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ASYMPTOTIC EXPANSIONS FOR POWER-EXPONENTIAL MOMENTS OF HITTING TIMES FOR NONLINEARLY PERTURBED SEMI-MARKOV PROCESSES

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Abstract. New algorithms for construction of asymptotic expansions for exponential and power-exponential moments of hitting times for nonlinearly perturbed semi-Markov processes are presented. The algorithms are based on special techniques of sequential phase space reduction and the systematical use of operational calculus for Laurent asymptotic expansions applied to moments of hitting times for perturbed semi-Markov processes. These algorithms have a universal character. They can be applied to nonlinearly perturbed semi-Markov processes with an arbitrary asymptotic communicative structure of a phase space. Asymptotic expansions are given in two forms, without and with explicit bounds for remainders. The algorithms are computationally effective, due to a recurrent character of the corresponding computational procedures.

1. Introduction

We present new algorithms for construction of asymptotic expansions, without and with explicit upper bounds for remainders, for exponential and power-exponential moments of hitting times for nonlinearly perturbed semi-Markov processes with finite phase spaces.

Hitting times are also known under such names as first-rare-event times, first passage times, and absorption times, in theoretical studies, and as lifetimes, failure times, extinction times, etc., in applications. These random functionals and their moments play an important role in the theory of semi-Markov processes. We refer to books [1]–[12] and [15]–[20] containing results related to asymptotic expansions for perturbed Markov chains and semi-Markov processes, including results concerning hitting times, as well as their applications to asymptotic analysis of reliability, queuing, bio-stochastic systems, information networks, and other models of perturbed stochastic processes and systems. Also, we would like to mention the recent paper [14], where one can find a comprehensive bibliography of works in the area and the corresponding bibliographical remarks.

We consider models where the phase space for embedded Markov chains of pre-limiting perturbed semi-Markov processes is one class of communicative states, while it can asymptotically split in one or several closed classes of communicative states and, possibly, a class of transient states.

The initial perturbation conditions are formulated in the forms of Laurent asymptotic expansions for power-exponential moments of transition times for perturbed semi-Markov processes given in two alternative forms, without or with explicit upper bounds

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for remainders. The algorithms are based on special time-space screening procedures for sequential phase space reduction and algorithms for recalculation of asymptotic expansions and upper bounds for remainders, which constitute perturbation conditions for the semi-Markov processes with reduced phase spaces. The final asymptotic expansions for exponential and power-exponential moments of hitting times for nonlinearly perturbed semi-Markov processes are given in the form of Laurent asymptotic expansions, without or with explicit upper bounds for remainders.

The present paper continues the line of research of the book [4] and the authors' recent works [14] and [15]. The book [4] contains a detailed presentation of results related to the asymptotic analysis of quasi-stationary distributions for nonlinearly perturbed semi-Markov processes, where the power-exponential moments of hitting times play the central role. In this book, asymptotic expansions for power-exponential moments have been obtained for the nonsingularly perturbed semi-Markov processes with the simple asymptotic communicative structure of the set of nonabsorbing states, which, in this case, consists of one communicative class plus possibly a class of transient states. However, the method (based on asymptotic analysis of generalized matrix inverses) used in this book does not work well for the more complex model of singularly perturbed semi-Markov processes, where the set of nonabsorbing states has a more complex asymptotic structure and can asymptotically split in several closed communicative classes of states plus possibly a class of transient states. In this case, moments of hitting times can be asymptotically unbounded functions of perturbation parameter due to the presence of asymptotically absorbing states or subsets of states. Their asymptotic analysis, with the use of the generalized matrix inverses, becomes rather intricate. Also, the only asymptotic expansions with remainders given in the standard form of $o(\cdot)$ have been given in this book. In works [14] and [15], asymptotic expansions are obtained for singularly perturbed semi-Markov processes, with remainders without and with explicit upper bounds for remainders, but only for simpler power moments of hitting times.

In the present paper, we get asymptotic expansions for more complex power-exponential moments of hitting times for nonlinearly and singularly perturbed semi-Markov processes. An important novelty of results presented in the paper is that the corresponding asymptotic expansions are obtained with remainders given not only in the standard form of $o(\cdot)$, but, also, in a more advanced form, with explicit power-type upper bounds for remainders asymptotically uniform with respect to the perturbation parameter. The latter asymptotic expansions for power-exponential moments of hitting times for nonlinearly perturbed semi-Markov processes were not known before.

The corresponding computational algorithms have a universal character. They can be applied to perturbed semi-Markov processes with an arbitrary asymptotic communicative structure of phase spaces and are computationally effective due to the recurrent character of computational procedures.

2. Laurent asymptotic expansions

Let $A(\varepsilon)$ be a real-valued function defined on an interval $(0, \varepsilon_0]$ for some $0 < \varepsilon_0 \leq 1$, and given on this interval by a Laurent asymptotic expansion,

$$A(\varepsilon) = a_{h_A}\varepsilon^{h_A} + \cdots + a_{k_A}\varepsilon^{k_A} + o_A\left(\varepsilon^{k_A}\right),$$

where

- **(a)** $-\infty < h_A \leq k_A < \infty$ are integers,
- **(b)** coefficients $a_{h_A}, \ldots, a_{k_A}$ are real numbers,
- **(c)** the function $o_A\left(\varepsilon^{k_A}\right)/\varepsilon^{k_A} \to 0$ as $\varepsilon \to 0$.

We refer to the Laurent asymptotic expansion $A(\varepsilon)$ as an (h_A, k_A)-expansion. We also refer to $A(\varepsilon)$ as an $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$-expansion, if additionally

(d) $\left|o_A\left(\varepsilon^{k_A}\right)\right| \leq G_A \varepsilon^{k_A+\delta_A}$ for $0 < \varepsilon \leq \varepsilon_A$, where

(e) $0 < \delta_A \leq 1, 0 < G_A < \infty$, and $0 < \varepsilon_A \leq \varepsilon_0$.

We say that the Laurent asymptotic expansion $A(\varepsilon)$ is pivotal if it is known that $a_{h_A} \neq 0$.

It is also useful to mention that a constant a can be interpreted as the function $A(\varepsilon) \equiv a$. Thus, 0 can be represented, for any integer $-\infty < h \leq k < \infty$, as the (h, k)-expansion, $0 = 0\varepsilon^h + \cdots + 0\varepsilon^k + o\left(\varepsilon^k\right)$, with remainder $o(\varepsilon^k) \equiv 0$. Also, 1 can be represented, for any integer $0 \leq k < \infty$, as the $(0, k)$-expansion, $1 = 1 + 0\varepsilon + \cdots + 0\varepsilon^k + o\left(\varepsilon^k\right)$, with remainder $o\left(\varepsilon^k\right) \equiv 0$.

Let us consider three Laurent asymptotic expansions,

$$A(\varepsilon) = a_{h_A}\varepsilon^{h_A} + \cdots + a_{k_A}\varepsilon^{k_A} + o_A\left(\varepsilon^{k_A}\right), \qquad B(\varepsilon) = b_{h_B}\varepsilon^{h_B} + \cdots + b_{k_B}\varepsilon^{k_B} + o_B\left(\varepsilon^{k_B}\right),$$

and

$$C(\varepsilon) = c_{h_C}\varepsilon^{h_C} + \cdots + c_{k_C}\varepsilon^{k_C} + o_C\left(\varepsilon^{k_C}\right)$$

defined on the interval $(0, \varepsilon_0]$.

Let us denote $F_A = \max_{h_A \leq i \leq k_A} |a_i|$, $F_B = \max_{h_B \leq i \leq k_B} |b_i|$, $F_C = \max_{h_C \leq i \leq k_C} |c_i|$.

The following lemma presents operational rules for Laurent asymptotic expansions. The corresponding proofs can be found in the authors' works [14] and [15].

Lemma 1. *The following operational rules take place for Laurent asymptotic expansions:*

(i) *If $A(\varepsilon)$ is an (h_A, k_A)-expansion and c is a constant, then $C(\varepsilon) = cA(\varepsilon)$ is an (h_C, k_C)-expansion such that:*

(a) $h_C = h_A$, $k_C = k_A$;

(b) $c_{h_C+r} = c a_{h_C+r}$, $r = 0, \ldots, k_C - h_C$.

This expansion is pivotal if and only if $c_{h_C} = c a_{h_A} \neq 0$.

(ii) *Also, if $A(\varepsilon)$ is an $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$-expansion, then $C(\varepsilon)$ is an $(h_C, k_C, \delta_C, G_C, \varepsilon_C)$-expansion such that:*

(a) $\delta_C = \delta_A$;

(b) $G_C = |c| G_A$;

(c) $\varepsilon_C = \varepsilon_A$.

(iii) *If $A(\varepsilon)$ is an (h_A, k_A)-expansion and $B(\varepsilon)$ is an (h_B, k_B)-expansion, then*

$$C(\varepsilon) = A(\varepsilon) + B(\varepsilon)$$

is an (h_C, k_C)-expansion such that:

(a) $h_C = h_A \wedge h_B$, $k_C = k_A \wedge k_B$;

(b) $c_{h_C+r} = a_{h_C+r} + b_{h_C+r}$, $r = 0, \ldots, k_C - h_C$, *where* $a_{h_C+r} = 0$ *for* $0 \leq r < h_A - h_C$ *and* $b_{h_C+r} = 0$ *for* $0 \leq r < h_B - h_C$.

This expansion is pivotal if and only if $c_{h_C} = a_{h_C} + b_{h_C} \neq 0$.

(iv) *Also, if $A(\varepsilon)$ is an $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$-expansion and $B(\varepsilon)$ is an $(h_B, k_B, \delta_B, G_B, \varepsilon_B)$-expansion, then $C(\varepsilon)$ is an $(h_C, k_C, \delta_C, G_C, \varepsilon_C)$-expansion such that:*

(a) $\delta_C = \delta_A \wedge \delta_B$;

(b) $G_C = G_A + F_A(k_A - k_C) + G_B + F_B(k_B - k_C)$;

(c) $\varepsilon_C = \varepsilon_A \wedge \varepsilon_B$.

(v) *If $A(\varepsilon)$ is an (h_A, k_A)-expansion and $B(\varepsilon)$ is an (h_B, k_B)-expansion, then*

$$C(\varepsilon) = A(\varepsilon) \cdot B(\varepsilon)$$

is an (h_C, k_C)-expansion such that:

(a) $h_C = h_A + h_B$, $k_C = (k_A + h_B) \wedge (k_B + h_A)$;

(b) $c_{h_C+r} = \sum_{0 \leq i \leq r} a_{h_A+i} b_{h_B+r-i}$, $r = 0, \ldots, k_C - h_C$.

This expansion is pivotal if and only if $c_{h_C} = a_{h_A} b_{h_B} \neq 0$.

(vi) *Also, if* $A(\varepsilon)$ *is an* $(h_A,k_A,\delta_A,G_A,\varepsilon_A)$*-expansion and* $B(\varepsilon)$ *is an* $(h_B,k_B,\delta_B,G_B,\varepsilon_B)$*-expansion, then* $C(\varepsilon)$ *is an* $(h_C,k_C,\delta_C,G_C,\varepsilon_C)$*-expansion such that:*
(a) $\delta_C = \delta_A \wedge \delta_B$;
(b)
$$G_C = F_A F_B (k_A - h_A + 1)(k_B - h_B + 1) + G_A F_B (k_B - h_B + 1) + G_B F_A (k_A - h_A + 1) + G_A G_B;$$
(c) $\varepsilon_C = \varepsilon_A \wedge \varepsilon_B$.

(vii) *If* $A(\varepsilon)$ *is an* (h_A, k_A)*-expansion and* $B(\varepsilon)$ *is a pivotal* (h_B, k_B)*-expansion such that* $B(\varepsilon) \neq 0, \varepsilon \in (0, \varepsilon_0]$*, then* $C(\varepsilon) = A(\varepsilon)/B(\varepsilon)$ *is an* (h_C, k_C)*-expansion such that:*
(a) $h_C = h_A - h_B$, $k_C = (k_A - h_B) \wedge (k_B - 2h_B + h_A)$;
(b) $c_{h_C+r} = b_{h_B}^{-1}\big(a_{h_A+r} - \sum_{1\leq i\leq r} b_{h_B+i}c_{h_C+r-i}\big)$, $r = 0, \ldots, k_C - h_C$.
This expansion is pivotal if and only if $c_{h_D} = a_{h_A}/b_{h_B} \neq 0$.

(viii) *Also, if* $A(\varepsilon)$ *is an* $(h_A, k_A, \delta_A, G_A, \varepsilon_A)$*-expansion and* $B(\varepsilon)$ *is a pivotal* $(h_B, k_B, \delta_B, G_B, \varepsilon_B)$*-expansion, then* $C(\varepsilon)$ *is an* $(h_C, k_C, \delta_C, G_C, \varepsilon_C)$*-expansion such that:*
(a) $\delta_C = \delta_A \wedge \delta_B$;
(b)
$$G_C = \left(\frac{|b_{h_B}|}{2}\right)^{-1} \times \big(F_A(k_A - k_A \wedge (h_A + k_B - h_B)) + G_A + F_B F_D (k_B - h_B + 1)(k_D - h_D + 1) + G_B F_D (k_D - h_D + 1)\big);$$
(c) $\varepsilon_C = \varepsilon_A \wedge \varepsilon_B \wedge \big(|b_{h_B}|/(2(F_B(k_B - h_B) + G_B))\big)^{1/\delta_B}$.

3. Perturbed semi-Markov processes

Let $\mathbb{X} = \{0, \ldots, m\}$ and let $(\eta_{\varepsilon,n}, \kappa_{\varepsilon,n})$, $n = 0, 1, \ldots,$ be, for every $\varepsilon \in (0, \varepsilon_0]$, a Markov renewal process, i.e., a homogeneous Markov chain with phase space $\mathbb{X} \times [0, \infty)$, an initial distribution $\bar{p}_\varepsilon = \langle p_{\varepsilon,i} = \mathsf{P}\{\eta_{\varepsilon,0} = i, \kappa_{\varepsilon,0} = 0\} = \mathsf{P}\{\eta_{\varepsilon,0} = i\},\ i \in \mathbb{X}\rangle$, and transition probabilities, defined for $(i, s), (j, t) \in X \times [0, \infty)$,
$$Q_{\varepsilon,ij}(t) = \mathsf{P}\big\{\eta_{\varepsilon,1} = j, \kappa_{\varepsilon,1} \leq t \ / \ \eta_{\varepsilon,0} = i, \kappa_{\varepsilon,0} = s\big\}.$$

Note that the above transition probabilities do not depend on the variable s. In this case, the random sequence $\eta_{\varepsilon,n}$ is also a homogeneous (embedded) Markov chain with phase space $\mathbb{X}$ and transition probabilities, defined for $i, j \in \mathbb{X}$,
$$p_{ij}(\varepsilon) = Q_{\varepsilon,ij}(\infty) = \mathsf{P}\{\eta_{\varepsilon,1} = j \,/\, \eta_{\varepsilon,0} = i\}.$$

The following communication condition plays an important role:

A: There exist sets $\mathbb{Y}_i \subseteq \mathbb{X}$, $i \in \mathbb{X}$, such that:
(a) probabilities $p_{ij}(\varepsilon) > 0$, $j \in \mathbb{Y}_i$, $i \in \mathbb{X}$, for $\varepsilon \in (0, \varepsilon_0]$;
(b) probabilities $p_{ij}(\varepsilon) = 0$, $j \in \overline{\mathbb{Y}}_i$, $i \in \mathbb{X}$, for $\varepsilon \in (0, \varepsilon_0]$;
(c) there exist, for every pair of states $i, j \in \mathbb{X}$, an integer $n_{ij} \geq 1$ and a chain of states $i = l_{ij,0}, l_{ij,1}, \ldots, l_{ij,n_{ij}} = j$ such that $l_{ij,1} \in \mathbb{Y}_{l_{ij,0}}, \ldots, l_{ij,n_{ij}} \in \mathbb{Y}_{l_{ij,n_{ij}-1}}$.

We refer to sets $\mathbb{Y}_i$, $i \in \mathbb{X}$, as transition sets. Condition **A** implies that all sets $\mathbb{Y}_i \neq \varnothing$, $i \in \mathbb{X}$, and that the phase space $\mathbb{X}$ of Markov chain $\eta_{\varepsilon,n}$ is one class of communicative states for every $\varepsilon \in (0, \varepsilon_0]$.

The following condition excludes instant transitions:

B: $Q_{\varepsilon,ij}(0) = 0$, $i, j \in \mathbb{X}$, for every $\varepsilon \in (0, \varepsilon_0]$.

Let us now introduce a semi-Markov process $\eta_\varepsilon(t) = \eta_{\varepsilon,\nu_\varepsilon(t)}$, $t \geq 0$, where

$$\nu_\varepsilon(t) = \max(n \geq 0\colon \zeta_{\varepsilon,n} \leq t)$$

is a number of jumps in the time interval $[0,t]$ and $\zeta_{\varepsilon,n} = \kappa_{\varepsilon,1} + \cdots + \kappa_{\varepsilon,n}$, $n = 0,1,\ldots,$ are sequential moments of jumps for the semi-Markov process $\eta_\varepsilon(t)$.

Let us introduce transition power-exponential moments of transition times, for $\varrho \geq 0$, $k = 0,1,\ldots,$ $i,j \in \mathbb{X}$,

$$\phi_{ij}(k,\varrho,\varepsilon) = \mathsf{E}_i \kappa_{\varepsilon,1}^k e^{\varrho\kappa_{\varepsilon,1}} \mathrm{I}(\eta_{\varepsilon,1} = j) = \int_0^\infty t^k e^{\varrho t}\, Q_{\varepsilon,ij}(dt). \tag{1}$$

Here and henceforth, the notation P_i and E_i is used for conditional probabilities and expectations under condition $\eta_{\varepsilon,0} = i$.

Conditions **A(a)–(b)** and **B** imply that, for every $\varepsilon \in (0,\varepsilon_0]$, moments

$$\phi_{ij}(k,\varrho,\varepsilon) \in (0,\infty]$$

for $\varrho \geq 0$, $k = 0,1,\ldots,$ $j \in \mathbb{Y}_i$, $i \in \mathbb{X}$, and $\phi_{ij}(k,\varrho,\varepsilon) = 0$ for $\varrho \geq 0$, $k = 0,1,\ldots,$ $j \in \overline{\mathbb{Y}}_i$, $i \in \mathbb{X}$.

Let us assume that the following condition holds for some $\rho_\circ > 0$:

$\mathbf{C}_{\rho_\circ}$: $\phi_{ij}(0,\rho_\circ,\varepsilon) < \infty$, $j \in \mathbb{Y}_i, i \in \mathbb{X}$, for $\varepsilon \in (0,\varepsilon_0]$.

Obviously condition $\mathbf{C}_{\rho_\circ}$ implies that moments $\phi_{ij}(k,\varrho,\varepsilon) < \infty$ for any $0 \leq \varrho < \rho_\circ$, $k = 0,1,\ldots, j \in \mathbb{Y}_i, i \in \mathbb{X}$.

It is appropriate to mention two important particular cases.

If $Q_{\varepsilon,ij}(t) = \mathrm{I}(t \geq 1)p_{ij}(\varepsilon)$, $t \geq 0$, $i,j \in \mathbb{X}$, then $\eta_\varepsilon(t) = \eta_{\varepsilon,[t]}$, $t \geq 0$, is a discrete time homogeneous Markov chain embedded in continuous time. In this case,

$$\phi_{ij}(k,\varrho,\varepsilon) = e^\varrho p_{ij}(\varepsilon) < \infty$$

for $\varrho > 0$, $i,j \in \mathbb{X}$.

If $Q_{\varepsilon,ij}(t) = \left(1 - e^{-\lambda_i(\varepsilon)t}\right) p_{ij}(\varepsilon)$, $t \geq 0$, $i,j \in \mathbb{X}$ (here, $0 < \lambda_i(\varepsilon) < \infty$, $i \in \mathbb{X}$), then $\eta_\varepsilon(t)$, $t \geq 0$, is a continuous time homogeneous Markov chain. In this case,

$$\phi_{ij}(k,\varrho,\varepsilon) = \frac{k\lambda_i(\varepsilon)}{(\lambda_i(\varepsilon) - \varrho)^{k+1}}\, p_{ij}(\varepsilon) < \infty$$

for $\varrho < \lambda_i(\varepsilon)$, $i,j \in \mathbb{X}$.

Let us define the hitting time for the semi-Markov process $\eta_\varepsilon(t)$ to the state 0 (of course, this state can be replaced by any other state $i \in \mathbb{X}$),

$$\tau_{\varepsilon,0} = \sum_{n=1}^{\nu_{\varepsilon,0}} \kappa_{\varepsilon,n}, \quad \text{where } \nu_{\varepsilon,0} = \min(n \geq 1\colon \eta_{\varepsilon,n} = 0). \tag{2}$$

The object of our interest are power-exponential moments for hitting times, for $\varrho \geq 0$, $k = 0,1,\ldots,$ $i \in \mathbb{X}$,

$$\Phi_i(k,\varrho,\varepsilon) = \mathsf{E}_i \tau_{\varepsilon,0}^k e^{\varrho\tau_{\varepsilon,0}}. \tag{3}$$

Condition $\mathbf{C}_{\rho_\circ}$ does not imply that exponential moments $\Phi_i(0,\rho_\circ,\varepsilon)$ are finite.

Necessary and sufficient conditions of finiteness for exponential moments of hitting times are given in terms of so-called test-functions in [4] and [13].

We refer to functions $v(i), i \in \mathbb{X}$, defined on the space $\mathbb{X}$ and taking value in the interval $[0,\infty)$ as test-functions.

Let us introduce a condition:

$\mathbf{D}_{\rho_\circ}$: There exists, for every $\varepsilon \in (0, \varepsilon_0]$, a test-function $v_{\varepsilon,\rho_\circ}(i)$, $i \in \mathbb{X}$, such that the following test inequalities hold:

$$v_{\varepsilon,\rho_\circ}(i) \geq \phi_{i0}(0, \rho_\circ, \varepsilon) + \sum_{j \in \mathbb{X}, j \neq 0} \phi_{ij}(0, \rho_\circ, \varepsilon) v_{\varepsilon,\rho_\circ}(j), \qquad i \in \mathbb{X}. \tag{4}$$

Lemma 2. *Let conditions* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}_{\rho_\circ}$ *hold. Then, the exponential moments* $\Phi_{i0}(0, \rho_\circ, \varepsilon) < \infty$, $i \in \mathbb{X}$, *for* $\varepsilon \in (0, \varepsilon_0]$ *if and only if condition* $\mathbf{D}_{\rho_\circ}$ *holds. In this case, inequalities* $\Phi_{i0}(0, \rho_\circ, \varepsilon) \leq v_{\varepsilon,\rho_\circ}(i)$, $i \in \mathbb{X}$, *hold for* $\varepsilon \in (0, \varepsilon_0]$, *and the exponential moments* $\Phi_{i0}(0, \rho_\circ, \varepsilon)$, $i \in \mathbb{X}$, *are, for every* $\varepsilon \in (0, \varepsilon_0]$, *the unique solution for the following system of linear equations:*

$$\Phi_{i0}(0, \rho_\circ, \varepsilon) = \phi_{i0}(0, \rho_\circ, \varepsilon) + \sum_{j \in \mathbb{X}, j \neq 0} \phi_{ij}(0, \rho_\circ, \varepsilon) \Phi_{j0}(0, \rho_\circ, \varepsilon), \qquad i \in \mathbb{X}. \tag{5}$$

In what follows, we always assume that conditions $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}_{\rho_\circ}$, and $\mathbf{D}_{\rho_\circ}$ hold.

It is obvious that $\Phi_{i0}(k, \varrho, \varepsilon) \leq L_{k,\rho_0 - \varrho} \Phi_{i0}(0, \rho_\circ, \varepsilon) < \infty$ for $0 \leq \varrho < \rho_\circ$, $k = 0, 1, \dots,$ $i \in \mathbb{X}$, where $L_{k,\rho_0 - \varrho} = \sup_{x \geq 0} x^k e^{-(\rho_\circ - \varrho)x} < \infty$.

Let us assume that the following perturbation condition, based on Laurent asymptotic expansions, holds for some integer $d \geq 0$ and real $0 < \rho < \rho_\circ$:

$\mathbf{E}_{d,\rho}$: $\phi_{ij}(k, \rho, \varepsilon) = \sum_{l = h^-_{ij}[k,\rho]}^{h^+_{ij}[k,\rho]} g_{ij}[k, \rho, l] \varepsilon^l + o_{k,\rho,ij}\big(\varepsilon^{h^+_{ij}[k,\rho]}\big)$, $\varepsilon \in (0, \varepsilon_0]$, for $k = 0, \dots, d$, $j \in \mathbb{Y}_i$, $i \in \mathbb{X}$, where

- **(a)** $-\infty < h^-_{ij}[k, \rho] \leq h^+_{ij}[k, \rho] < \infty$ are integers, coefficients
$$g_{ij}[k, \rho, l], \qquad l = h^-_{ij}[k, \rho], \dots, h^+_{ij}[k, \rho],$$
are real numbers, and $g_{ij}\big[k, \rho, h^-_{ij}[k, \rho]\big] > 0$ for $k = 0, \dots, d$, $j \in \mathbb{Y}_i$, $i \in \mathbb{X}$;
- **(b)** the function $o_{k,\rho,ij}\big(\varepsilon^{h^+_{ij}[k,\rho]}\big)/\varepsilon^{h^+_{ij}[k,\rho]} \to 0$ as $\varepsilon \to 0$ for $k = 0, \dots, d$, $j \in \mathbb{Y}_i$, $i \in \mathbb{X}$.

We refer here to the book [4], where the asymptotic expansions appearing in condition $\mathbf{E}_{d,\rho}$ are explicitly given for the cases of discrete and continuous time Markov chains.

If $\eta_{\varepsilon,0} \neq 0$, then the first hitting time $\tau_{\varepsilon,0} \geq \tau_\varepsilon = \sum_{n=1}^{\mu_\varepsilon} \kappa_{\varepsilon,n}$, where

$$\mu_\varepsilon = \max(n \geq 0 \colon \eta_{\varepsilon,n} \neq \eta_{\varepsilon,0}).$$

This inequality implies that, for $\varrho \geq 0$, $i \neq 0$, and $\varepsilon \in (0, \varepsilon_0]$,

$$\Phi_i(0, \varrho, \varepsilon) \geq \mathsf{E}_i e^{\varrho \tau_\varepsilon} = \sum_{n \geq 1} \phi_{ii}(0, \varrho, \varepsilon)^{n-1} \sum_{j \neq i} \phi_{ij}(0, \varrho, \varepsilon) = \frac{\sum_{j \neq i} \phi_{ij}(0, \varrho, \varepsilon)}{1 - \phi_{ii}(0, \varrho, \varepsilon)}. \tag{6}$$

Thus, condition $\mathbf{D}_{\rho_\circ}$ implies that the following inequalities should also hold for $\varepsilon \in (0, \varepsilon_0]$:

$$\phi_{ii}(0, \rho_\circ, \varepsilon) < 1, \qquad i \neq 0. \tag{7}$$

Condition $\mathbf{E}_{d,\rho}$ and inequalities (7) imply that the following condition should also hold:

$\mathbf{F}_\rho$: For every $i \neq 0$, either
- (a) $h^-_{ii}[0, \rho] > 0$, or
- (b) $h^-_{ii}[0, \rho] = 0$ and $g_{ii}[0, \rho, h^-_{ii}[0, \rho]] < 1$, or
- (c) $h^-_{ii}[0, \rho] = 0$, $g_{ii}[0, \rho, h^-_{ii}[0, \rho]] = 1$, $h^+_{ii}[0, \rho] \geq 1$

and there are nonzero terms in the sequence $g_{ii}[0,\rho,1],\ldots,g_{ii}[0,\rho,h^+_{ii}[0,\rho]]$, moreover, the first such term, say $g_{ii}[0,\rho,l_i]$, where $1 \leq l_i \leq h^+_{ii}[0,\rho]$, is a negative number.

It is useful to note that proposition **(i)** of Lemma 1 and conditions $\mathbf{E}_{d,\rho}$ and $\mathbf{F}_\rho$ imply that the function

$$
\begin{aligned}
1-\phi_{ii}(0,\rho,\varepsilon) &= 1+\sum_{l=h^-_{ii}[0,\rho]}^{h^+_{ii}[0,\rho]} -g_{ii}[0,\rho,l]\varepsilon^l - o_{0,\rho,ii}\left(\varepsilon^{h^+_{ii}[0,\rho]}\right) \\
&= \sum_{l=\bar{h}^-_{ii}[0,\rho]}^{\bar{h}^+_{ii}[0,\rho]} \bar{g}_{ii}[0,\rho,l]\varepsilon^l + \bar{o}_{0,\rho,ii}\left(\varepsilon^{\bar{h}^+_{ii}[0,\rho]}\right), \qquad \varepsilon\in(0,\varepsilon_0],
\end{aligned} \tag{8}
$$

is, for every $i\in\mathbb{Y}_i$, $i\neq 0$, a pivotal Taylor asymptotic expansion, with parameters $\bar{h}^-_{ii}[0,\rho]$ equal to 0 if alternative (a) or (b) takes place, or l_i if alternative (c) takes place in condition $\mathbf{F}_\rho$, with $\bar{h}^+_{ii}[0,\rho]=h^+_{ii}[0,\rho]$, and with the corresponding coefficients and remainder determined in an obvious way by relation (8). Note also that $1-\phi_{ii}(0,\rho,\varepsilon)\equiv 1$ for $i\in\overline{\mathbb{Y}}_i$, $i\neq 0$.

Conditions $\mathbf{E}_{d,\rho}$ and $\mathbf{F}_\rho$ guarantee that there exists $\varepsilon'_0\in(0,\varepsilon_0]$ such that the function $\phi_{ii}(0,\varrho,\varepsilon)$ given by the asymptotic expansion appearing in condition satisfies, for every $i\neq 0$ and $\varepsilon\in(0,\varepsilon'_0]$, the inequality $0<\phi_{ii}(0,\rho,\varepsilon)<1$. For simplicity, we just assume that $\varepsilon'_0=\varepsilon_0$.

In the case where Laurent asymptotic expansions with explicit upper bounds for remainders are the objects of interest, the assumption $\mathbf{E}_{d,\rho}$ **(b)** imposed on the remainders $o_{k,ij}\big(\varepsilon^{h^+_{ij}[k,\rho]}\big)$ should be replaced by the following stronger condition:

- $\mathbf{G}_{d,\rho}$: $\big|o_{k,\rho,ij}\big(\varepsilon^{h^+_{ij}[k,\rho]}\big)\big| \leq G_{ij}[k,\rho]\varepsilon^{h^+_{ij}[k,\rho]+\delta_{ij}[k,\rho]}$, $0<\varepsilon\leq\varepsilon_{ij}[k,\rho]$, for $k=0,\ldots,d$, $j\in\mathbb{Y}_i$, $i\in\mathbb{X}$, where $0<\delta_{ij}[k,\rho]\leq 1$, $0\leq G_{ij}[k,\rho]<\infty$, $0<\varepsilon_{ij}[k,\rho]\leq\varepsilon_0$ for $k=0,\ldots,d$, $j\in\mathbb{Y}_i$, $i\in\mathbb{X}$.

It is also useful to note that, in this case, the above $\big(\bar{h}^-_{ii}[0,\rho],\bar{h}^+_{ii}[0,\rho]\big)$-expansion for the function $1-\phi_{ii}(0,\rho,\varepsilon)$ is an $\big(\bar{h}^-_{ii}[0,\rho],\bar{h}^+_{ii}[0,\rho],\delta_{ii}[k,\rho],G_{ii}[k,\rho],\varepsilon_{ii}[k,\rho]\big)$-expansion for $i\neq 0$.

Condition $\mathbf{E}_{d,\rho}$ does not imply that there exist limits, $\lim_{\varepsilon\to 0}p_{ij}(\varepsilon)$, $i,j\in\mathbb{X}$. However, any sequence $\varepsilon_n\to 0$ as $n\to\infty$ obviously contains a subsequence $\varepsilon_{n_N}\to 0$ as $N\to\infty$ such that there exist limits, $\lim_{N\to 0}p_{ij}(\varepsilon_{n_N})=p_{ij}(0)$, $i,j\in\mathbb{X}$. Matrix $\|p_{ij}(\varepsilon)\|$ is stochastic for every $\varepsilon\in(0,\varepsilon_0]$, and, thus, matrix $\|p_{ij}(0)\|$ is also stochastic. It is possible that matrix $\|p_{ij}(0)\|$ has more zero elements than matrices $\|p_{ij}(\varepsilon)\|$. Therefore, a Markov chain $\eta_{0,n}$, with the phase space $\mathbb{X}$ and the matrix of transition probabilities $\|p_{ij}(0)\|$, can be nonergodic, and its phase space $\mathbb{X}$ can consist of one or several closed classes of communicative states plus, possibly, a class of transient states.

4. Reduced semi-Markov processes

In what follows, we assume that conditions $\mathbf{A}$–$\mathbf{D}_{\rho_\circ}$ hold.

Let us choose some state $r\neq 0$ and consider the reduced phase space ${}_r\mathbb{X}=\mathbb{X}\setminus\{r\}$, with the state r excluded from the phase space $\mathbb{X}$.

We define the sequential moments of hitting space ${}_r\mathbb{X}$ by the embedded Markov chain ${}_r\xi_{\varepsilon,n}=\min(k>{}_r\xi_{\varepsilon,n-1},\,\eta_{\varepsilon,k}\in{}_r\mathbb{X})$, $n=1,2,\ldots$, where ${}_r\xi_{\varepsilon,0}=0$, and the random sequence

$$({}_r\eta_{\varepsilon,n}, {}_r\kappa_{\varepsilon,n}) = \Big(\eta_{\varepsilon,{}_r\xi_{\varepsilon,n}}, \sum_{k={}_r\xi_{\varepsilon,n-1}+1}^{{}_r\xi_{\varepsilon,n}} \kappa_{\varepsilon,k}\Big), \quad n = 1, 2, \ldots, \qquad ({}_r\eta_{\varepsilon,0}, {}_r\kappa_{\varepsilon,0}) = (\eta_{\varepsilon,0}, 0).$$

This sequence is also a Markov renewal process with the phase space $\mathbb{X} \times [0, \infty)$, the initial distribution ${}_r\bar{p}_\varepsilon = \langle {}_rp_{\varepsilon,i} = p_{\varepsilon,i}, i \in \mathbb{X}\rangle$, and transition probabilities, defined for $(i, s), (j, t) \in \mathbb{X} \times [0, \infty)$,

$${}_rQ_{\varepsilon,ij}(t) = \mathsf{P}\big\{{}_r\eta_{\varepsilon,1} = j, {}_r\kappa_{\varepsilon,1} \leq t \;/\; {}_r\eta_{\varepsilon,0} = i, {}_r\kappa_{\varepsilon,0} = s\big\}.$$

Obviously, transition probabilities ${}_rQ_{\varepsilon,ir}(t) = 0$ for $i \in \mathbb{X}$, $t \geq 0$.

The transition probabilities ${}_rQ_{\varepsilon,ij}(t)$ are expressed via the transition probabilities $Q_{\varepsilon,ij}(t)$ by the following formula for $i \in \mathbb{X}$, $j \in {}_r\mathbb{X}$, $t \geq 0$:

$$\begin{aligned} {}_rQ_{\varepsilon,ij}(t) &= \mathsf{P}_i\,\{\eta_{\varepsilon,1} = j, \kappa_{\varepsilon,1} \leq t\} \\ &\quad + \sum_{n=0}^{\infty} \mathsf{P}_i\big\{\eta_{\varepsilon,1} = r, \eta_{\varepsilon,k+1} = r, 1 \leq k \leq n, \eta_{\varepsilon,n+2} = j, \\ &\qquad\qquad \kappa_{\varepsilon,1} + \cdots + \kappa_{\varepsilon,n+2} \leq t\big\} \\ &= Q_{\varepsilon,ij}(t) + \sum_{n=0}^{\infty} Q_{\varepsilon,ir}(t) * Q^{*n}_{\varepsilon,rr}(t) * Q_{\varepsilon,rj}(t). \end{aligned} \tag{9}$$

Here, the symbol $*$ is used to denote the corresponding variant of convolution for the above semi-Markov transition probabilities.

The above formula directly implies the following formula for transition probabilities of the embedded Markov chain ${}_r\eta_{\varepsilon,n}$ for $i \in \mathbb{X}$, $j \in {}_r\mathbb{X}$:

$${}_rp_{ij}(\varepsilon) = {}_rQ_{\varepsilon,ij}(\infty) = p_{ij}(\varepsilon) + p_{ir}(\varepsilon)\,\frac{p_{rj}(\varepsilon)}{1 - p_{rr}(\varepsilon)}. \tag{10}$$

The transition distributions for the Markov chain ${}_r\eta_{\varepsilon,n}$ are concentrated on the reduced phase space ${}_r\mathbb{X}$, i.e., for every $i \in \mathbb{X}$,

$$\begin{aligned} \sum_{j\in{}_r\mathbb{X}} {}_rp_{ij}(\varepsilon) &= \sum_{j\in{}_r\mathbb{X}} p_{ij}(\varepsilon) + p_{ir}(\varepsilon) \sum_{j\in{}_r\mathbb{X}} \frac{p_{rj}(\varepsilon)}{1 - p_{rr}(\varepsilon)} \\ &= \sum_{j\in{}_r\mathbb{X}} p_{ij}(\varepsilon) + p_{ir}(\varepsilon) = 1. \end{aligned} \tag{11}$$

If the initial distribution $\bar{p}_\varepsilon$ is concentrated on the phase space ${}_r\mathbb{X}$, i.e., $p_{\varepsilon,r} = 0$, then the random sequence $({}_r\eta_{\varepsilon,n}, {}_r\kappa_{\varepsilon,n}), n = 0, 1, \ldots,$ can also be considered as a Markov renewal process with the reduced phase ${}_r\mathbb{X} \times [0, \infty)$, the initial distribution

$${}_r\bar{p}_\varepsilon = \big\langle p_{\varepsilon,i} = \mathsf{P}\{{}_r\eta_{\varepsilon,0} = i, {}_r\kappa_{\varepsilon,0} = 0\} = \mathsf{P}\{{}_r\eta_{\varepsilon,0} = i\}, \, i \in {}_r\mathbb{X}\big\rangle,$$

and transition probabilities ${}_rQ_{\varepsilon,ij}(t)$, $t \geq 0$, $i, j \in {}_r\mathbb{X}$.

If the initial distribution $\bar{p}$ is not concentrated on the phase space ${}_r\mathbb{X}$, i.e., $p_{\varepsilon,r} > 0$, then the random sequence $({}_r\eta_{\varepsilon,n}, {}_r\kappa_{\varepsilon,n})$, $n = 0, 1, \ldots$, can be considered as a Markov renewal process with so-called transition period.

Respectively, one can define the transformed semi-Markov process

$${}_r\eta_\varepsilon(t) = {}_r\eta_{\varepsilon,{}_r\nu_\varepsilon(t)}, \qquad t \geq 0, \tag{12}$$

where ${}_r\nu_\varepsilon(t) = \max(n \geq 0\colon {}_r\zeta_{\varepsilon,n} \leq t)$ is a number of jumps at time interval $[0, t]$ for $t \geq 0$, and ${}_r\zeta_{\varepsilon,n} = {}_r\kappa_{\varepsilon,1} + \cdots + {}_r\kappa_{\varepsilon,n}$, $n = 0, 1, \ldots$, are sequential moments of jumps for the semi-Markov process ${}_r\eta_\varepsilon(t)$.

If the initial distribution $\bar{p}_\varepsilon$ is concentrated on the phase space ${}_r\mathbb{X}$, then the process ${}_r\eta_\varepsilon(t)$ can be considered as a standard semi-Markov process with the reduced phase ${}_r\mathbb{X}$, the initial distribution ${}_r\bar{p}_\varepsilon = \langle {}_rp_i = \mathsf{P}\{{}_r\eta_\varepsilon(0) = i\}, i \in {}_r\mathbb{X}\rangle$, and transition probabilities ${}_rQ_{\varepsilon,ij}(t)$, $t \geq 0$, $i, j \in {}_r\mathbb{X}$.

According to the above remarks, we can refer to the process ${}_r\eta_\varepsilon(t)$ as a reduced semi-Markov process.

If the initial distribution $\bar{p}_\varepsilon$ is not concentrated on the phase space ${}_r\mathbb{X}$, then the process ${}_r\eta_\varepsilon(t)$ can be interpreted as a reduced semi-Markov process with transition period.

Let us introduce the following sets for $i, r \in \mathbb{X}$:

$$(13)\qquad \mathbb{Y}_{ir}^+ = \{j \in {}_r\mathbb{X}\colon j \in \mathbb{Y}_i\} \quad \text{and} \quad \mathbb{Y}_{ir}^- = \begin{cases} \{j \in {}_r\mathbb{X}\colon j \in \mathbb{Y}_r\}, & \text{if } r \in \mathbb{Y}_i, \\ \varnothing, & \text{if } r \notin \mathbb{Y}_i, \end{cases}$$

and

$$(14)\qquad {}_r\mathbb{Y}_i = \mathbb{Y}_{ir}^- \cup \mathbb{Y}_{ir}^+, \qquad i \in \mathbb{X}.$$

It is readily seen that, for every $r \neq 0$, condition **A** holds for the reduced Markov chains ${}_r\eta_{\varepsilon,n}$, with the phase space ${}_r\mathbb{X}$. In this case, ${}_r\mathbb{Y}_i$, $i \in {}_r\mathbb{X}$, are the corresponding transition sets.

Condition **A** implies that $p_{rr}(\varepsilon) \in [0, 1)$, $r \neq 0$, $\varepsilon \in (0, \varepsilon_0]$.

This relation and formulas (10)–(11) imply that transition probabilities ${}_rp_{rj}(\varepsilon) > 0$, $j \in {}_r\mathbb{Y}_r = \mathbb{Y}_r \setminus \{r\}$, for $\varepsilon \in (0, \varepsilon_0]$, or ${}_rp_{rj}(\varepsilon) = 0$, $j \in {}_r\overline{\mathbb{Y}}_r$, for $\varepsilon \in (0, \varepsilon_0]$.

Thus, if ${}_r\eta_{\varepsilon,n}$ is a reduced Markov chain with transition period, then the set ${}_r\mathbb{X}$ is a closed class of communicative states, while r is a transient state for every $\varepsilon \in (0, \varepsilon_0]$.

Obviously, condition **B** also holds for the reduced semi-Markov processes ${}_r\eta_\varepsilon(t)$.

Taking into account that ${}_r\xi_{\varepsilon,1}$ is a Markov time for the Markov renewal process $(\eta_{\varepsilon,n}, \kappa_{\varepsilon,n})$, we can write down the following system of stochastic equalities for every $i, j \in {}_r\mathbb{X}$:

$$(15)\qquad \begin{cases} {}_r\kappa_{\varepsilon,i,1}\mathrm{I}({}_r\eta_{\varepsilon,i,1} = j) \overset{d}{=} \kappa_{\varepsilon,i,1}\mathrm{I}(\eta_{\varepsilon,i,1} = j) \\ \qquad\qquad + (\kappa_{\varepsilon,i,1} + {}_r\kappa_{\varepsilon,r,1})\mathrm{I}(\eta_{\varepsilon,i,1} = r)\mathrm{I}({}_r\eta_{\varepsilon,r,1} = j), \\ {}_r\kappa_{\varepsilon,r,1}\mathrm{I}({}_r\eta_{\varepsilon,r,1} = j) \overset{d}{=} \kappa_{\varepsilon,r,1}\mathrm{I}(\eta_{\varepsilon,r,1} = j) \\ \qquad\qquad + (\kappa_{\varepsilon,r,1} + {}_r\kappa_{\varepsilon,r,1})\mathrm{I}(\eta_{\varepsilon,r,1} = r)\mathrm{I}({}_r\eta_{\varepsilon,r,1} = j), \end{cases}$$

where:

(a) $(\eta_{\varepsilon,i,1}, \kappa_{\varepsilon,i,1})$ is a random vector, which takes values in the space $\mathbb{X} \times [0, \infty)$ and has distribution $\mathsf{P}\{\eta_{\varepsilon,i,1} = j, \kappa_{\varepsilon,i,1} \leq t\} = Q_{ij}(t)$, $j \in \mathbb{X}$, $t \geq 0$, for every $i \in \mathbb{X}$;

(b) $({}_r\eta_{\varepsilon,i,1}, {}_r\kappa_{\varepsilon,i,1})$ is a random vector which takes values in the space ${}_r\mathbb{X} \times [0, \infty)$ and has distribution $\mathsf{P}\{{}_r\eta_{\varepsilon,i,1} = j, {}_r\kappa_{\varepsilon,i,1} \leq t\} = \mathsf{P}_i\{{}_r\eta_{\varepsilon,1} = j, {}_r\kappa_{\varepsilon,1} \leq t\} = {}_kQ_{ij}(t)$, $j \in {}_k\mathbb{X}$, $t \geq 0$, for every $i \in \mathbb{X}$;

(c) $(\eta_{\varepsilon,i,1}, \kappa_{\varepsilon,i,1})$ and $({}_r\eta_{\varepsilon,r,1}, {}_r\eta_{\varepsilon,r,1})$ are independent random vectors for every $i, r \in \mathbb{X}$.

Here, the symbol $\overset{d}{=}$ is used to show that random variables on the left- and right-hand sides of the corresponding equality have the same distribution.

Let us introduce transition power-exponential moments for $\varrho \geq 0$, $k = 0, 1, \ldots$, $r \neq 0$, $i \in \mathbb{X}$, $j \in {}_r\mathbb{X}$:

$$(16)\qquad {}_r\phi_{ij}(k, \varrho, \varepsilon) = \int_0^\infty t^k e^{\varrho t}\, {}_rQ_{\varepsilon,ij}(dt).$$

By computing exponential moments in stochastic relations (15) we get, for every $0 \leq \varrho \leq \rho_0$, $r \neq 0$, $i, j \in {}_r\mathbb{X}$, and $\varepsilon \in (0, \varepsilon_0]$, the following system of linear equations for the exponential moments ${}_r\phi_{rj}(0,\varrho,\varepsilon)$, ${}_r\phi_{ij}(0,\varrho,\varepsilon)$:

$$\begin{cases} {}_r\phi_{rj}(0,\varrho,\varepsilon) = \phi_{rj}(0,\varrho,\varepsilon) + \phi_{rr}(0,\varrho,\varepsilon)\, {}_r\phi_{rj}(0,\varrho,\varepsilon), \\ {}_r\phi_{ij}(0,\varrho,\varepsilon) = \phi_{ij}(0,\varrho,\varepsilon) + \phi_{ir}(0,\varrho,\varepsilon)\, {}_r\phi_{rj}(0,\varrho,\varepsilon). \end{cases} \tag{17}$$

It is possible that the moment $\phi_{rr}(0,\rho,\varepsilon)$ or $\phi_{ir}(0,\rho,\varepsilon)$ equals 0, while the moment $\phi_{j0}(0,\rho,\varepsilon)$ equals $+\infty$ in relation (17). In such cases, one should set the product $0\cdot\infty$ to be 0 when calculating the products in the right-hand side of equality (17).

However, inequality (7) and relation (17) imply that ${}_r\phi_{ij}(0,\rho_0,\varepsilon) < \infty$ for every $r \neq 0$, $i \in \mathbb{X}$, $j \in {}_r\mathbb{X}$, and $\varepsilon \in (0,\varepsilon_0]$.

Thus, relation (17) yields the following formulas for the moments ${}_r\phi_{rj}(0,\rho,\varepsilon)$ and ${}_r\phi_{ij}(0,\rho,\varepsilon)$ for every $0 \leq \varrho \leq \rho_0, r \neq 0, i,j \in {}_r\mathbb{X}$:

$$\begin{cases} {}_r\phi_{rj}(0,\rho,\varepsilon) = \dfrac{\phi_{rj}(0,\rho,\varepsilon)}{1-\phi_{rr}(0,\rho,\varepsilon)}, \\ {}_r\phi_{ij}(0,\rho,\varepsilon) = \phi_{ij}(0,\rho,\varepsilon) + \dfrac{\phi_{ir}(0,\rho,\varepsilon)\phi_{rj}(0,\rho,\varepsilon)}{1-\phi_{rr}(0,\rho,\varepsilon)}. \end{cases} \tag{18}$$

It is useful to note that the second formula in relation (18) reduces to the first one, if we assign $i = r$ in this formula.

Thus, condition $\mathbf{C}_{\rho_\circ}$ holds for the reduced semi-Markov processes ${}_r\eta_\varepsilon(t)$.

Obviously, ${}_r\phi_{ij}(k,\varrho,\varepsilon) \leq L_{k,\rho_0-\varrho}\, {}_r\phi_{ij}(0,\rho_\circ,\varepsilon)$ for $0 \leq \varrho < \rho_\circ$, $k = 0,1,\dots, r \neq 0$, $i \in \mathbb{X}$, $j \in {}_r\mathbb{X}$, and $\varepsilon \in (0,\varepsilon_0]$.

Also, it is easily seen that for every $0 \leq \varrho < \rho_\circ$, $k = 1,\dots,$ $r \neq 0$, $i \in \mathbb{X}$, $j \in {}_r\mathbb{X}$, and $\varepsilon \in (0,\varepsilon_0]$, the function ${}_r\phi_{ij}(0,\varrho,\varepsilon)$ has a derivative of order k, and it is the function ${}_r\phi_{ij}(k,\varrho,\varepsilon)$.

Therefore, we can differentiate equations (17) and get the following system of linear equations for every $0 \leq \varrho < \rho_\circ$, $k = 1,\dots,$ $r \neq 0$, $i \in \mathbb{X}$, $j \in {}_r\mathbb{X}$, and $\varepsilon \in (0,\varepsilon_0]$:

$$\begin{cases} {}_r\phi_{rj}(k,\varrho,\varepsilon) = {}_r\lambda_{rj}(k,\varrho,\varepsilon) + \phi_{rr}(0,\varrho,\varepsilon)\, {}_r\phi_{rj}(k,\varrho,\varepsilon), \\ {}_r\phi_{ij}(k,\varrho,\varepsilon) = {}_r\lambda_{ij}(k,\varrho,\varepsilon) + \phi_{ir}(0,\varrho,\varepsilon)\, {}_r\phi_{rj}(k,\varrho,\varepsilon), \end{cases} \tag{19}$$

where

$${}_r\lambda_{ij}(k,\varrho,\varepsilon) = \phi_{ij}(k,\varrho,\varepsilon) + \sum_{l=0}^{k-1} \binom{k}{l} \phi_{ir}(k-l,\varrho,\varepsilon)\, {}_r\phi_{rj}(l,\varrho,\varepsilon). \tag{20}$$

Relation (19) yields the following formulas for moments ${}_r\phi_{rj}(k,\varrho,\varepsilon)$ and ${}_r\phi_{ij}(k,\varrho,\varepsilon)$, which can be used for every $0 \leq \varrho < \rho_\circ$, $k = 0,1,\dots,$ $r \neq 0$, $i,j \in {}_r\mathbb{X}$, and $\varepsilon \in (0,\varepsilon_0]$:

$$\begin{cases} {}_r\phi_{rj}(k,\varrho,\varepsilon) = \dfrac{{}_r\lambda_{rj}(k,\varrho,\varepsilon)}{1-\phi_{rr}(0,\varrho,\varepsilon)}, \\ {}_r\phi_{ij}(k,\varrho,\varepsilon) = {}_r\lambda_{ij}(k,\varrho,\varepsilon) + \dfrac{\phi_{ir}(0,\varrho,\varepsilon)\, {}_r\lambda_{rj}(k,\varrho,\varepsilon)}{1-\phi_{rr}(0,\varrho,\varepsilon)}. \end{cases} \tag{21}$$

Formulas (21) have recurrent character since expressions for functions ${}_r\lambda_{rj}(k,\varrho,\varepsilon)$, ${}_r\lambda_{ij}(k,\varrho,\varepsilon)$ include functions ${}_r\phi_{rj}(l,\varrho,\varepsilon)$, $l = 0,1,\dots,k-1$.

For $k = 0$, formulas (21) reduce to formulas (18).

Let us define the hitting times for the reduced semi-Markov processes for $r \neq 0$,

$${}_r\tau_{\varepsilon,0} = \sum_{n=1}^{{}_r\nu_{\varepsilon,0}} {}_r\kappa_{\varepsilon,n}, \quad \text{where} \quad {}_r\nu_{\varepsilon,0} = \min(n \geq 1 \colon {}_r\eta_{\varepsilon,n} = 0), \tag{22}$$

and the corresponding power-exponential moments for $\varrho \geq 0$, $k = 0,1,\dots,$ $i \in \mathbb{X}$,

$${}_r\Phi_i(k,\varrho,\varepsilon) = \mathsf{E}_i\, {}_r\tau_{\varepsilon,0}^k\, e^{\varrho\, {}_r\tau_{\varepsilon,0}}. \tag{23}$$

For every $\varepsilon \in (0, \varepsilon_0]$, the semi-Markov processes $\eta_\varepsilon(t)$ and ${}_r\eta_\varepsilon(t)$ and, in the sequel, the hitting times $\tau_{\varepsilon,0}$ and ${}_r\tau_{\varepsilon,0}$ are defined on the same probability space. This space can, however, be different for different ε.

Moreover, the following proposition follows from the fact that hitting of state $j \in {}_r\mathbb{X}$ by the semi-Markov process ${}_r\eta_\varepsilon(t)$ can occur only at moments of hitting space ${}_r\mathbb{X}$ by the semi-Markov process $\eta_\varepsilon(t)$. Its proof can be found, for example, in [14] and [15].

Lemma 3. *For every state $r \neq 0$ and $\varepsilon \in (0, \varepsilon_0]$, the hitting times $\tau_{\varepsilon,0}$ and ${}_r\tau_{\varepsilon,0}$ to the state 0 for semi-Markov processes $\eta_\varepsilon(t)$ and ${}_r\eta_\varepsilon(t)$, respectively, coincide.*

According to Lemma 3, $\tau_{\varepsilon,0}$ and ${}_r\tau_{\varepsilon,0}$ are, in fact, the same random variable defined in two different forms in terms, respectively, of processes $\eta_\varepsilon(t)$ and ${}_r\eta_\varepsilon(t)$. The following lemma, which is an obvious corollary of Lemma 3, plays an important role in what follows.

Lemma 4. *The exponential moments*

$$ {}_r\Phi_{i0}(0, \varrho, \varepsilon) = \Phi_{i0}(0, \varrho, \varepsilon) < \infty $$

for any $0 \leq \varrho \leq \rho_\circ$, $r \neq 0$, $i \in \mathbb{X}$, and $\varepsilon \in (0, \varepsilon_0]$, and the power-exponential moments ${}_r\Phi_{i0}(k, \varrho, \varepsilon) = \Phi_{i0}(k, \varrho, \varepsilon) < \infty$ for any $0 \leq \varrho < \rho_\circ$, $k = 0, 1, \ldots$, $r \neq 0$, $i \in \mathbb{X}$, and $\varepsilon \in (0, \varepsilon_0]$.

Let us summarize the above remarks.

Lemma 5. *Conditions* $\mathbf{A}$–$\mathbf{D}_{\rho_\circ}$ *hold for the semi-Markov processes $\eta_\varepsilon(t)$, and also hold for the reduced semi-Markov processes ${}_r\eta_\varepsilon(t)$.*

Since condition $\mathbf{D}_{\rho_\circ}$ holds for reduced semi-Markov processes ${}_r\eta_\varepsilon(t)$, the following inequalities also hold for $\varepsilon \in (0, \varepsilon_0]$:

(24) $$ {}_r\phi_{ii}(0, \rho_\circ, \varepsilon) < 1, \qquad i \neq 0, r. $$

5. Asymptotic expansions for power-exponential moments of hitting times

Let us now describe algorithms for the construction of asymptotic expansions for power-exponential moments of hitting times.

The proofs of Theorems 1 and 2 presenting these algorithms are based on recurrent application of operational rules for Laurent asymptotic expansions given in Lemma 1 to the reduced semi-Markov processes constructed with the use of the recurrent time-space screening procedures of phase space reduction described below. In fact, one should correctly describe which functions, in which order, and which operational rules should be applied for getting the corresponding expansions (their parameters, coefficients, and parameters of upper bounds for remainders) as well as to indicate some particular cases where the corresponding computational steps should be modified. This is exactly what is done in the proofs of Theorems 1 and 2. An explicit writing down of the corresponding operational formulas representing the recurrent algorithms described below (which could be given as corollaries of the above theorems) would, in fact, replicate the above proofs in the formal form, require implementation of a huge number of intermediate notation, take too much space, etc., but would not add any new essential information about the corresponding algorithms. That is why the decision was made to just say in each theorem that the description of the corresponding algorithm is given in its proof. This makes formulations slightly unusual. But, as we think, this is the most compact way for presentation of the corresponding asymptotic results and algorithms.

Theorem 1. *The following propositions take place:*

(i) *If conditions* $\mathbf{A}$–$\mathbf{F}_\rho$ *hold for the semi-Markov processes* $\eta_\varepsilon(t)$, *then these conditions also hold for the reduced semi-Markov processes* ${}_r\eta_\varepsilon(t)$ *for every* $r \neq 0$. *The corresponding pivotal* $({}_rh^-_{ij}[k,\rho],\ {}_rh^+_{ij}[k,\rho])$*-expansions for the mixed power-exponential moments* ${}_r\phi_{ij}(k,\rho,\varepsilon), k = 0,\dots,d, j \in {}_r\mathbb{Y}_i, i \in \mathbb{X}$, *are given by the algorithm described below in the proof of the theorem.*

(ii) *If, additionally, condition* $\mathbf{G}_{d,\rho}$ *holds for the semi-Markov processes* $\eta_\varepsilon(t)$, *then this condition also holds for the reduced semi-Markov processes* ${}_r\eta_\varepsilon(t)$. *In this case, the above* $\big({}_rh^-_{ij}[k,\rho], {}_rh^+_{ij}[k,\rho]\big)$*-expansions are also the pivotal*

$$\big({}_rh^-_{ij}[k,\rho], {}_rh^+_{ij}[k,\rho], {}_r\delta_{ij}[k,\rho], {}_rG_{ij}[k,\rho]{}_r\varepsilon_{ij}[k,\rho]\big)\text{-expansions,}$$

with parameters ${}_r\delta_{ij}[k,\rho]$, ${}_rG_{ij}[k,\rho]$, ${}_r\varepsilon_{ij}[k,\rho]$ *given by the algorithm described below in the proof of the theorem.*

Proof. Lemma 5 implies that conditions $\mathbf{A}$–$\mathbf{D}_{\rho_\circ}$ hold for the semi-Markov processes ${}_r\eta_\varepsilon(t)$, with the same parameter ε_0 as for the semi-Markov processes $\eta_\varepsilon(t)$, and the transition sets ${}_r\mathbb{Y}_i$, $i \in {}_r\mathbb{X}$, given by relation (14).

In order to prove that condition $\mathbf{E}_{d,\rho}$ also holds for semi-Markov processes ${}_r\eta_\varepsilon(t)$, with the same parameter ε_0 and the transition sets ${}_r\mathbb{Y}_i$, $i \in {}_r\mathbb{X}$, given by relation (14), let us construct the corresponding asymptotic expansions appearing in this condition.

Let $r \neq 0$, $i \in \mathbb{X}$, and $j, r \in \mathbb{Y}_i \cap \mathbb{Y}_r$.

At the initial step, we construct the asymptotic expansions for exponential moments ${}_r\phi_{rj}(1,\rho,\varepsilon)$ and ${}_r\phi_{ij}(0,\rho,\varepsilon)$ using formulas (18) and the corresponding asymptotic expansions appearing in condition $\mathbf{E}_{d,\rho}$.

First, proposition **(vi)** (the multiplication rule) of Lemma 1 should be applied to the product $\phi_{ir}(0,\rho,\varepsilon)\,\phi_{rj}(0,\rho,\varepsilon)$.

Second, proposition **(vii)** (the division rule) of Lemma 1 should be applied to the quotient

$$\frac{\phi_{ir}(0,\rho,\varepsilon)\phi_{rj}(0,\rho,\varepsilon)}{1-\phi_{rr}(0,\rho,\varepsilon)}.$$

Here, the asymptotic expansion for the function $1 - \phi_{rr}(0,\rho,\varepsilon)$ given in relation (8) should be used.

Third, proposition **(iii)** (the summation rule) of Lemma 1 should be applied to the sum

$$\phi_{ij}(0,\rho,\varepsilon) + \frac{\phi_{ir}(0,\rho,\varepsilon)\phi_{rj}(0,\rho,\varepsilon)}{1-\phi_{rr}(0,\rho,\varepsilon)}.$$

If $j \notin \mathbb{Y}_i$, then $\phi_{ij}(0,\rho,\varepsilon) \equiv 0$; if $j \notin \mathbb{Y}_r$, then $\phi_{rj}(0,\rho,\varepsilon) \equiv 0$; if $r \notin \mathbb{Y}_i$, then $\phi_{ir}(0,\rho,\varepsilon) \equiv 0$; and if $r \notin \mathbb{Y}_r$, then $1 - \phi_{rr}(0,\rho,\varepsilon) \equiv 1$. In these cases, the above algorithm is readily simplified.

According to Lemma 1, the $\big({}_rh^-_{ij}[0,\rho], {}_rh^+_{ij}[0,\rho]\big)$-expansions

$$(25)\qquad {}_r\phi_{ij}(0,\rho,\varepsilon) = \sum_{l={}_rh^-_{ij}[0,\rho]}^{{}_rh^+_{ij}[0,\rho]} {}_rg_{ij}[0,\rho,l]\varepsilon^l + {}_ro_{0,\rho,ij}\left(\varepsilon^{{}_rh^+_{ij}[0,\rho]}\right)$$

yielded by the above algorithm for $r \neq 0$, $i \in \mathbb{X}$, $j \in {}_r\mathbb{Y}_i$ are pivotal.

Steps of the algorithm described above should be recurrently repeated for $k = 1,\dots,d$.

Let us assume that the corresponding pivotal asymptotic expansions for power-exponential moments ${}_r\phi_{rj}(l,\rho,\varepsilon)$, ${}_r\phi_{ij}(l,\rho,\varepsilon)$, $l = 0,\dots,k-1$, have already been constructed with the use of formulas (20)–(21). In this case, the asymptotic expansions for moments ${}_r\phi_{rj}(k,\rho,\varepsilon)$, ${}_r\phi_{ij}(k,\rho,\varepsilon)$ can be constructed using the above asymptotic expansions, formulas (20)–(21), and the corresponding asymptotic expansions appearing in condition $\mathbf{E}_{d,\rho}$, in the following way.

First, propositions **(i)** (the multiplication by constant rule) and **(v)** (the multiplication rule) of Lemma 1 should be applied to the products $\binom{k}{l}\phi_{qr}(k-l,\rho,\varepsilon)\,{}_r\phi_{rj}(l,\rho,\varepsilon)$ for $l = 0,\dots,k-1$ and $q = i, r$.

Second, proposition **(iii)** (the summation rule) of Lemma 1 should be recurrently applied to the sum

$$\begin{aligned}{}_r\lambda_{qj}(n,k,\rho,\varepsilon) &= \phi_{qj}(k,\rho,\varepsilon) + \sum_{l=0}^{n}\binom{k}{l}\phi_{qr}(k-l,\rho,\varepsilon)\,{}_r\phi_{rj}(l,\rho,\varepsilon)\\ &= {}_r\lambda_{qj}(n-1,k,\rho,\varepsilon) + \binom{k}{n}\phi_{qr}(k-n,\rho,\varepsilon)\,{}_r\phi_{rj}(n,\rho,\varepsilon)\end{aligned}$$

for $n = 1,\dots,k-1$, in order to get the asymptotic expansion for the sum

$${}_r\lambda_{qj}(k,\varrho,\varepsilon) = {}_r\lambda_{qj}(k-1,k,\rho,\varepsilon) = \phi_{qj}(k,\varrho,\varepsilon) + \sum_{l=0}^{k-1}\binom{k}{l}\phi_{q,r}(k-l,\varrho,\varepsilon)\,{}_r\phi_{rj}(l,\varrho,\varepsilon)$$

for $q = i, r$.

Third, proposition **(v)** (the multiplication rule) of Lemma 1 should be applied to the product $\phi_{ir}(0,\varrho,\varepsilon)\,{}_r\lambda_{rj}(k,\varrho,\varepsilon)$.

Fourth, proposition **(vii)** (the division rule) of Lemma 1 should be applied to the quotient

$$\frac{\phi_{ir}(0,\varrho,\varepsilon)\,{}_r\lambda_{rj}(k,\varrho,\varepsilon)}{1-\phi_{rr}(0,\varrho,\varepsilon)}.$$

Here, the asymptotic expansion for the function $1-\phi_{rr}(0,\rho,\varepsilon)$ given in relation (8) should be used.

Fifth, proposition **(i)** (the summation rule) of Lemma 1 should be applied to the sum

$${}_r\lambda_{ij}(k,\varrho,\varepsilon) + \frac{\phi_{ir}(0,\varrho,\varepsilon)\,{}_r\lambda_{rj}(k,\varrho,\varepsilon)}{1-\phi_{rr}(0,\varrho,\varepsilon)}.$$

As was already mentioned above, the five steps of the above algorithm should be recurrently repeated for $k = 1, 2, \dots, d$.

If $j \notin \mathbb{Y}_i$, then $\phi_{ij}(k,\rho,\varepsilon) \equiv 0$, $k = 0,\dots,d$; if $j \notin \mathbb{Y}_r$, then $\phi_{rj}(k,\rho,\varepsilon) \equiv 0$, $k = 0,\dots,d$; if $r \notin \mathbb{Y}_i$, then $\phi_{ir}(k,\rho,\varepsilon) \equiv 0$, $k = 0,\dots,d$; and if $r \notin \mathbb{Y}_r$, then $\phi_{rr}(k,\rho,\varepsilon) \equiv 0$, $k = 1,\dots,d$ and $1-\phi_{rr}(0,\rho,\varepsilon) \equiv 1$. In these cases, the above recurrent algorithm is readily simplified.

Note that the parameter ε_0 does not change in the multiplication and summation steps as well as in the division step, since $1-\phi_{rr}(0,\rho,\varepsilon) > 0$, $\varepsilon \in (0,\varepsilon_0]$.

According to Lemma 1, the $\left({}_rh^-_{ij}[k,\rho], {}_rh^+_{ij}[k,\rho]\right)$-expansions

$$(26)\qquad {}_r\phi_{ij}(k,\rho,\varepsilon) = \sum_{l={}_rh^-_{ij}[k,\rho]}^{{}_rh^+_{ij}[k,\rho]} {}_rg_{ij}[k,\rho,l]\varepsilon^l + {}_ro_{k,\rho,ij}\left(\varepsilon^{{}_rh^+_{ij}[k,\rho]}\right)$$

yielded by the above recurrent algorithm for $k = 1,\dots,d$, $r \neq 0$, $i \in \mathbb{X}$, $j \in {}_r\mathbb{Y}_i$ are pivotal.

It remains to note that condition $\mathbf{E}_{d,\rho}$ and inequalities (24) imply that condition $\mathbf{F}_\rho$ also holds for the reduced semi-Markov process ${}_r\eta_\varepsilon(t)$ for every $r \neq 0$.

This completes the proof of proposition **(i)** of Theorem 1.

In order to prove proposition **(ii)** of Theorem 1, one should repeat the same sequence of recurrent steps described above and, additionally, apply to every intermediate asymptotic expansion obtained with the use of the operational rules given in propositions **(i)**, **(iii)**, **(v)**, or **(vii)** of Lemma 1 the corresponding additional operational rules given, respectively, in propositions **(ii)**, **(vi)**, **(vi)**, or **(viii)**, for computing parameters of the corresponding upper bounds for remainders. □

Remark 1. It is worth noting that the above algorithm yields the asymptotic expansions for mixed power-exponential moments ${}_r\phi_{ij}(k, \rho, \varepsilon)$ for $k = 1, \ldots, d$, $r \neq 0$, $i \in \mathbb{X}$, $j \in {}_r\mathbb{Y}_i$, i.e., for the corresponding transition characteristics of the reduced semi-Markov processes ${}_r\eta_\varepsilon(t)$ with transition period defined in Section 4.

Let us choose some state $q \neq 0$ and let $\bar{r}_{q,m} = \langle r_{q,0}, \ldots, r_{q,m}\rangle = \langle r_{q,0}, \ldots, r_{q,m}\rangle$ be a permutation of the sequence $\langle 0, \ldots, m\rangle$ such that $r_{q,m-1} = q$, $r_{q,m} = 0$, and let $\bar{r}_{q,n} = \langle r_{q,0}, \ldots, r_{q,n}\rangle$, $n = 0, \ldots, m$, be the corresponding chain of growing sequences of states from space $\mathbb{X}$.

Theorem 2. *The following propositions take place:*

(i) *Let conditions* $\mathbf{A}$–$\mathbf{F}_\rho$ *hold for the semi-Markov processes* $\eta_\varepsilon(t)$*. Then, for every* $i \in \mathbb{X}$*, the pivotal* $\big(\dot{h}^-_{i0}[k, \rho], \dot{h}^+_{i0}[k, \rho]\big)$*-expansions for the power-exponential moments of hitting times* $\Phi_{i0}(k, \rho, \varepsilon)$*,* $k = 1, \ldots, d$*,* $i = q, 0$*, are given, for every* $q \neq 0$*, by the recurrent algorithm based on the sequential exclusion of states* $r_{q,0}, \ldots, r_{q,m-2}, q$ *from the phase space* $\mathbb{X}$ *of the processes* $\eta_\varepsilon(t)$*. This algorithm is described below in the proof of the theorem. The above* $\big(\dot{h}^-_{i0}[k, \rho], \dot{h}^+_{i0}[k, \rho]\big)$*-expansions are invariant with respect to any permutation* $\bar{r}_{q,m} = \langle r_{q,0}, \ldots, r_{q,m-2}, q, 0\rangle$ *of the sequence* $\langle 0, \ldots, m\rangle$*.*

(ii) *If, additionally, condition* $\mathbf{G}_{d,\rho}$ *holds for the semi-Markov processes* $\eta_\varepsilon(t)$*, then the above* $\big(\dot{h}^-_{i0}[k, \rho], \dot{h}^+_{i0}[k, \rho]\big)$*-expansions for the power-exponential moments of hitting times* $\Phi_{i0}(k, \rho, \varepsilon)$*,* $k = 1, \ldots, d$*,* $i = q, 0$*, also are, for every* $q \neq 0$*, pivotal* $\big(\dot{h}^-_{i0}[k, \rho], \dot{h}^+_{i0}[k, \rho], {}_r\dot{\delta}_{i0}[k, \rho], {}_r\dot{G}_{i0}[k, \rho], {}_r\dot{\varepsilon}_{i0}[k, \rho]\big)$*-expansions, with parameters* ${}_{\bar{r}_{i,m}}\delta_{i0}[k, \rho]$*,* ${}_{\bar{r}_{i,m}}G_{i0}[k, \rho]$*,* ${}_{\bar{r}_{i,m}}\varepsilon_{i0}[k, \rho]$ *given by the algorithm described below in the proof of the theorem.*

Proof. Let us exclude the state $r_{i,0}$ from the phase space of the semi-Markov process $\eta_\varepsilon(t)$ using the time-space screening procedure described in Section 4. Let $\eta_{\varepsilon, \bar{r}_{q,0}}(t) = {}_{r_{q,0}}\eta_\varepsilon(t)$ be the corresponding reduced semi-Markov process, with the phase space

$$
{}_{\bar{r}_{q,0}}\mathbb{X} = \mathbb{X} \setminus \{r_{q,0}\}.
$$

The above procedure can be repeated. The state $r_{q,1}$ can be excluded from the phase space of the semi-Markov process $\eta_{\varepsilon, \bar{r}_{q,0}}(t)$. Let $\eta_{\varepsilon, \bar{r}_{q,1}}(t) = {}_{r_{q,1}}\eta_{\varepsilon, \bar{r}_{q,0}}(t)$ be the corresponding reduced semi-Markov process, with the phase space ${}_{\bar{r}_{q,1}}\mathbb{X} = \mathbb{X} \setminus \{r_{q,0}, r_{q,1}\}$. By continuing the above procedure for states $r_{q,2}, \ldots, r_{q,n}$, we construct the reduced semi-Markov process $\eta_{\varepsilon, \bar{r}_{q,n}}(t) = {}_{r_{q,n}}\eta_{\varepsilon, \bar{r}_{q,n-1}}(t)$. This semi-Markov process has the phase space ${}_{\bar{r}_{q,n}}\mathbb{X} = \mathbb{X} \setminus \{r_{q,0}, r_{q,1}, \ldots, r_{q,n}\}$.

Let ${}_{\bar{r}_{q,n}}\mathbb{Y}_i$, $i \in {}_{\bar{r}_{q,n-1}}\mathbb{X}$, and ${}_{\bar{r}_{q,n}}\phi_{ij}(k, \rho, \varepsilon)$, $j \in {}_{\bar{r}_{q,n}}\mathbb{Y}_i$, $i \in {}_{\bar{r}_{q,n-1}}\mathbb{X}$, be, respectively, the transition sets and transition power-exponential moments for the process

$$
\eta_{\varepsilon, \bar{r}_{q,n}}(t) = {}_{r_{q,n}}\eta_{\varepsilon, \bar{r}_{q,n-1}}(t)
$$

defined in the same way as the transition sets ${}_r\mathbb{Y}_i$, $i \in \mathbb{X}$, and the transition power-exponential moments ${}_r\phi_{ij}(k, \rho, \varepsilon)$, $j \in {}_r\mathbb{Y}_i$, $i \in \mathbb{X}$, for the process ${}_r\eta_\varepsilon(t)$.

Theorem 1 implies, by induction, that conditions $\mathbf{A}$–$\mathbf{F}_\rho$ hold for the reduced semi-Markov processes $\eta_\varepsilon(t), \eta_{\varepsilon, \bar{r}_{q,0}}(t), \ldots, \eta_{\varepsilon, \bar{r}_{q,n}}(t)$.

Thus, the recurrent application of the algorithm described in Theorem 1 to processes $\eta_{\varepsilon,\bar{r}_{q,0}}(t), \dots, \eta_{\varepsilon,\bar{r}_{q,n}}(t)$ let us construct the pivotal Laurent asymptotic expansions for transition power-exponential moments ${}_{\bar{r}_{q,n}}\phi_{ij}(k,\rho,\varepsilon)$, $j \in {}_{\bar{r}_{q,n}}\mathbb{Y}_i$, $i \in {}_{\bar{r}_{q,n-1}}\mathbb{X}$.

Let us take $n = m - 1$. In this case, the semi-Markov process $\eta_{\varepsilon,\bar{r}_{q,m-1}}(t)$ has the phase space ${}_{\bar{r}_{q,m-1}}\mathbb{X} = \{0\}$, which is a one-state set. Also, the space ${}_{\bar{r}_{q,m-2}}\mathbb{X} = \{q, 0\}$ is a two-state set.

By Lemma 4, the power-exponential moments of hitting times, $\Phi_{i0}(k,\rho,\varepsilon)$, coincide for the semi-Markov processes $\eta_\varepsilon(t)$, $\eta_{\varepsilon,\bar{r}_{q,0}}(t)$, $\dots, \eta_{\varepsilon,\bar{r}_{q,m-1}}(t)$ for every $k = 0, \dots, d$, $i = q, 0$.

Also, for the reduced semi-Markov process $\eta_{\varepsilon,\bar{r}_{q,m-1}}(t) = {}_q\eta_{\varepsilon,\bar{r}_{q,m-2}}(t)$, the exponential moment $\Phi_{i0}(k,\rho,\varepsilon) = {}_{\bar{r}_{q,m-1}}\phi_{i0}(k,\rho,\varepsilon)$ for every $k = 0, \dots, d$, $i = q, 0$.

Thus, the recurrent algorithm of sequential phase space reduction described above allows us to construct, for $k = 1, \dots, d$, $i = q, 0$, the pivotal $(\dot{h}^-_{i0}[k,\rho], \dot{h}^+_{i0}[k,\rho])$-expansions

$$\Phi_{i0}(k,\rho,\varepsilon) = \sum_{l=\dot{h}^-_{i0}[k,\rho]}^{\dot{h}^+_{ij}[k,\rho]} \dot{g}_{i0}[k,\rho,l]\varepsilon^l + \dot{o}_{k,\rho,0j}\left(\varepsilon^{\dot{h}^+_{i0}[k,\rho]}\right). \tag{27}$$

The above Laurent asymptotic expansions coincide with the corresponding Laurent asymptotic expansions for the transition power-exponential moments ${}_{\bar{r}_{q,m-1}}\phi_{i0}(k,\rho,\varepsilon)$.

The summation and multiplication operational rules for Laurent asymptotic expansions presented in propositions **(iii)** and **(v)** of Lemma 1 possess commutative, associative, and distributive properties, which should be understood as identities for the corresponding Laurent asymptotic expansions, i.e., identities for the corresponding parameters h, k, coefficients and remainders of functions represented in two alternative forms in the corresponding functional identities. We refer to works of the authors [14,15] for the corresponding details.

This makes it possible to prove that the Laurent asymptotic expansions for power-exponential moments ${}_{\bar{r}_{q,m-1}}\phi_{i0}(k,\rho,\varepsilon)$ are invariant with respect to any permutation $\bar{r}_{q,m} = \langle r_{q,0}, \dots, r_{q,m-2}, q, 0\rangle$ of the sequence $\langle 0, \dots, m\rangle$.

This legitimates the notation (with omitted index ${}_{\bar{r}_{q,m-1}}$) used for parameters, coefficients, and remainders in the asymptotic expansions (27).

Let $0 \leq n \leq m - 2$ and let $\bar{r}'_{q,n} = \langle r_{q,0}, \dots, r'_{q,n}\rangle$ be a permutation of the sequence $\bar{r}_{q,n}$.

The corresponding reduced semi-Markov process $\eta_{\varepsilon,\bar{r}'_{q,n}}(t)$ is constructed as the sequence of states for the initial semi-Markov process $\eta_\varepsilon(t)$ at sequential moments of its hitting into the same reduced phase space

$${}_{\bar{r}'_{q,n}}\mathbb{X} = \mathbb{X} \setminus \{r'_{q,0}, \dots, r'_{q,n}\} = {}_{\bar{r}_{q,n}}\mathbb{X} = \mathbb{X} \setminus \{r_{q,1}, \dots, r_{q,n}\}.$$

The times between sequential jumps of the reduced semi-Markov process $\eta_{\varepsilon,\bar{r}'_{q,n}}(t)$ are the times between sequential instants of hitting the above reduced phase space by the initial semi-Markov process $\eta_\varepsilon(t)$.

This obviously implies that the transition power-exponential moment ${}_{\bar{r}_{q,n}}\phi_{ij}(k,\rho,\varepsilon)$ is, for every $k = 0, \dots, d$, $j \in {}_{\bar{r}_{q,n}}\mathbb{Y}_i$, $i \in {}_{\bar{r}_{q,n-1}}\mathbb{X}$, $n = 0, \dots, m-1$, invariant (as functions of ε) with respect to any permutation $\bar{r}'_{q,n}$ of the sequence $\bar{r}_{q,n}$.

Moreover, as follows from the recurrent algorithms presented above, the transition power-exponential moment ${}_{\bar{r}_{q,n}}\phi_{ij}(k,\rho,\varepsilon)$ is a rational function of the initial transition power-exponential moment $\phi_{ij}(k,\rho,\varepsilon)$, $j \in \mathbb{Y}_i$, $i \in \mathbb{X}$ (quotients of sums of products for some of these moments).

By using identity arithmetical transformations (disclosure of brackets, imposition of a common factor out of the brackets, bringing a fractional expression to a common denominator, permutation of summands or multipliers, elimination of expressions with equal absolute values and opposite signs in the sums, and elimination of equal expressions in quotients) the rational functions ${}_{\bar{r}_{q,n}}\phi_{ij}(k,\rho,\varepsilon)$ can be transformed, respectively, into the rational functions ${}_{\bar{r}'_{q,n}}\phi_{ij}(k,\rho,\varepsilon)$ and vice versa.

In fact, one should only check this for the case where the permutation $\bar{r}'_{q,n}$ is obtained from the sequence $\bar{r}_{q,n}$ by exchange of a pair of neighbor states $r_{q,l}$ and $r_{q,l+1}$ for some $0 \leq l \leq n-1$. Then, the proof can be repeated for a pair of neighbor states for the sequence $\bar{r}'_{q,n}$, etc. In this way, the proof can be expanded to the case of an arbitrary permutation $\bar{r}'_{q,n}$ of the sequence $\bar{r}_{q,n}$. The above-mentioned poof of pairwise permutation invariance involves processes ${}_{\bar{r}_{q,l-1}}\eta_\varepsilon(t)$ (for the moment, we denote as

$$ {}_{\bar{r}_{q,-1}}\eta_\varepsilon(t) = \eta_\varepsilon(t) $$

the initial semi-Markov process), ${}_{\bar{r}_{q,l}}\eta_\varepsilon(t)$, and ${}_{\bar{r}_{q,l+1}}\eta_\varepsilon(t)$. It is absolutely analogous for $0 \leq l \leq n-1$. Taking this into account, we just show how this proof can be accomplished for the case $l = 0$.

The transition exponential moments ${}_{\bar{r}_{q,1}}\phi_{ij}(0,\rho,\varepsilon)$ and ${}_{\bar{r}'_{q,1}}\phi_{ij}(0,\rho,\varepsilon)$ for the sequences $\bar{r}_{q,1} = \langle r_0, r_1\rangle$ and $\bar{r}'_{q,1} = \langle r_1, r_0\rangle$ (here, $i, j \neq r_0, r_1$) can be transformed into the same symmetric (with respect to r_0 and r_1) rational function of the corresponding exponential moments, using the identity arithmetical transformations listed above:

(28)

$$
\begin{aligned}
{}_{\bar{r}_{q,1}}\phi_{ij}(0,\rho,\varepsilon) &= {}_{r_0}\phi_{ij}(0,\rho,\varepsilon) + {}_{r_0}\phi_{ir_1}(0,\rho,\varepsilon)\frac{{}_{r_0}\phi_{r_1 j}(0,\rho,\varepsilon)}{1 - {}_{r_0}\phi_{r_1 r_1}(0,\rho,\varepsilon)} \\
&= \phi_{ij}(0,\rho,\varepsilon) + \phi_{ir_0}(0,\rho,\varepsilon)\frac{\phi_{r_0 j}(0,\rho,\varepsilon)}{1 - \phi_{r_0 r_0}(0,\rho,\varepsilon)} \\
&\quad + (\phi_{ir_1}(0,\rho,\varepsilon) + \phi_{ir_0}(0,\rho,\varepsilon)\frac{\phi_{r_0 r_1}(0,\rho,\varepsilon)}{1 - \phi_{r_0 r_0}(0,\rho,\varepsilon)}) \\
&\quad \times \frac{(\phi_{r_1 j}(0,\rho,\varepsilon) + \phi_{r_1 r_0}(0,\rho,\varepsilon)\frac{\phi_{r_0 j}(0,\rho,\varepsilon)}{1-\phi_{r_0 r_0}(0,\rho,\varepsilon)})}{1 - \phi_{r_1 r_1}(0,\rho,\varepsilon) - \phi_{r_1 r_0}(0,\rho,\varepsilon)\frac{\phi_{r_0 r_1}(0,\rho,\varepsilon)}{1-\phi_{r_0 r_0}(0,\rho,\varepsilon)}} \\
&= \phi_{ij}(0,\rho,\varepsilon) \\
&\quad + \frac{\phi_{ir_0}(0,\rho,\varepsilon)\phi_{r_0 j}(0,\rho,\varepsilon)(1 - \phi_{r_1 r_1}(0,\rho,\varepsilon))}{(1 - \phi_{r_0 r_0}(0,\rho,\varepsilon))(1 - \phi_{r_1 r_1}(0,\rho,\varepsilon)) - \phi_{r_0 r_1}(0,\rho,\varepsilon)\phi_{r_1 r_0}(0,\rho,\varepsilon)} \\
&\quad + \frac{\phi_{ir_0}(0,\rho,\varepsilon)\phi_{r_0 r_1}(0,\rho,\varepsilon)\phi_{r_1 j}(0,\rho,\varepsilon)}{(1 - \phi_{r_0 r_0}(0,\rho,\varepsilon))(1 - \phi_{r_1 r_1}(0,\rho,\varepsilon)) - \phi_{r_0 r_1}(0,\rho,\varepsilon)\phi_{r_1 r_0}(0,\rho,\varepsilon)} \\
&\quad + \frac{\phi_{ir_1}(0,\rho,\varepsilon)\phi_{r_1 j}(0,\rho,\varepsilon)(1 - \phi_{r_0 r_0}(0,\rho,\varepsilon))}{(1 - \phi_{r_0 r_0}(0,\rho,\varepsilon))(1 - \phi_{r_1 r_1}(0,\rho,\varepsilon)) - \phi_{r_0 r_1}(0,\rho,\varepsilon)\phi_{r_1 r_0}(0,\rho,\varepsilon)} \\
&\quad + \frac{\phi_{ir_1}(0,\rho,\varepsilon)\phi_{r_1 r_0}(0,\rho,\varepsilon)\phi_{r_0 j}(0,\rho,\varepsilon)}{(1 - \phi_{r_0 r_0}(0,\rho,\varepsilon))(1 - \phi_{r_1 r_1}(0,\rho,\varepsilon)) - \phi_{r_0 r_1}(0,\rho,\varepsilon)\phi_{r_1 r_0}(0,\rho,\varepsilon)} \\
&= {}_{r_1}\phi_{ij}(0,\rho,\varepsilon) + {}_{r_1}\phi_{ir_0}(0,\rho,\varepsilon)\frac{{}_{r_1}\phi_{r_0 j}(0,\rho,\varepsilon)}{1 - {}_{r_1}\phi_{r_0 r_0}(0,\rho,\varepsilon)} \\
&= {}_{\bar{r}'_{q,1}}\phi_{ij}(0,\rho,\varepsilon).
\end{aligned}
$$

The above proof for the power-exponential moments ${}_{\bar{r}_{q,n}}\phi_{ij}(k,\rho,\varepsilon)$ is analogous.

Due to commutative, associative, and distributive properties of operations rules for Laurent asymptotic expansions, the above arithmetical transformations do not affect the corresponding asymptotic expansions for functions ${}_{\bar{r}_{q,n}}\phi_{ij}(k,\rho,\varepsilon)$ and, thus, these expansions are invariant with respect to any permutation $\bar{r}'_{q,n}$ of the sequence $\bar{r}_{q,n}$.

Therefore, the Laurent asymptotic expansions for the transition power-exponential moments ${}_{\bar{r}_{q,n}}\phi_{ij}(k,\rho,\varepsilon)$ and ${}_{\bar{r}'_{q,n}}\phi_{ij}(k,\rho,\varepsilon)$, given by the recurrent algorithm of sequential phase space reduction described above, are identical.

We refer to the authors' book [15], where one can find an analogous proof, concerning the invariance property of the corresponding Laurent asymptotic expansions for transition power moments for hitting times, presented in a more detailed form.

The recurrent algorithm described above for construction of the Laurent asymptotic expansions for power-exponential moments $\Phi_{i0}(k,\rho,\varepsilon)$, $k = 0,\ldots,d$, $i = q,0$, can be repeated for every $q \neq 0$.

This completes the proof of proposition **(i)** of Theorem 2.

In order to prove proposition **(ii)** of Theorem 2, one should repeat the same sequence of recurrent steps described above and, additionally, apply to every intermediate asymptotic expansion obtained above with the use of the operational rules given in propositions **(i)**, **(iii)**, **(v)**, and **(vii)** of Lemma 1 the corresponding additional operational rules given, respectively, in propositions **(ii)**, **(vi)**, **(vi)**, and **(viii)** of Lemma 1, for computing parameters of the corresponding upper bounds for remainders.

Unfortunately, the summation and multiplication operational rules for Laurent asymptotic expansions with explicit upper bound for remainders, presented in propositions **(iv)** and **(vi)** of Lemma 1, possess commutative but not associative and distributive properties. This makes the parameters ${}_{\bar{r}_{q,m}}\delta_{i0}[k,\rho]$, ${}_{\bar{r}_{q,m}}G_{i0}[k,\rho]$, ${}_{\bar{r}_{q,m}}\varepsilon_{i0}[k,\rho]$, $k = 0,\ldots,d$, $i = q,0$, given by the algorithm described below dependent on the choice of the sequence $\bar{r}_{q,m}$ for $q \neq 0$. □

Remark 2. Formulas for parameter δ_C given in Lemma 1 imply, however, that the following explicit inequalities take place for any sequence of states $\bar{r}_{q,m}$ and $k = 0,\ldots,d$, $i = q,0$, $q \neq 0$:

$$(29) \qquad {}_{\bar{r}_{i,m}}\delta_{i0}[k,\rho] \geq \delta^*[k,\rho] = \min_{j\in\mathbb{Y}_i, i\in\mathbb{X}, n=0,\ldots,k} \delta_{ij}[n,\rho].$$

We would like to note that, despite bulky forms, the algorithms for computing coefficients in the asymptotic expansions and parameters for the upper bound for remainders presented in Theorems 1 and 2 are computationally effective due to their recurrent character.

In conclusion, we would like to mention again that the power-exponential moments, which are interesting objects themselves, play the central role in studies of so-called quasi-stationary phenomena in stochastic systems. These phenomena describe the behavior of stochastic systems with random lifetimes. The core of the quasi-stationary phenomenon is that one can observe something that resembles a stationary behavior of the system before the lifetime goes to the end. The corresponding quasi-stationary distribution can be expressed via the exponential moments of sojourn times and the first-order power-exponential moments of return times, with parameter ρ, which is the characteristic root for the distributions of the corresponding return times. Related formulas, comments, and examples of applications to asymptotic analysis of perturbed queuing systems and bio-stochastic systems can be found in the book [4]. Also, related numerical examples can be found in the book [15]. The asymptotic expansions for quasi-stationary distributions of nonlinearly perturbed semi-Markov processes do involve higher-order power-exponential moments of return times and asymptotic expansions for these moments. We hope to publish the corresponding asymptotic results for nonlinearly and singularly perturbed semi-Markov processes in the near future.

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