbol{h}^0 = \boldsymbol{h}\left(F^0\right)$ and $\boldsymbol{h}\left(F^0\right) \in H_{\mathcal{D}}$, where F^0 is a solution of the conditional extremum problem

$$\widetilde{\Delta}(F) = -\Delta\left(\boldsymbol{h}\left(F^0\right); F\right) \to \inf, \qquad F\in\mathcal{D},$$

$$\begin{aligned}\Delta&\left(\boldsymbol{h}\left(F^0\right); F\right)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(i\lambda)^n\left[\sum_{j=0}^{N+\mu n}\left(\left(D^0_{N+\mu n}\right)^{-1}\boldsymbol{b}_{N+\mu n}\right)_j e^{ij\lambda}\right]^{\top}}{\left(1-e^{i\lambda\mu}\right)^n}\left(F^0(\lambda)\right)^{-1}\\ &\quad\times F(\lambda)\left(F^0(\lambda)\right)^{-1}\frac{\overline{(-i\lambda)^n\left[\sum_{j=0}^{N+\mu n}\left(\left(D^0_{N+\mu n}\right)^{-1}\boldsymbol{b}_{N+\mu n}\right)_j e^{ij\lambda}\right]}}{\left(1-e^{-i\lambda\mu}\right)^n}\,d\lambda.\end{aligned} \tag{27}$$

The latter problem is equivalent to the unconditional extremum problem

$$\Delta_{\mathcal{D}}(F) = \widetilde{\Delta}(F) + \delta(F\mid\mathcal{D}) \to \inf, \tag{28}$$

where $\delta(F\mid\mathcal{D})$ is the indicator function of the set $\mathcal{D}$. The solution F^0 of the unconditional extremum problem is characterized by the condition $0\in\partial\Delta_{\mathcal{D}}(F^0)$ which is necessary and sufficient for the function F^0 to belong to the set of minimums of the functional $\Delta_{\mathcal{D}}(F)$ (see [15, 24]). The expression $\partial\Delta_{\mathcal{D}}\left(F^0\right)$ here denotes the subdifferential of the functional $\Delta_{\mathcal{D}}(F)$ at the point $F = F^0$.

The expression (27) for the functional $\Delta\big(\boldsymbol{h}\left(F^0\right); F\big)$ is convenient for applying the Lagrange multipliers method in the extremum problem (28). Following the Lagrange multipliers method and using the expressions for subdifferentials of indicator functions for sets of the admissible densities, one can find relations that determine the least favorable spectral densities and minimax spectral characteristics of the optimal estimate in the case of some specific sets of admissible spectral densities.

5. Least favorable spectral densities in the class $\mathcal{D}_{0,n}^{-}$

Consider the problem of the minimax estimation of a functional $\widehat{A}_N\boldsymbol{\xi}$ by using the observations $\boldsymbol{\xi}(j)$, $j\in\mathbb{Z}\setminus\{0,1,\dots,N\}$, under the condition that the spectral density $F(\lambda)$ of a stationary increment $\boldsymbol{\xi}^{(n)}(m,\mu)$ of order n belongs to the set

$$\mathcal{D}_{0,n}^{-}=\left\{F(\lambda)\;\middle|\;\frac{1}{2\pi}\int_{-\pi}^{\pi}\lambda^{2n}\left[F(\lambda)\right]^{-1}d\lambda=P\right\},$$

where $P=\{p_{ij}\}_{i,j=1}^{T}$ is a given matrix.

The condition $0\in\partial\Delta_{\mathcal{D}}\left(F^0\right)$ for $\mathcal{D}=\mathcal{D}_{0,n}^{-}$ implies the following relation for the least favorable spectral density in a given set:

$$\frac{1}{|1-e^{i\lambda\mu}|^{2n}}\left[\sum_{j=0}^{N+\mu n}\left(\left(D^0_{N+\mu n}\right)^{-1}\boldsymbol{b}_{N+\mu n}\right)_j e^{ij\lambda}\right]\overline{\times\left[\sum_{j=0}^{N+\mu n}\left(\left(D^0_{N+\mu n}\right)^{-1}\boldsymbol{b}_{N+\mu n}\right)_j e^{ij\lambda}\right]^{\top}}=\boldsymbol{\alpha}\overline{\boldsymbol{\alpha}^{\top}}, \tag{29}$$

where $\boldsymbol{\alpha}=\{\alpha_p\}_{p=1}^{T}$ is the vector of Lagrange multipliers.

Denote by $\boldsymbol{s}^{(\alpha)}(j)$, $j=0,\dots,N+\mu n$, the vector of dimension T, where

$$\boldsymbol{s}^{(\alpha)}(k\mu)=\boldsymbol{\alpha}(-1)^k\binom{n}{k},\quad k=0,\dots,n,\qquad \boldsymbol{s}^{(\alpha)}(j)=0,\quad j\neq k\mu.$$

Put $\boldsymbol{s}^{(\alpha)}_{N+\mu n}=\left\{\boldsymbol{s}^{(\alpha)}(j)\right\}_{j=0}^{N+\mu n}$ (this is a vector of dimension $(N+\mu n+1)T$). Then we derive the following system of equations from relation (29):

$$D^0_{N+\mu n}\boldsymbol{s}^{(\alpha)}_{N+\mu n}=\boldsymbol{b}_{N+\mu n}. \tag{30}$$

Note that the elements of the vector $\boldsymbol{s}^{(\alpha)}_{N+\mu n}$ are such that $\boldsymbol{s}^{(\alpha)}(j)=0$ for $j>\mu n$ and $\boldsymbol{s}^{(\alpha)}(j)=(-1)^{\mu n}\boldsymbol{s}^{(\alpha)}(\mu n-j)$ for $j\leqslant\mu n$. The entries of the matrix $D^0_{N+\mu n}$ are such that $D^0(j)=D^0(-j)$, $j=0,\dots,N+\mu n$. The following conditions are sufficient for system (30) to be solvable:

$$\boldsymbol{b}(j)=(-1)^{\mu n}\boldsymbol{b}(\mu n-j),\qquad j=0,\dots,\mu n. \tag{31}$$

The constraints imposed on the functions constituting the class $\mathcal{D}_{0,n}^{-}$ yield the equation

$$\sum_{j=-N-\mu n}^{N+\mu n}\frac{D^0(|j|)}{2\pi}\int_{-\pi}^{\pi}e^{ij\lambda}\left|1-e^{i\lambda\mu}\right|^{2n}d\lambda=P. \tag{32}$$

Therefore the least favorable spectral density satisfies the relation

$$\frac{\lambda^{2n}}{|1-e^{i\lambda\mu}|^{2n}}\left[F^0(\lambda)\right]^{-1}=\sum_{j=-N-\mu n}^{N+\mu n}D^0(|j|)e^{ij\lambda}, \tag{33}$$

where equalities (30) and (32) hold for the matrices $D^{(0)}(j), j = 0, \dots, N + \mu n$. The class of the least favorable densities is of the form

$$\mathcal{R} = \left\{ F^0(\lambda) \in \mathcal{D}_{0,n}^- \colon F^0(\lambda) = \frac{\lambda^{2n}}{|1 - e^{i\lambda\mu}|^{2n}} \left(\sum_{j=-N-\mu n}^{N+\mu n} D^0(j) e^{ij\lambda} \right)^{-1} \right\}. \tag{34}$$

Therefore we proved the following result.

Theorem 5.1. *Let a vector random sequence* $\{\boldsymbol{\xi}(m), m \in \mathbb{Z}\}$ *determine a stationary increment of order* n. *Assume that the matrices* $D^{(0)}(j)$, $j = 0, \dots, N + \mu n$, *satisfy equalities* (29), (30), (31), *and* (32).

If the sequence $\boldsymbol{a}(0), \dots, \boldsymbol{a}(N)$ *satisfies condition* (31), *then the set of the least favorable in the class* $\mathcal{D}_{0,n}^-$ *spectral densities used to construct the optimal linear estimate of the functional* $A_N\boldsymbol{\xi}$ *by observations of the sequence* $\boldsymbol{\xi}(m)$ *at points* $m \in \mathbb{Z} \setminus \{0, 1, \dots, N\}$ *is of the form* (34). *The minimax spectral characteristic of the optimal estimate is evaluated according to equality* (18).

6. Concluding remarks

The problem of the mean square optimal linear estimation of the functional

$$A_N\xi = \sum_{k=0}^{N} a(k)\xi(k)$$

is studied in the paper. The functional depends on unknown values of the random sequence $\xi(k)$ with periodically stationary increments. An estimate is constructed from observations of this sequence at the points belonging to the set $\mathbb{Z} \setminus \{0, 1, \dots, N\}$.

Both methods of estimation, the classical and minimax (robust), are used for the case of spectral definiteness where the spectral density of the sequence is known as well as for the case of spectral indefiniteness where the spectral density of the sequence is unknown but a set of admissible densities is specified. In particular, we found the expressions for the mean square error and for spectral characteristic of the optimal estimate of the functional for the case of a known spectral density. If the spectral density is unknown but a class of admissible spectral densities is specified, then we derive a relation that determines the least favorable spectral densities and minimax spectral characteristics.

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