Теорія Ймовір. та Матем. Статист.
Вип. 97, 2017

Theor. Probability and Math. Statist.
No. 97, 2018, Pages 201–209
https://doi.org/10.1090/tpms/1057
Article electronically published on February 21, 2019

ACCURACY AND RELIABILITY OF A MODEL OF AN ISOTROPIC AND HOMOGENEOUS GAUSSIAN RANDOM FIELD IN THE SPACE $C(\mathbb{T})$

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Abstract. The accuracy and reliability of a model of an isotropic homogeneous random field are studied in the space $C(\mathbb{T})$.

1. Introduction

Computer simulation is developing along with the development of computer technologies. Numerical simulation of stochastic processes and random fields is widely used nowadays in various fields of natural and social science, in particular in meteorology, radio engineering, sociology, and financial mathematics, as well as in testing different technical systems. Computer simulation became an effective tool allowing one to understand the essence of natural phenomena and to predict consequences of human activity and its impact on the environment.

A variety of methods for simulation of stochastic processes and random fields were developed by Mikhaĭlov and his collaborators [12]–[16]. G. O. Mikhĭlov, in particular, proposed the method of partition and randomization of the spectrum, the most popular method of simulation for stationary processes. A no less significant contribution to the development of methods of simulation was done by M. I. Yadrenko and his students [17, 18, 22–24].

The question about the accuracy and reliability of a model and the rate of approximation of a stochastic process or random field in various metrics is as important as the question of simulation itself. This question has been studied by Yu. V. Kozachenko and his students (see [4–8, 11]).

A mean-square continuous real-valued isotropic homogeneous Gaussian random field in $\mathbb{R}^2$ is studied in the current paper. Like the papers [11, 20, 21], a model for such a field is constructed by using a modified method of partition and randomization of the spectrum. In doing so we apply the representation of an isotropic and homogeneous random field proposed by M. I. Yadrenko in his monograph [23].

This paper is a continuation of [19]. One of the main results of the current paper is a bound for the probability of deviations in the uniform metric between a field and its model in a compact set $\mathbb{T}$. More precisely, we find a bound for the probability

$$\mathsf{P}\left\{\sup_{(t,x)\in\mathbb{T}}\left|X(t,x)-\widehat{X}(t,x)\right|>\varepsilon\right\},$$

2010 *Mathematics Subject Classification.* Primary 60G15; Secondary 60G07.

Key words and phrases. Gaussian random fields, isotropic homogeneous random field, simulation, accuracy and reliability.

where $X(t,x)$, $(t,x)\in\mathbb{R}^2$, is a random field and $\widehat{X}(t,x)$ is its model. The bound for the distribution of the deviation between a field and its model in the space $C(\mathbb{T})$ is derived from the corresponding bounds obtained in the paper [19]. In addition, the accuracy and reliability of the model are studied in the current paper.

The paper is organized as follows. Main results of the paper are described in Section 1. Section 2 contains necessary definitions and auxiliary results of the theory of sub-Gaussian random variables. Some results of the papers [19] are also given in Section 2. A bound for the probability of the deviation between an isotropic homogeneous random field and its model is found in Section 3. In addition, the reliability and accuracy of the model in the space $C(\mathbb{T})$ are also studied in Section 3. Section 4 summarizes the main results of the paper.

2. Auxiliary notions and results

Definition 2.1 ([1])**.** A random variable χ is called sub-Gaussian if there exists a constant $a\ge 0$ such that

$$\mathsf{E}\exp\{\lambda\chi\}\le\exp\left\{\frac{a^2\lambda^2}{2}\right\}$$

for all $\lambda\in\mathbb{R}$.

The space of all sub-Gaussian random variables defined in a standard probability space $\{\Omega,\mathbf{B},\mathsf{P}\}$ is denoted by $\operatorname{Sub}(\Omega)$. Note that $\operatorname{Sub}(\Omega)$ is a Banach space with respect to the norm $\tau(\chi)=\sup_{\lambda\neq 0}\left[2\ln\mathsf{E}\exp\{\lambda\chi\}/\lambda^2\right]^{1/2}$.

Definition 2.2 ([1])**.** A random field $X=\{X(u,v),u\in\mathbb{R},v\in\mathbb{R}\}$ is called sub-Gaussian if $X(u,v)\in\operatorname{Sub}(\Omega)$ for all $u,v\in\mathbb{R}$ and $\sup_{u,v\in\mathbb{R}}\tau(X(u,v))<\infty$.

Definition 2.3 ([23])**.** A random field $X=\{X(z),z\in\mathbb{R}^2\}$ is called homogeneous in the wide sense in $\mathbb{R}^2$ if $\mathsf{E}\,X(z)=\text{const}$, $z\in\mathbb{R}^2$, and

$$\mathsf{E}\,X(z)X(w)=B(z-w)=\int_{\mathbb{R}^2}e^{i(\lambda,z-w)}\,dF(\lambda),\qquad z,w\in\mathbb{R}^2.$$

Definition 2.4 ([23])**.** Let $SO(2)$ denote the group of rotations in $\mathbb{R}^2$ about the origin. A homogeneous random field $X(z),z\in\mathbb{R}^2$, is called isotropic if

$$\mathsf{E}\,X(z)X(w)=\mathsf{E}\,X(gz)X(gw)$$

for all elements g of the group $SO(2)$ and for all $z,w\in\mathbb{R}^2$.

Let $X=\{X(u,v),u\in\mathbb{R},v\in\mathbb{R}\}$ be a mean-square continuous real-valued isotropic homogeneous Gaussian random field in $\mathbb{R}^2$. We assume that $\mathsf{E}\,X(u,v)=0$. Similarly to the case of a complex-valued random field (see [23]) one can easily obtain a representation of the field $X(t,x)$ with (t,x) the polar coordinates, that is, $t\in\mathbb{R}^+$ and $x\in[0,2\pi]$. Namely

$$X(t,x)=\sum_{k=1}^{\infty}\cos(kx)\int_0^\infty J_k(t\lambda)\,d\eta_{1,k}(\lambda)+\sum_{k=1}^{\infty}\sin(kx)\int_0^\infty J_k(t\lambda)\,d\eta_{2,k}(\lambda),\tag{1}$$

where $\eta_{i,k}(\lambda)$, $i=1,2$, $k=1,2,\dots$, are independent Gaussian processes with independent increments, $\mathsf{E}\,\eta_{i,k}(\lambda)=0$, $\mathsf{E}(\eta_{i,k}(b)-\eta_{i,k}(c))^2=F(b)-F(c)$, $b>c$, $F(\lambda)$ is the spectral function of the field, and $J_k(u)=\frac{1}{\pi}\int_0^\pi\cos(k\varphi-u\sin\varphi)\,d\varphi$ is the Bessel function of the first kind, $k=1,2,\dots$.

Consider a partition $L=\{\lambda_0,\dots,\lambda_N\}$ of the set $[0,\infty)$ such that $\lambda_0=0$, $\lambda_l<\lambda_{l+1}$, $\lambda_{N-1}=\Lambda$, $\lambda_N=\infty$, and $C=\max_{0<l\le N-2}\lambda_{l+1}/\lambda_l<\infty$.

Then

$$\widehat{X}(t,x) = \sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \eta_{1,k,l} J_k(t\zeta_l) + \sum_{k=1}^{M} \sin(kx) \sum_{l=0}^{N-1} \eta_{2,k,l} J_k(t\zeta_l) \tag{2}$$

is treated as a model of the field $X(t,x)$, where $\eta_{i,k,l} = \int_{\lambda_l}^{\lambda_{l+1}} d\eta_{i,k}(\lambda)$, $i = 1,2$, and $\eta_{i,k,l}$ are independent Gaussian random variables such that $\mathsf{E}\,\eta_{i,k,l} = 0$, $\mathsf{E}\,\eta_{i,k,l}^2 = F(\lambda_{l+1}) - F(\lambda_l) = b_l^2$, $b_l^2 > 0$, and ζ_l, $l = 0,\dots,N-2$, are independent random variables that do not depend on $\eta_{i,k,l}$ and are distributed in the intervals $[\lambda_l, \lambda_{l+1}]$ according to the distribution functions

$$F_l(\lambda) = \mathsf{P}\{\zeta_l < \lambda\} = \frac{F(\lambda) - F(\lambda_l)}{F(\lambda_{l+1}) - F(\lambda_l)},$$

$\zeta_{N-1} = \Lambda$. If $b_l^2 = 0$, then $\zeta_l = 0$ with probability one. For simplicity we suppose that $b_l^2 > 0$, $l = 0,1,\dots,N-1$.

It is shown in the paper [20] that $\widehat{X}(t,x)$ and $X(t,x) - \widehat{X}(t,x)$ are sub-Gaussian random fields.

Put

$$\chi_M(t,x) = X(t,x) - \widehat{X}(t,x), \qquad 0 \le t \le T, \quad 0 \le x \le 2\pi, \tag{3}$$

and let

$$\sigma_0 = \sup_{\substack{0\le t\le T\\ 0\le x\le 2\pi}} \tau\big(\chi_M(t,x)\big)$$

and

$$\sigma(h) = \sup_{\substack{|t-s|\le h\\ |x-y|\le h}} \tau\big(\chi_M(t,x) - \chi_M(s,y)\big),$$

where $0 \le t,s \le T$ and $0 \le x,y \le 2\pi$.

Proposition 2.1 ([19]). *Let $X(t,x)$ and $\widehat{X}(t,x)$ be defined in (1) and (2), respectively. Assume that a partition $L = \{\lambda_0,\dots,\lambda_N\}$ of the set $[0,\infty)$ is such that $\lambda_l < \lambda_{l+1}$ and $\lambda_{l+1} - \lambda_l = \frac{\Lambda}{N-1}$, $l = 0,\dots,N-2$. If*

$$\int_0^\infty \lambda^{2\alpha}\, dF(\lambda) < \infty$$

for some $\frac12 < \alpha \le 1$, then

$$\begin{aligned}\sigma_0 \le \Bigg[&\frac{4^{2(1-\alpha)+1} T^{2\alpha} \pi^{2\alpha} M}{2\alpha - 1} \left(2\alpha - \frac{1}{M^{2\alpha-1}}\right) \left(\frac{\Lambda}{N-1}\right)^{2\alpha} \\ &\times \left(F(\Lambda) + \left(\frac{3T}{2}\right)^{2\alpha} \int_0^\Lambda \lambda^{2\alpha}\, dF(\lambda) \right) \\ &+ 8M^2\big(F(+\infty) - F(\Lambda)\big) \\ &\qquad + \frac{2^{2(1-\alpha)+1} T^{2\alpha} \pi^{2\alpha}}{(2\alpha-1) M^{2\alpha-1}} \int_0^\infty \lambda^{2\alpha}\, dF(\lambda) \Bigg]^{\frac12}.\end{aligned}$$

Proposition 2.2 ([19]). *Let $X(t,x)$ and $\widehat{X}(t,x)$ be defined in (1) and (2), respectively, and let*

$$\sigma(h) = \sup_{\substack{|t-s|\le h\\ |x-y|\le h}} \tau\big(\chi_M(t,x) - \chi_M(s,y)\big),$$

where $\chi_M(t,x)$ *is given by* (3). *Assume that a partition* $L=\{\lambda_0,\dots,\lambda_N\}$ *of the interval* $[0,\infty)$ *is such that* $\lambda_l<\lambda_{l+1}$ *and* $\lambda_{l+1}-\lambda_l=\frac{\Lambda}{N-1}$. *If*

$$\int_0^\infty \lambda^{2\nu}\,dF(\lambda)<\infty$$

for some $\nu>\frac{1}{2}$, *then*

$$\sigma(h)\le\frac{C_1}{\left(\ln\left(\frac{1}{h}+1\right)\right)^\delta},$$

where

$$\begin{aligned}
C_1=\Bigg[&2\cdot 4^{2(2-\alpha)}\left(\frac{\delta}{\alpha}\right)^{2\delta}\left(\frac{\pi}{2}\right)^{2\alpha}\frac{M}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)\left(\frac{\Lambda}{N-1}\right)^{2\alpha}\\
&\times\left(F(\Lambda)+\left[\left(\frac{3T}{4}\right)^{2\alpha}+(1+2^{\alpha+1})T^{2\alpha}+\left(\frac{3T^2\Lambda}{2}\right)^{2\alpha}\right]\int_0^\Lambda\lambda^{2\alpha}\,dF(\lambda)\right)\\
&+9\cdot 4^{4-2\alpha}M^2\left(\frac{\delta}{\alpha}\right)^{2\delta}\left(\int_\Lambda^\infty|\lambda-\Lambda|^{2\alpha}\,dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}\big(F(+\infty)-F(\Lambda)\big)\right)\\
&+4^{4-2\alpha}T^{2\alpha}\pi^{2\alpha}M\left(\sum_{k=1}^{M}\frac{\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}}{k^{2\alpha}}\right)\left(\frac{\Lambda}{N-1}\right)^{2\alpha}\\
&\times\left(F(\Lambda)+\left(\frac{3T}{2}\right)^{2\alpha}\int_0^\Lambda\lambda^{2\alpha}\,dF(\lambda)\right)\\
&+16M(F(+\infty)-F(\Lambda))\sum_{k=1}^{M}\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}\\
&+\frac{4^{3-2\alpha}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}}\\
&\times\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_0^\infty\lambda^{2\alpha}\,dF(\lambda)+(2T)^{2\alpha}\left(\frac{\delta}{\beta}\right)^{2\delta}\int_0^\infty\lambda^{2\nu}\,dF(\lambda)\right)\\
&+2^{4-\alpha}T^{2\alpha}\pi^{2\alpha}\int_0^\infty\lambda^{2\alpha}\,dF(\lambda)\sum_{k=M+1}^{\infty}\frac{\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}}{k^{2\alpha}}\Bigg]^{\frac{1}{2}},
\end{aligned}\tag{4}$$

$\frac{1}{2}<\alpha\le 1$, $\frac{\alpha}{\delta}\le 1$, $\delta>0$, *and* $0<\beta\le 1$.

Definition 2.5. Let $\mathbb{T}=\{0\le t\le T,\ 0\le x\le 2\pi\}$. We say that a random field $\widehat{X}(t,x)$ approximates a Gaussian field $X(t,x)$ with reliability $1-\gamma$, $0<\gamma<1$, and accuracy $q>0$ in the space $C(\mathbb{T})$ if there exists a partition L such that

$$\mathbf{P}\left\{\sup_{t\in\mathbb{T}}\left|X(t,x)-\widehat{X}(t,x)\right|>q\right\}\le\gamma.$$

Theorem 2.1. *Let* $\mathbb{R}^2$, $\mathbb{T}=\{t=(t_1,t_2)\colon 0\le t_i\le T, i=1,2\}$, $T>0$, $d(t,s)=\max_{1\le i\le 2}|t_i-s_i|$, *and* $X=\{X(t),t\in\mathbb{T}\}$ *be a sub-Gaussian random field. Assume that* $\sup_{d(t,s)\le h}\tau\big(X(t)-X(s)\big)\le\sigma(h)$, *where* $\sigma(h)$ *is a continuous decreasing function such that* $\sigma(h)\to 0$ *as* $h\to 0$ *and*

$$\int_0^{\varepsilon_0}\sqrt{-\frac{1}{2}\ln\left(\sigma^{(-1)}(\varepsilon)\right)}\,d\varepsilon<\infty,$$

where $\varepsilon_0=\sup_{t\in\mathbb{T}}\left(\mathsf{E}|X(t)|^2\right)^{1/2}<\infty$ *and* $\sigma^{(-1)}(\varepsilon)$ *denotes the inverse function to* $\sigma(\varepsilon)$.

Then

$$\mathsf{P}\left\{\sup_{t\in\mathbb{T}}|X(t)| > u\right\} \le 2\widetilde{A}(u,\theta)$$

for all $0 < \theta < 1$ *and* $u > \frac{2\widetilde{I}(\theta\varepsilon_0)}{\theta(1-\theta)}$, *where*

$$\widetilde{A}(u,\theta) = \exp\left\{-\frac{1}{2\varepsilon_0^2}\left(u(1-\theta) - \frac{2}{\theta}\widetilde{I}(\theta\varepsilon_0)\right)^2\right\},$$

$$\widetilde{I}(v) = \int_0^v \left(2\ln\left(\frac{T}{2\sigma^{(-1)}(\varepsilon)} + 1\right)\right)^{\frac{1}{2}} d\varepsilon.$$

Theorem 2.1 is a particular case of Theorem 8 of [10] (also see [9]).

3. Mainstream

Theorem 3.1. *Let a model* $\widehat{X}(t,x)$ *be constructed from a partition* L *such that* $q > \frac{2\widetilde{I}(\theta\varepsilon_0)}{\theta(1-\theta)}$, $0 < \theta < 1$, *and*

$$2\exp\left\{-\frac{1}{2\varepsilon_0^2}\left(q(1-\theta) - \frac{2}{\theta}\widetilde{I}(\theta\varepsilon_0)\right)^2\right\} \le \gamma,$$

where $\varepsilon_0 = \sup_{0\le t\le T}\tau(\chi_M(t,x)) = \sigma_0$ *and* $\chi_M(t,x)$ *is defined by* (3). *Further let* $\widetilde{I}(\theta\varepsilon_0) \le \widehat{I}(\theta\varepsilon_0)$, *where*

$$\widehat{I}(\theta\varepsilon_0) = \int_0^{\theta\varepsilon_0} \sqrt{2\ln\left(\frac{T}{2}\left(\exp\left\{\left(\frac{C_1}{\varepsilon}\right)^{1/\delta}\right\} - 1\right) + 1\right)}\, d\varepsilon,$$

C_1 *is defined by* (4), $T > 2\pi$, $\frac{1}{2} < \alpha \le 1$, $\frac{\alpha}{\delta} \le 1$, $\delta > 0$, $0 < \beta \le 1$, *and* $\nu > \frac{1}{2}$.

Then the model $\widehat{X}(t,x)$ *approximates the Gaussian random field* $X(t,x)$ *with reliability* $1-\gamma$, $0 < \gamma < 1$, *and accuracy* $q > 0$ *in the space* $C(\mathbb{T})$.

Proof. According to Theorem 2.1

$$\mathsf{P}\left\{\sup_{t\in\mathbb{T}}|\chi_M(t,x)| > q\right\} \le 2\exp\left\{-\frac{1}{2\varepsilon_0^2}\left(q(1-\theta) - \frac{2}{\theta}\widetilde{I}(\theta\varepsilon_0)\right)^2\right\}$$

for $q > \frac{2\widetilde{I}(\theta\varepsilon_0)}{\theta(1-\theta)}$, $0 < \theta < 1$, where

$$\widetilde{I}(\theta\varepsilon_0) = \int_0^{\theta\varepsilon_0} \left(2\ln\left(\frac{T}{2\sigma^{(-1)}(\varepsilon)} + 1\right)\right)^{\frac{1}{2}} d\varepsilon, \qquad \sigma(h) = \sup_{\substack{|t-s|\le h\\|x-y|\le h}} \tau\big(\chi_M(t,x) - \chi_M(s,y)\big).$$

Proposition 2.2 with $\sigma(h)$ implies

$$\sigma^{(-1)}(h) = \frac{1}{\exp\left\{\left(\frac{C_1}{h}\right)^{1/\delta}\right\} - 1},$$

where $\frac{1}{2} < \alpha \le 1$, $\frac{\alpha}{\delta} \le 1$, $\delta > 0$, $0 < \beta \le 1$, $\nu > \frac{1}{2}$, and C_1 is defined by (4). Then

$$\widetilde{I}(\theta\varepsilon_0) \le \int_0^{\theta\varepsilon_0} \sqrt{2\ln\left(\frac{T}{2}\left(\exp\left\{\left(\frac{C_1}{\varepsilon}\right)^{1/\delta}\right\} - 1\right) + 1\right)}\, d\varepsilon = \widehat{I}(\theta\varepsilon_0)$$

and $\widehat{I}(\theta\varepsilon_0)$ can be arbitrarily small with a certain choice of parameters M, Λ, and N. More precisely, given an accuracy and reliability we choose M in such a way that the fifth and sixth terms in (4) are arbitrarily small. Then using this value of M we choose Λ such that the second and fourth terms in (4) are arbitrarily small. Finally, with M and Λ fixed as above we choose N such that the first and third terms in (4) are arbitrarily small. Note that not only is C_1 arbitrarily small for M, Λ, and N chosen above but also ε_0 defined in Proposition 2.1 is arbitrarily small. This means that there exists a partition L such that

$$2\exp\left\{-\frac{1}{2\varepsilon_0^2}\left(q(1-\theta)-\frac{2}{\theta}\widetilde{I}(\theta\varepsilon_0)\right)^2\right\}\le\gamma.$$

This together with Definition 2.5 implies that the model $\widehat{X}(t,x)$ constructed above approximates the field $X(t,x)$ with reliability $1-\gamma$, $0<\gamma<1$, and accuracy $q>0$ in the space $C(\mathbb{T})$. □

Example. For an isotropic homogeneous Gaussian random field consider a model $\widehat{X}(t,x)$ constructed according to equality (2). Put

$$F(\lambda)=\begin{cases}1-\frac{1}{\lambda^4}, & \text{if } \lambda\ge 1,\\ 0, & \text{if } \lambda<1.\end{cases}$$

Now we estimate the constants C_1 and ε_0. Represent both constants as sums of three terms as follows:

$$C_1=\left(C_I+C_{II}+C_{III}\right)^{\frac{1}{2}},$$

where

$$\begin{aligned}
C_I&=\frac{4^{3-2\alpha}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}}\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_0^\infty\lambda^{2\alpha}\,dF(\lambda)+(2T)^{2\alpha}\left(\frac{\delta}{\beta}\right)^{2\delta}\int_0^\infty\lambda^{2\nu}\,dF(\lambda)\right)\\
&\quad+2^{4-\alpha}T^{2\alpha}\pi^{2\alpha}\int_0^\infty\lambda^{2\alpha}\,dF(\lambda)\sum_{k=M+1}^{\infty}\frac{\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}}{k^{2\alpha}},\\
C_{II}&=9\cdot 4^{4-2\alpha}M^2\left(\frac{\delta}{\alpha}\right)^{2\delta}\left(\int_\Lambda^\infty|\lambda-\Lambda|^{2\alpha}\,dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}\big(F(+\infty)-F(\Lambda)\big)\right)\\
&\quad+16M\big(F(+\infty)-F(\Lambda)\big)\sum_{k=1}^{M}\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta},\\
C_{III}&=2\cdot 4^{2(2-\alpha)}\left(\frac{\delta}{\alpha}\right)^{2\delta}\left(\frac{\pi}{2}\right)^{2\alpha}\frac{M}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)\left(\frac{\Lambda}{N-1}\right)^{2\alpha}\\
&\quad\times\left(F(\Lambda)+\left[\left(\frac{3T}{4}\right)^{2\alpha}+(1+2^{\alpha+1})T^{2\alpha}+\left(\frac{3T^2\Lambda}{2}\right)^{2\alpha}\right]\int_0^\Lambda\lambda^{2\alpha}\,dF(\lambda)\right)\\
&\quad+4^{4-2\alpha}T^{2\alpha}\pi^{2\alpha}M\left(\sum_{k=1}^{M}\frac{\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}}{k^{2\alpha}}\right)\left(\frac{\Lambda}{N-1}\right)^{2\alpha}\\
&\quad\times\left(F(\Lambda)+\left(\frac{3T}{2}\right)^{2\alpha}\int_0^\Lambda\lambda^{2\alpha}dF(\lambda)\right),
\end{aligned}$$

and

$$\varepsilon_0=\left(\varepsilon_I+\varepsilon_{II}+\varepsilon_{III}\right)^{\frac{1}{2}}.$$

Here

$$\varepsilon_I = \frac{2^{2(1-\alpha)+1} T^{2\alpha} \pi^{2\alpha}}{(2\alpha-1) M^{2\alpha-1}} \int_0^\infty \lambda^{2\alpha}\, dF(\lambda),$$

$$\varepsilon_{II} = 8M^2 \big(F(+\infty) - F(\Lambda)\big),$$

$$\varepsilon_{III} = \frac{4^{2(1-\alpha)+1} T^{2\alpha} \pi^{2\alpha} M}{2\alpha-1} \left(2\alpha - \frac{1}{M^{2\alpha-1}}\right) \left(\frac{\Lambda}{N-1}\right)^{2\alpha} \times \left(F(\Lambda) + \left(\frac{3T}{2}\right)^{2\alpha} \int_0^\Lambda \lambda^{2\alpha}\, dF(\lambda)\right).$$

Choose $\alpha = 1$, $\beta = \frac{1}{2}$, $\delta = 1$, $\nu = \frac{3}{2}$, and $T = 1$. After simple algebra we obtain

$$C_I = \frac{784\pi^2}{3M} + 16\pi^2 \sum_{k=M+1}^{\infty} \frac{\big(\ln(k^2+e)\big)^2}{k^2},$$

$$C_{II} = \frac{336M^2}{\Lambda^2} + \frac{16M}{\Lambda^4} \sum_{k=1}^{M} \left(\ln\left(k^2+e\right)\right)^2,$$

$$C_{III} = 8\pi^2(2M-1)\left(\frac{\Lambda}{N-1}\right)^2 \left(\frac{9}{2}\Lambda^2 - \frac{89}{8\Lambda^2} - \frac{1}{\Lambda^4} + \frac{61}{8}\right) + 16\pi^2 M \left(\frac{\Lambda}{N-1}\right)\left(\frac{11}{2} - \frac{9}{2\Lambda^2} - \frac{1}{\Lambda^4}\right) \sum_{k=1}^{M} \frac{\left(\ln\left(k^2+e\right)\right)^2}{k^2}.$$

Now we fix particular values of the accuracy and reliability for a model that approximates the field. Namely $q = 0.06$ and $1-\gamma = 0.99$. Also let $\theta = \frac{1}{2}$. Then Theorem 3.1 implies

$$2\exp\left\{-\frac{1}{2\varepsilon_0^2}\left(0.06 \cdot \frac{1}{2} - 4\widehat{I}\left(\frac{\varepsilon_0}{2}\right)\right)^2\right\} \le 0.01,$$

where

$$\widehat{I}\left(\frac{\varepsilon_0}{2}\right) = \int_0^{\frac{\varepsilon_0}{2}} \sqrt{2\ln\left(\frac{1}{2}\left(\exp\left\{\left(\frac{C_1}{\varepsilon}\right)\right\} - 1\right) + 1\right)}\, d\varepsilon = \int_0^{\frac{\varepsilon_0}{2}} \sqrt{2\ln\left(\frac{1}{2}\exp\left\{\frac{C_1}{\varepsilon}\right\} + \frac{1}{2}\right)}\, d\varepsilon,$$

that is,

$$2\exp\left\{-\frac{1}{2\varepsilon_0^2}\left(0.03 - 4\int_0^{\frac{\varepsilon_0}{2}} \sqrt{2\ln\left(\frac{1}{2}\exp\left\{\frac{C_1}{\varepsilon}\right\} + \frac{1}{2}\right)}\, d\varepsilon\right)^2\right\} \le 0.01.$$

One can check numerically that the latter inequality holds if $\widehat{C}_1 = 15.79$ and $\hat{\varepsilon}_0 = 0.97$. In other words,

$$(C_I + C_{II} + C_{III})^{\frac{1}{2}} \le \widehat{C}_1$$

and

$$(\varepsilon_I + \varepsilon_{II} + \varepsilon_{III})^{\frac{1}{2}} \le \hat{\varepsilon}_0.$$

Without loss of generality assume that

$$C_I \le \widehat{C}_1^2/3, \qquad C_{II} \le \widehat{C}_1^2/3, \qquad C_{III} \le \widehat{C}_1^2/3$$

and

$$\varepsilon_I \le \hat{\varepsilon}_0^2/3, \qquad \varepsilon_{II} \le \hat{\varepsilon}_0^2/3, \qquad \varepsilon_{III} \le \hat{\varepsilon}_0^2/3.$$

Solving the inequalities for C_I and ε_I with respect to M we find two values of M, say M_1 and M_2. Then we choose $M = \max\{M_1, M_2\}$. With this value of M we solve the inequalities for C_{II} and ε_{II} with respect to Λ. As Λ we take the maximal solution of these inequalities. Substituting M and Λ just found in the inequalities for C_{III} and ε_{III} we evaluate N in a similar fashion.

Using an appropriate software one can easily solve the inequalities mentioned above and find values of all parameters of interest. Then one can construct the corresponding model for an isotropic homogeneous Gaussian random field.

4. Concluding remarks

This paper is a continuation of research initiated in [19]. In the current paper, bounds are found for the deviation between an isotropic and homogeneous random field and its model in the metric of the space $C(\mathbb{T})$. These bounds allow us to study the accuracy and reliability of the model constructed according to a modified method of partition and randomization of the spectrum.

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Received 16/AUG/2016

Translated by S. V. KVASKO