Теорія Ймовір. та Матем. Статист.
Вип. 98, 2018

Theor. Probability and Math. Statist.
No. 98, 2019, Pages 115–126
https://doi.org/10.1090/tpms/1066
Article electronically published on August 19, 2019

# SCHRÖDINGER EQUATION WITH GAUSSIAN POTENTIAL

UDC 519.21

Y. HU

Abstract. This paper studies the Schrödinger equation with fractional Gaussian noise potential of the form $\Delta u(x) = u(x) \diamond \dot{W}(x)$, $x \in D$, $u(x) = \phi(x)$, $x \in \partial D$, where $\Delta$ is the Laplacian on the $d$-dimensional Euclidean space $\mathbb{R}^d$, $D \subseteq \mathbb{R}^d$ is a given domain with smooth boundary $\partial D$, $\phi$ is a given nice function on the boundary $\partial D$, and $\dot{W}$ is the fractional Gaussian noise of Hurst parameters $(H_1, \dots, H_d)$ and $\diamond$ denotes the Wick product. We find a family of distribution spaces $(\mathbb{W}_\lambda, \lambda > 0)$, with the property $\mathbb{W}_\lambda \subseteq \mathbb{W}_\mu$ when $\lambda \le \mu$, such that under the condition $\sum_{i=1}^d H_i > d-2$, the solution exists uniquely in $\mathbb{W}_{\lambda_0}$ when $\lambda_0$ is sufficiently large and the solution is not in $\mathbb{W}_{\lambda_1}$ when $\lambda_1$ is sufficiently small.

## 1. Introduction

Let $\big(W(x) = W^H(x_1, \dots, x_d), x = (x_1, \dots, x_d)^T \in \mathbb{R}^d\big)$ be a fractional Brownian field on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. The expectation on $(\Omega, \mathcal{F}, \mathsf{P})$ is denoted by $\mathsf{E}$. This means that $W(x)$ is a mean zero Gaussian field with covariance given by

$$\mathsf{E}\left[W(x)W(y)\right] = \left(\frac{1}{2}\right)^d \prod_{i=1}^d \left(|x_i|^{2H_i} + |y_i|^{2H_i} - |x_i - y_i|^{2H_i}\right) \quad \text{for all } x, y \in \mathbb{R}^d. \tag{1.1}$$

In this paper we shall fix the Hurst parameters $H = (H_1, \dots, H_d)$ with $H_i > \frac{1}{2}$, $i = 1, 2, \dots, d$. For notational simplicity we omit the dependence on $H$. Consider the following stochastic Poisson equation with multiplicative fractional Gaussian noise $\dot{W}(x) = \frac{\partial^d}{\partial x_1 \dots \partial x_d} W(x)$:

$$\begin{cases} \Delta U(x) = u(x) \diamond \dot{W}_H(x), & x \in D; \\ U(x) = \phi(x), & x \in \partial D, \end{cases} \tag{1.2}$$

where $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplacian, $D \subseteq \mathbb{R}^d$ is a given bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial D$, $\phi$ is a nice function defined on the boundary $\partial D$, and $\diamond$ denotes the Wick product (see, e.g., [13]), which is related to the Skorohod integral (see (3.2) below for the definition of the solution to (1.2)).

Stochastic partial differential equations driven by fractional noises or general Gaussian noises have been studied by many researchers. There is an enormous amount of references. Among the investigations published recently there is a well-studied equation relevant to our equation (1.2) which is the parabolic Anderson model $\partial_t u(t,x) = \Delta u(t,x) + \kappa u \diamond \dot{W}$. The moment bounds, various kinds of asymptotic behaviours as

2010 *Mathematics Subject Classification.* Primary 60G15, 60G22, 46F25.
*Key words and phrases.* Fractional Brownian field, fractional Gaussian noise, Schrödinger equation, distribution spaces, chaos expansion, Poisson equation, multiplicative noise.

$t \to \infty$, intermittency, and so on are known for this model. We refer the interested readers to [8, 9] and in particular to the references therein for details.

However, the stationary counterpart (1.2) of the parabolic Anderson model has received much less attention. When the noise is additive, namely for the stochastic Poisson equation $\Delta U(x) = \dot{W}(x)$ there has been some studies in [11, 12], where it is proved that when $\sum_{i=1}^d > d-2$, the solution exists uniquely in $L^2 = L^2(\Omega, \mathcal{F}, \mathsf{P})$. In [19], a nonlinear equation but still with additive noise of the form $\Delta u + f(u) = \dot{W}$ is studied, and still under the condition $\sum_{i=1}^d > d - 2$. The paper [20] may be considered to be relevant to more general nonadditive noise, where the equation of the form $\partial_{tt} z_t = \sigma(z_t)\dot{x}_t$, $t \in [0, 1]$ is studied. Here $x$ is a given Hölder continuous function. However, it is in the framework of rough path theory and is in one dimension.

In this paper we are mainly concerned with the stochastic Poisson equation (1.2) with *multiplicative noise* in general dimension ($d \geq 2$). Following the terminology of [2] we call equation (1.2) the *Schrödinger equation with fractional Gaussian potential*. We shall construct a family of distribution spaces $\mathbb{W}_\lambda$ satisfying $\mathbb{W}_\lambda \subseteq \mathbb{W}_\mu$ for $0 < \lambda \leq \mu < \infty$, such that the solution exists uniquely in $\mathbb{W}_\lambda$ when $\lambda$ is sufficiently large, and it is not in $\mathbb{W}_\lambda$ when $\lambda$ is sufficiently small. As a consequence the solution is not square integrable.

In the proof of our main theorem we use the Hardy–Littlewood-type inequality obtained by Mémin, Mishura, and Valkeila [16]. The use of this inequality can also simplify the proofs in [11] and [12].

Here is the organization of the paper. In Section 2 we briefly present some preliminary material that is needed in this paper. In particular, we shall recall the chaos expansion, define the stochastic integral, and introduce some distribution spaces. In Section 3 we state the main results of the paper in one theorem and in Section 4 we present the proof of the main results. When the dimension $d = 1$, the problem becomes simpler. In this case not only can we consider the above Dirichlet boundary condition, we can also study "initial"-type condition. We present a brief discussion of a one-dimensional case in Section 5.

## 2. Preliminaries

Recall that $\big(W(x), x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d\big)$ is a fractional Brownian field of the Hurst parameter $H = (H_1, \ldots, H_d)$ on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Namely, $W$ is a mean zero Gaussian random field with covariance given by (1.1). Here and in what follows we use $^T$ to denote the transpose of a vector or matrix. Then, formally the fractional Gaussian noise $\dot{W}$ has the following covariance structure:

$$\mathsf{E}\left[\dot{W}(x)\dot{W}(y)\right] = \phi(x, y) := \prod_{i=1}^d \phi_{H_i}(x_i, y_i) \quad \text{for all } x, y \in \mathbb{R}^d, \tag{2.1}$$

where

$$\phi_{H_i}(u, v) = H_i(2H_i - 1)|u - v|^{2H_i - 2}, \quad u, v \in \mathbb{R}. \tag{2.2}$$

We suppose that $\mathcal{F} = \sigma\left(W(x), x \in \mathbb{R}^d\right)$. For simplicity of the presentation, we assume that $H_i > \frac{1}{2}$ for all $i = 1, \ldots, d$. First, we will follow [7] (see also [18]) to define the stochastic integral with respect to $W$. For one parameter case ($d = 1$) there has been extensive studies on such stochastic integral; see, for example, [1, 6, 17].

Let $D$ be a bounded domain in $\mathbb{R}^d$ and let $\mathcal{S}$ be the set of all smooth functions from $D$ to $\mathbb{R}$ with compact support. For any two functions $f$ and $g$ in $\mathcal{S}$, we define their scalar product as

$$\langle f, g \rangle_{\mathcal{H}} = \int_D f(x)g(y)\phi(x, y)\, dx\, dy.$$

Let $\mathcal{H}$ denote the Hilbert space obtained by completing $\mathcal{S}$ with respect to the above scalar product. As is well-known $\mathcal{H}$ contains genuine distributions (generalized functions).

For any $f \in \mathcal{S}$, we can define the stochastic integral $\int_D f(x)\,dW(x)$ by the integration by parts in the following way (since $f$ has compact support in $D$):

$$\int_D f(x)\,dW(x) = (-1)^d \int_D W(x)\,\frac{\partial^d}{\partial x_1 \dots \partial x_d} f(x)\,dx.$$

It is easy to verify that for any $f, g \in \mathcal{S}$, we have

$$\mathsf{E}\left[\int_D f(x)\,dW(x)\right] = 0, \tag{2.3}$$

$$\mathsf{E}\left[\int_D f(x)\,dW(x) \int_D g(x)\,dW(x)\right] = \langle f, g\rangle_{\mathcal{H}}, \tag{2.4}$$

where (2.4) follows from the integration by parts and is called the isometry formula. For any $f \in \mathcal{H}$, there exists a sequence $f_n \in \mathcal{S}$ such that $f_n \to f$ in $\mathcal{H}$. By the isometry (2.4) we see that $\int_D f_n(x)dW(x)$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathsf{P})$ and hence has a limit. It can be shown routinely that this limit is independent of the choice of sequence $f_n$ and is called the stochastic integral of $f$, denoted by $\int_D f(x)dW(x)$. This stochastic integral also satisfies the properties (2.3)–(2.4).

Let

$$H_n(x) = e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-x^2/2}, \qquad n = 0, 1, 2, \dots,$$

be the Hermite polynomials, which constitute an orthogonal basis of

$$L^2\left(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx\right).$$

Let $e_1, \dots, e_n, \dots \in \mathcal{S}$ be an orthonormal basis of $\mathcal{H}$. Denote $\tilde{e}_n = \int_D e_n(x)\,dW(x)$. Then $\tilde{e}_1, \dots, \tilde{e}_n, \dots$ are i.i.d. standard normal random variables. We define the multiple integral of $e_k^{\otimes n}$ as

$$I_n\left(e_k^{\otimes n}\right) = H_n(\tilde{e}_k).$$

We can extend $I_n$ to be a linear mapping from $\mathcal{H}^{\otimes n}$, the $n$-fold tensor product space of $\mathcal{H}$, to $L^2(D, \mathcal{F}, \mathsf{P})$. (This can be achieved by the polarization technique (see [7]). Notice that we also have $I_n(f) = I_n(\hat{f})$, where $\hat{f}$ denotes the symmetrization of $f \in \mathcal{H}^{\otimes n}$.) It is well known that for any $f \in \mathcal{H}^{\otimes n}$ and $g \in \mathcal{H}^{\otimes m}$, we have the following Itô isometry formula:

$$\mathsf{E}\big(I_n(f) I_m(g)\big) = \begin{cases} n!\,\langle \hat{f}, \hat{g}\rangle_{\mathcal{H}^{\otimes n}} & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \tag{2.5}$$

An element $f \in \mathcal{H}^{\otimes n}$ can be viewed as a (generalized) function of $n \times d$ variables $f(\mathbf{x}_1, \dots, \mathbf{x}_n)$, $\mathbf{x}_i \in \mathbb{R}^d$. ($\mathcal{H}^{\otimes n}$ may contain distributions. It is the completion of the smooth functions of the form $f(\mathbf{x}_1, \dots, \mathbf{x}_n)$ defined on $D^n$ with respect to the scalar product of $\mathcal{H}^{\otimes n}$.) If $f \in \mathcal{H}^{\otimes n}$ we also write $I_n(f) = \int_{D^n} f(\mathbf{x}_1, \dots, \mathbf{x}_n)\,dW(\mathbf{x}_1) \dots dW(\mathbf{x}_n)$. When $f_n$ is a genuine function the above multiple integral can be approximated by using the Wick product (see, e.g., [7, 13]). In the remaining part of the paper, we use $x$ to represent $\mathbf{x}$ so that $x_1, \dots, x_n$ are $d$-dimensional vectors.

From the chaos expansion theorem ([7]) we know that for any square integrable random variable $F \in L^2(D, \mathcal{F}, \mathsf{P})$, there are $f_n \in \mathcal{H}^{\otimes n}$, $n = 0, 1, 2, \dots$, such that

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{2.6}$$

and

$$\mathsf{E}\left(F^2\right) = \sum_{n=0}^{\infty} n!\, \|f_n\|^2_{\mathcal{H}^{\otimes n}}. \tag{2.7}$$

Motivated by the Hida and Kondratiev distribution spaces for nonlinear Wiener functionals ([5, 14, 15]) we introduce the following distribution spaces.

**Definition 2.1.** For any $\alpha > 0$ and $\lambda > 0$, we say $F \in \mathcal{W}_{\alpha,\lambda}$ if

$$F = \sum_{n=0}^{\infty} I_n(f_n) \quad \text{and} \quad \|F\|^2_{\alpha,\lambda} := \sum_{n=0}^{\infty} (n!)^{1-\alpha} \lambda^{-n} \|f_n\|^2_{\mathcal{H}^{\otimes n}} < \infty. \tag{2.8}$$

*Remark* 2.2. (1) $\mathcal{W}_{0,1} = L^2 (:= L^2(\Omega, \mathcal{F}, \mathsf{P}))$.

(2) For $\alpha > 0$ and $\lambda > 0$, $\mathcal{W}_{\alpha,\lambda}$ is a space of distributions. It can also be defined as the dual of the test functional space $\mathcal{W}_{-\alpha,-\lambda}$ (which is a subspace of $L^2(\Omega, \mathcal{F}, \mathsf{P})$).

(3) It is clear that for any positive $\lambda_1$ and $\lambda_2$, if $\alpha_1 < \alpha_2$, then $\mathcal{W}_{\alpha_1,\lambda_1} \subseteq \mathcal{W}_{\alpha_2,\lambda_2}$. It is also clear that for any $0 < \alpha_1 \le \alpha_2 < \infty$ and $0 < \lambda_1 \le \lambda_2 < \infty$, we have

$$\mathcal{W}_{\alpha_1,\lambda_1} \subseteq \mathcal{W}_{\alpha_2,\lambda_2}.$$

(4) Our space $\mathcal{W}_{\alpha,\lambda}$ is more regular than the Hida distribution spaces or the Kondratiev distribution spaces ([5, 14, 15]), where $f_n$ are allowed to be more general distributions. We require that $f_n \in \mathcal{H}^{\otimes n}$. In particular this means that in the chaos expansion of $F$, each chaos $I_n(f_n)$ of $F$ is required to be square integrable. Our space $\mathcal{W}_{\alpha,\lambda}$ is less regular than the Meyer–Watanabe distribution space which requires $\sum_{n=0}^{\infty} (n!) n^{-\alpha} \|f_n\|^2_{\mathcal{H}^{\otimes n}} < \infty$ for some $\alpha > 0$.

The case when $\alpha = 1$ is given particular attention in this paper. We give the following definition.

**Definition 2.3.** We define the distribution spaces

$$\mathbb{W}_\lambda = \mathcal{W}_{1,\lambda}$$

and

$$\mathbb{W} = \bigcup_{\lambda > 0} \mathbb{W}_\lambda.$$

It is easy to see that $L^2 = \mathcal{W}_{0,1} \subset W_\lambda$ for any $\lambda > 0$ and $L^2$ is a true subset of $\mathbb{W}_\lambda$ for any $\lambda > 0$.

**Definition 2.4.** Let $F(x) = \sum_{n=0}^{\infty} I_n(f_n(x))$, $x \in D$. Assume that there is a $\lambda > 0$ such that for any $x \in D$, $F(x) \in \mathbb{W}_\lambda$ (notice that $\lambda$ is independent of $x \in D$). For any $x \in D$, $f_n(x)$ can be viewed as a (generalized) function $f_n(x; x_1, \dots, x_n)$ on $D^n$ for all $x \in D$. $F(x)$ is called *Skorohod integrable* if

(1) $\tilde{f}(x_1, \dots, x_{n+1}) = f_n(x_{n+1}; x_1, \dots, x_n) \in \mathcal{H}^{\otimes(n+1)}$,

(2) $\sum_{n=1}^{\infty} I_{n+1}(\tilde{f}_n)$ is in $\mathbb{W}$. Namely, $\sum_{n=1}^{\infty} I_{n+1}(\tilde{f}_n)$ is convergent in $\mathbb{W}_\lambda$ for some $\lambda \in (0, \infty)$.

We denote $\int_D F(x) dW(x) = \sum_{n=1}^{\infty} I_{n+1}(\tilde{f}_n)$.

## 3. Main Result

Now we return to the equation (1.2). Let the domain $D \subseteq \mathbb{R}^d$ be bounded with boundary $\partial D$. Let $G(x, y)$ be the associated Green's function. First we make the following assumptions.

**Hypothesis 3.1.** *There is a positive constant $C$ such that*

$$|G(x,y)| \le \begin{cases} C|x-y|^{2-d} & \forall\, x,y\in D \quad \text{when } d\ge 3;\\ C\log|x-y| & \forall\, x,y\in D \quad \text{when } d=2. \end{cases} \tag{3.1}$$

**Hypothesis 3.2.** *There are another constant $c>0$ and a nonempty open subset $D_0\subseteq D$ such that*

$$G(x,y) \le \begin{cases} -c|x-y|^{2-d} & \forall\, x,y\in D_0 \quad \text{when } d\ge 3;\\ c\log|x-y| & \forall\, x,y\in D_0 \quad \text{when } d=2. \end{cases}$$

When $D$ has smooth boundary $\partial D$, it is known that Hypothesis 3.1 always holds.

*Remark* 3.3. Hypothesis 3.1 is about the boundedness of the absolute value of $G$ when $x$ and $y$ are in the whole domain $D$. Hypothesis 3.2 requires that $G$ is bounded above by a negative function (so $G$ is negative) when $x$ and $y$ are in a compact subset of $G$.

**Example 3.4.** If $D=B_R(0)$ is the $d$-dimensional ball of center 0 and radius $R$, then the corresponding Green's function has the following form (see [4, Proposition 1.22]):

$$G(x,y)=\begin{cases} \frac{1}{(n-2)\omega_d}\left(-|x-y|^{2-n}+\left|\frac{R}{|x|}x-\frac{|x|}{R}y\right|^{2-n}\right) & \text{when } d\ge 3;\\ \frac{1}{2\pi}\left(\log|x-y|-\log\left|\frac{R}{|x|}x-\frac{|x|}{R}y\right|\right) & \text{when } d=2, \end{cases}$$

where $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$. If $|x|,|y|\le R/2$, then

$$\left|\frac{R}{|x|}x-\frac{|x|}{R}y\right| \ge \left|\frac{R}{|x|}|x|-\frac{|x|}{R}|y|\right| \ge \frac{3R}{4}.$$

Thus when $d\ge 3$, we have

$$G(x,y)\le \frac{1}{(n-2)\omega_d}\left(-|x-y|^{2-n}+(3R/4)^{2-n}\right)\le -c|x-y|^{2-n}$$

when $x,y$ are in a certain neighbourhood of 0. In the same way we have when $d=2$,

$$G(x,y)\le c\log|x-y|$$

in a neighbourhood of 0.

By using the Green's function the solution to

$$\begin{cases} \Delta u(x)=f(x), & x\in D,\\ u(x)=\phi(x), & x\in\partial D, \end{cases}$$

can be represented by

$$u(x)=\int_D G(x,y)f(y)\,dy+\int_{\partial D}\frac{\partial G}{\partial n_y}(x,y)\phi(y)\,dS_y,$$

where $\frac{\partial}{\partial n_y}$ denotes the gradient along the normal direction and $dS_y$ denotes the surface measure on the boundary $\partial D$; the presence of $y$ means that we consider $x$ as a fixed parameter in the above derivation or integration.

Motivated by the above formula, we give the following definition about the solution to (1.2).

**Definition 3.5.** We say that a family of random variables $(u(x), x\in D)$ is a (mild) solution to (1.2) if the followings hold:

(1) For any $x\in D$, $G(x,\cdot)u(\cdot)$ is Skorohod integrable in $\mathbb{W}$.

(2) The following integral equation holds:

$$u(x) = \int_D G(x,y)u(y)\,dW(y) + \int_{\partial D} \frac{\partial G}{\partial n_y}(x,y)\phi(y)\,dS_y \quad \forall\, x \in D. \tag{3.2}$$

Denote

$$u_0(x) = \int_{\partial D} \frac{\partial G}{\partial n_y}(x,y)\phi(y)\,dS_y, \qquad x \in D.$$

**Hypothesis 3.6.** *Assume the boundary condition $\phi$ satisfies*

$$|u_0(x)| \le C \sum_{I=1}^{m} |x - a_i|^{-\alpha} \quad \forall\, x \in D \tag{3.3}$$

*for some points $a_1, \dots, a_m \in \bar{D}$ and for some $\alpha < d$. We also assume that $u_0(x)$ has the same sign for all $x \in D$.*

**Example 3.7.** If $D = B_R(0)$ is the ball of center 0 and radius $R$, then (see, e.g., [4, Corollary 1.23])

$$\frac{\partial G}{\partial n}(x,y) = \frac{R^2 - |x|^2}{\omega_d R|x-y|^d} \quad \forall\, x \in B_R(0),\ y \in \partial B_R(0).$$

In this case we can take the boundary condition $\phi$ to be the linear combination of the Dirac point mass measures (with respect to the surface measure $dS_y$):

$$\phi(y) = \sum_{i=1}^{m} \rho_i \delta(y - a_i),$$

where $\rho_1, \dots, \rho_m$ are real numbers and they have the same sign; $a_1, \dots, a_m \in \partial B_R(0)$. Thus

$$u_0(x) = \sum_{i=1}^{m} \rho_i \frac{R^2 - |x|^2}{\omega_d R|x - a_i|^d}.$$

Obviously, if $\rho_i$ are all positive, then $u_0(x)$ is always positive on the ball $B_R(0)$. Moreover, since

$$|x - a_i| \ge \big||a_i| - |x|\big| = R - |x|$$

we see

$$0 \le u_0(x) = \sum_{i=1}^{m} \rho_i \frac{(R - |x|)(R + |x|)}{\omega_d R|x - a_i|^{d-1}|x - a_i|} \le C \sum_{i=1}^{m} \frac{1}{|x - a_i|^{d-1}}.$$

This means that if $\phi$ is a linear combination of the Dirac point mass measures, then $u_0(x)$ satisfies (3.3) with $\alpha = d - 1$.

Now we state the main theorem of the paper.

**Theorem 3.8.** *Let $\sum_{i=1}^d H_i > d - 2$ and let Hypotheses* 3.1 *and* 3.6 *be satisfied. Then we have the following statements:*

(1) *There is a unique mild solution in $\mathbb{W}$ to the equation* (1.2).
(2) *There is a $\lambda_0 > 0$ such that for any $x \in D$, the solution $u(x)$ is in $\mathbb{W}_{\lambda_0}$.*
(3) *If, in addition, Hypothesis* 3.2 *is satisfied and $|u_0(x)| \ge c > 0$, then there is a $\lambda_1 > 0$ such that for any $x \in D$, the solution $u(x)$ is not in $\mathbb{W}_{\lambda_1}$.*

*Remark* 3.9. Let us emphasize that from part (3) of the above theorem we see that for any $x \in D$, the solution $u(x)$ is not square integrable.

## 4. Proof

In this section we give proof to the main theorem presented at the end of the previous section. We only deal with the case $d \geq 3$. The case $d = 2$ is completely analogous.

Without loss of generality we may assume that $0 \leq u_0(x) \leq C|x|^{-\alpha}$ for some $\alpha < d$. In fact, for the upper bound we shall bound $u_0(x)$ by $|u_0(x)|$, which satisfies

$$0 \leq |u_0(x)| \leq C|x|^{-\alpha}$$

by Hypothesis 3.6. The assumption that $u_0(x) \geq 0$ for some $\alpha < d$ is for the lower bound and we make this assumption at the beginning of the proof to simplify the presentation.

If $u$ is a mild solution to (1.2), namely, if $u$ satisfies

$$u(x) = u_0(x) + \int_D G(x, y)u(y)\, dW(y),$$

where

$$u_0(x) = \int_{\partial D} \frac{\partial G}{\partial n_y}(x, y)\phi(y)\, dS_y,$$

then we can use $u(y) = u_0(y) + \int_D G(y, z)u(z)\, dW(z)$ to substitute the above $u(y)$ to obtain

$$\begin{aligned} u(x) &= u_0(x) + \int_D G(x, y)u(y)\, dW(y) \\ &= u_0(x) + \int_D G(x, y)u_0(y)\, dW(y) + \int_{D^2} G(x, y)G(y, z)u(z)\, dW(y)\, dW(z). \end{aligned}$$

Continuing this way we obtain the chaos expansion of $u(x)$ as

$$u(x) = u_0(x) + \sum_{n=1}^{\infty} u_n = u_0(x) + \sum_{n=1}^{\infty} u_n, \tag{4.1}$$

where $u_n = I_n(f_n(x)) = I_n\big(\hat{f}_n(x)\big)$ and

$$f_n(x; y_1, \ldots, y_n) = G(x - y_1)G(y_1 - y_2) \ldots G(y_{n-1} - y_n)u_0(y_n) \tag{4.2}$$

or its symmetrization [7]:

$$\hat{f}_n(x; y_1, \ldots, y_n) = \frac{1}{n!} \sum_{\sigma} f_n\left(x; y_{\sigma(1)}, \ldots, y_{\sigma(n)}\right),$$

where $\sigma$ is a permutation of $\{1, 2, \ldots, n\}$.

To prove the theorem it suffices to prove parts (2) and (3) of the theorem. This can be achieved once we obtain the precise bound of $\mathsf{E}(u_n^2)$. Now we are going to compute $\mathsf{E}(u_n^2)$. Recalling the definition of $\phi(y, z) := \phi_H(y, z)$ defined by (2.1) and the Itô isometry (2.5) we have

$$\mathsf{E}\left(u_n^2\right) = n! \int_{D^{2n}} \hat{f}_n(x; y_1, \ldots, y_n)\hat{f}_n(x; z_1, \ldots, z_n) \prod_{i=1}^{n} \phi(y_i, z_i)\, dy\, dz.$$

Since $\phi$ is positive definite, we have

$$\begin{aligned} \mathsf{E}\left(u_n^2\right) &\leq n! \int_{D^{2n}} f_n(x; y_1, \ldots, y_n)f_n(x; z_1, \ldots, z_n) \prod_{i=1}^{n} \phi(y_i, z_i)\, dy\, dz \\ &\leq n! \int_{D^{2n}} |f_n(x; y_1, \ldots, y_n)| \cdot |f_n(x; z_1, \ldots, z_n)| \prod_{i=1}^{n} \phi(y_i, z_i)\, dy\, dz. \end{aligned}$$

Now using Hypothesis 3.1 and denoting $y_0 = x$ we have

$$\begin{aligned}\mathsf{E}\left(u_n^2\right) &\le n!\,C^n \int_{D^{2n}} \prod_{i=1}^n |y_{i-1}-y_i|^{2-d}|u_0(y_n)| \\ &\qquad\qquad \times \prod_{i=1}^n |z_{i-1}-z_i|^{2-d}|u_0(z_n)| \prod_{i=1}^n \phi(y_i,z_i)\,dy\,dz \\ &\le n!\,C^n \int_{D^{2n}} \prod_{i=1}^n |y_{i-1}-y_i|^{2-d}|y_n|^{-\alpha} \prod_{i=1}^n |z_{i-1}-z_i|^{2-d}|z_n|^{-\alpha} \prod_{i=1}^n \phi(y_i,z_i)\,dy\,dz.\end{aligned}$$

Using $a_1+a_2+\ldots+a_d \ge a_1^{\alpha_1}\ldots\alpha_d^{\alpha_d}$ for any positive $a_1,\ldots,a_d$ and any positive $\alpha_1,\ldots,\alpha_d$ such that $\alpha_1+\ldots+\alpha_d = 1$, we have

$$\begin{aligned}\mathsf{E}\left(u_n^2\right) \le n!\,C^n \int_{D^{2n}} &\prod_{j=1}^d \prod_{i=1}^n \left|y_{i-1}^j - y_i^j\right|^{(2-d)\alpha_j} \left|y_n^j\right|^{-\alpha/d} \\ &\times \prod_{i=1}^n \left|z_{i-1}^j - z_i^j\right|^{(2-d)\alpha_j} \left|z_n^j\right|^{-\alpha/d} \prod_{i=1}^n \phi(y_i,z_i)\,dy\,dz.\end{aligned}$$

Let $R>0$ be a positive number such that $D \subseteq [-R,R]^d$. Then

$$\begin{aligned}\mathsf{E}(u_n^2) &\le n!\,C^n \int_{[-R,R]^{2nd}} \prod_{j=1}^d \prod_{i=1}^n \left|y_{i-1}^j - y_i^j\right|^{(2-d)\alpha_j} \left|y_n^j\right|^{-\alpha/d} \\ &\qquad \times \prod_{i=1}^n \left|z_{i-1}^j - z_i^j\right|^{(2-d)\alpha_j} \left|z_n^j\right|^{-\alpha/d} \prod_{i=1}^n \phi(y_i,z_i)\,dy\,dz \\ &\le n!\,C^n \prod_{j=1}^d \Theta_j,\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}\Theta_j = \int_{[-R,R]^{2n}} &\prod_{i=1}^n \left|y_{i-1}^j - y_i^j\right|^{(2-d)\alpha_j} \left|y_n^j\right|^{-\alpha/d} \\ &\times \prod_{i=1}^n \left|z_{i-1}^j - z_i^j\right|^{(2-d)\alpha_j} \left|z_n^j\right|^{-\alpha/d} \prod_{i=1}^n \phi_{H_j}\left(y_i^j, z_i^j\right) dy^j\, dz^j.\end{aligned}$$

Now by the famous Hardy–Littlewood-type inequality obtained in Mémin, Mishura and Valkeila [16] (we use in fact the multidimensional version [10, Inequality (2.4)]) we have

$$\Theta_j \le C^n \left\{ \int_{[-R,R]^n} \prod_{i=1}^n \left|y_{i-1}^j - y_i^j\right|^{(2-d)\alpha_j/H_j} \left|y_n^j\right|^{-\alpha/(dH_j)}\, dy^j \right\}^{2H_j}.$$

For any $\beta_1,\beta_2 \in (-1,\infty)$ we have

$$\begin{aligned}\int_{-1}^1 |y|^{\beta_1}\,|x-y|^{\beta_2} &= x^{1+\beta_1+\beta_2} \int_{-x}^x |u|^{\beta_1}\,|1-u|^{\beta_2} \\ &\le |x|^{1+\beta_1+\beta_2} \int_{-1}^1 |u|^{\beta_1}\,|1-u|^{\beta_2}\,du \le C\,|x|^{1+\beta_1+\beta_2} \quad \forall\, x \in (0,1].\end{aligned} \tag{4.4}$$

Let

$$\frac{(2-d)\alpha_j}{H_j} > -1 \quad \text{or} \quad H_j \ge (d-2)\alpha_j. \tag{4.5}$$

The above inequality (4.4) is also true when $x$ is $[-1,0)$. If (4.5) is satisfied, then (4.4) yields

$$\Theta_j \le C_D^n$$

for some constant $C_D$ depending on the domain $D$. Therefore we have from (4.3)

$$\mathsf{E}(u_n^2) \le n!\, C_D^n. \tag{4.6}$$

If $\sum_{i=1}^d H_i > d-2$ we can find $\alpha_i \in (0,1)$, $i=1,\dots,d$, such that (4.5) holds true, which implies part (2) of the theorem.

Now we turn to prove part (3) of the theorem. We use

$$\begin{aligned}\mathsf{E}\left(u_n^2\right) &= n! \int_{D^{2n}} \hat{f}_n(x;y_1,\dots,y_n)\hat{f}_n(x;z_1,\dots,z_n) \prod_{i=1}^n \phi(y_i,z_i)\,dy\,dz \\ &= \sum_\sigma \int_{D^{2n}} f_n(x;y_1,\dots,y_n) f_n\left(x;z_{\sigma(1)},\dots,z_{\sigma(n)}\right) \prod_{i=1}^n \phi(y_i,z_i)\,dy\,dz.\end{aligned} \tag{4.7}$$

Since the Green's function is negative (see, e.g., [4, Proposition 1.21]) $f_n(x,y_1,\dots,y_n)$ and $f_n(x;z_{\sigma(1)},\dots,z_{\sigma(n)})$ have the same sign. Moreover, we also know from its explicit expression that $\phi_H(y_i,z_i)$ are also positive. This means that the integrand in (4.7):

$$f_n(x;y_1,\dots,y_n) f_n\left(x;z_{\sigma(1)},\dots,z_{\sigma(n)}\right) \prod_{i=1}^n \phi(y_i,z_i)$$

is always positive. Thus

$$\begin{aligned}\mathsf{E}\left(u_n^2\right) &\ge n! \min_\sigma \int_{D^{2n}} f_n(x;y_1,\dots,y_n) f_n\left(x;z_{\sigma(1)},\dots,z_{\sigma(n)}\right) \prod_{i=1}^n \phi(y_i,z_i)\,dy\,dz \\ &\ge n! \min_\sigma \int_{D_0^{2n}} f_n(x;y_1,\dots,y_n) f_n\left(x;z_{\sigma(1)},\dots,z_{\sigma(n)}\right) \prod_{i=1}^n \phi(y_i,z_i)\,dy\,dz \\ &\ge n!\, C^n \min_\sigma \int_{D_0^{2n}} \prod_{i=1}^n |y_{i-1}-y_i|^{2-d} \left|z_{\sigma(i-1)}-z_{\sigma(i)}\right|^{2-d} \prod_{i=1}^n \phi(y_i,z_i)\,dy\,dz.\end{aligned} \tag{4.8}$$

Since $D_0$ is a nonempty open set, it must contain a square $\{x; \max_{1\le i\le d} |x_i-a_i| \le \rho\}$ for some $a=(a_1,\dots,a_d)\in\mathbb{R}^d$ and $\rho>0$.

Denote the square

$$D_\rho = \left\{x;\ \max_{1\le i\le d} |x_i-a_i| \le \rho\right\}.$$

If $\rho$ is sufficiently small, then for $y_i \in D_\rho$,

$$|y_{i-1}-y_i| = |y_{i-1}-a-(y_i-a)| \le |y_{i-1}-a| + |y_i-a| \le 2\sqrt{d}\rho \le 1.$$

In the same way we see that if both $y_i$ and $z_i$ are in $D_\rho$, then

$$\phi(y_i,z_i) \ge C_H^n.$$

Then we have from (4.8)

$$\begin{aligned}\mathsf{E}\left(u_n^2\right) &\ge n!\, C^n \min_\sigma \int_{D_\rho^{2n}} \prod_{i=1}^n |y_{i-1}-y_i|^{2-d} \left|z_{\sigma(i-1)}-z_{\sigma(i)}\right|^{2-d} \prod_{i=1}^n \phi(y_i,z_i)\,dy\,dz \\ &\ge n!\, C^n \min_\sigma \int_{D_\rho^{2n}} dy\,dz = n!\, C_\rho^n,\end{aligned} \tag{4.9}$$

where $C_\rho$ is a strictly positive constant. This proves part (3) of Theorem 3.8.

## 5. One-dimensional case

In this section we shall discuss one-dimensional case $d = 1$. In this case the Schrödinger equation becomes

$$\begin{cases} u''(x) = u \diamond \dot{W}(x), & a < x < b, \\ u(a) = \lambda,\ u(b) = \mu, \end{cases} \tag{5.1}$$

where $a$, $b$, $\lambda$, $\mu$ are given constants and $W$ is a one parameter fractional Brownian motion of the Hurst parameter $H > \frac{1}{2}$. A direct integration yields that the mild solution satisfies

$$u(x) = \int_a^b G(x,y)u(y)\,dW(y) + u_0(x),$$

where

$$\begin{cases} G(x,y) = (x-y)I_{(a,x)}(y) - \frac{x-a}{b-a}(b-y), \\ u_0(x) = \lambda + \frac{x-a}{b-a}(\mu - \lambda). \end{cases}$$

It is easy to verify that Hypotheses 3.1, 3.2, and 3.6 are satisfied. Thus Theorem 3.8 holds true for the one-dimensional case as well. We can state the following.

**Theorem 5.1.** (1) *There is a $\lambda_0 > 0$ such that for any $x \in (a,b)$, the solution $u(x)$ to* (5.1) *is in $\mathbb{W}_{\lambda_0}$.*
(2) *If one of $\lambda$ or $\mu$ is nonzero, then there is a $\lambda_1 > 0$ such that for any $x \in D$, the solution $u(x)$ to* (5.1) *is not in $\mathbb{W}_{\lambda_1}$.*

On the other hand, we may replace the boundary conditions $u(a) = \lambda$ and $u(b) = \mu$ by initial conditions. Thus we are led to consider

$$\begin{cases} u''(x) = u \diamond \dot{W}(x), & a < x < \infty, \\ u(a) = \lambda, \quad u'(a) = \nu. \end{cases} \tag{5.2}$$

The mild solution is now given by

$$u(x) = \int_a^x G(x,y)u(y)\,dW(y) + u_0(x), \tag{5.3}$$

where

$$G(x,y) = x - y, \qquad u_0(x) = \nu(x-a) + \lambda. \tag{5.4}$$

Thus the unique mild solution to (5.2) is given by

$$u(x) = \sum_{n=0}^{\infty} I_n\big(f_n(x)\big),$$

where $f_n(x)$ is given by (denoting $x_0 = x$)

$$f_n(x, x_1, \ldots, x_n) = \prod_{i=1}^n G(x_{i-1}, x_i)u_0(x_n),$$

and

$$I_n\big(f_n(x)\big) = \int_{a<x_n \ldots <x_1<x} G(x-x_1)G(x_1,x_2)\ldots G(x_{n-1},x_n)u_0(x_n)\,dW(x_n)\ldots dW(x_1).$$

Before computing $\mathsf{E}\left[I_n(f_n(x))^2\right]$, let us state an elementary lemma from [8].

**Lemma 5.2.** *For any integer $n \geq 1$ let $\alpha_i \in (-1, \infty)$, $i = 1, 2, \dots, n$, and denote $|\alpha| = \sum_{i=1}^n \alpha_i$. Then there is a constant $c > 0$, independent of $n$, such that*

$$\int_{a<x_1<\dots<x_n<b} \prod_{i=1}^n (x_i - x_{i-1})^{\alpha_i} \, dx_1 \dots dx_n \leq \frac{c^n (b-a)^{|\alpha|+n}}{\Gamma(|\alpha| + n + 1)}, \tag{5.5}$$

*where by convention, we set $x_0 = a$.*

In fact, in [8] the above lemma is stated for $\alpha_i \in (-1, 1)$. But it is easy to see that the inequality holds true for all $\alpha_i \in (-1, \infty)$ which we shall need in the following.

From the inequality (4.3) we see

$$\begin{aligned}
\mathsf{E}\left[I_n(f_n(x))^2\right] &\leq n! \left( \int_{a<x_n\dots<x_1<x} |f_n(x, x_1, \dots, x_n)|^{1/H} \, dx_n \dots dx_1 \right)^{2H} \\
&\leq n! \left[|\nu| (b-a) + |\lambda|\right]^2 \left( \int_{a<x_n\dots<x_1<x} \left| \prod_{i=1}^n G(x_{i-1}, x_i) \right|^{1/H} dx_n \dots dx_1 \right)^{2H} \\
&= n! \left[|\nu| (b-a) + |\lambda|\right]^2 \left( \int_{a<x_n\dots<x_1<x} \prod_{i=1}^n |x_{i-1} - x_i|^{1/H} \, dx_n \dots dx_1 \right)^{2H} \\
&\leq n! \, C^n \left[|\nu| (b-a) + |\lambda|\right]^2 \left( \frac{(x-a)^{n(1+H)/H}}{\Gamma\left(\left(1 + \frac{1}{H}\right) n + 1\right)} \right)^{2H} \\
&\leq \frac{C^n (x-a)^{(2+2H)n} \left[|\nu| (b-a) + |\lambda|\right]^2}{\Gamma\big((1+2H)n + 1\big)},
\end{aligned} \tag{5.6}$$

where the last inequality follows from the Stirling formula. From the hypercontractivity [7], we have for any $p > 2$,

$$\begin{aligned}
\|u(x)\|_p &:= (\mathsf{E}|u(x)|^p)^{1/p} \\
&\leq \sum_{n=0}^\infty \|I_n(f_n(x))\|_p \\
&\leq \sum_{n=0}^\infty (p-1)^{n/2} \|I_n(f_n(x))\|_2 \\
&\leq \sum_{n=0}^\infty p^{n/2} \left[|\nu| (b-a) + |\lambda|\right] \frac{C^n (x-a)^{(1+H)n}}{\Gamma\left((\frac{1}{2} + H)n + 1\right)} \\
&\leq C \left[|\nu| (b-a) + |\lambda|\right] \exp\left[c(x-a)^{(2+2H)/(1+2H)} p^{1/(1+2H)}\right],
\end{aligned}$$

where the last inequality follows from the asymptotic property of the Mittag-Leffler function. Summarizing we have the following.

**Theorem 5.3.** *The mild solution to (5.2) is in $L^p$ for any $p \geq 1$ and there are universal positive constants $c$ and $C$, independent of $x$, $a$, $b$, and $p$ such that*

$$\mathsf{E}|u(x)|^p \leq C \left[|\nu| (b-a) + |\lambda|\right]^p \exp\left[c(x-a)^{(2+2H)/(1+2H)} p^{(2+2H)/(1+2H)}\right] \quad \forall \, p \geq 1. \tag{5.7}$$

## Bibliography

[1] F. Biagini, Y. Hu, B. Øksendal, and T. Zhang, *Stochastic Calculus for Fractional Brownian Motion and Applications*, Probab. Appl. (N. Y.), Springer-Verlag, London, 2008. MR2387368
[2] K. L. Chung and Z. X. Zhao, *From Brownian Motion to Schrödinger's Equation*, Grundlehren Math. Wiss., vol. 312, Springer-Verlag, Berlin, 1995. MR1329992
[3] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. MR1814364
[4] Q. Han and F. Lin, *Elliptic Partial Differential Equations*, 2nd ed., Courant Lect. Notes Math., vol. 1, Amer. Math. Soc., Providence, RI, 2011. MR2777537
[5] T. Hida, H.-H. Kuo, J. Potthoff, and L. Streit, *White Noise. An Infinite Dimensional Calculus*, Mathematics and Its Applications, vol. 253, Kluwer Academic Publishers Group, Dordrecht, 1993. MR1244577
[6] Y. Hu, *Integral transformations and anticipative calculus for fractional Brownian motions*, Mem. Amer. Math. Soc. **175** (2005), no. 825. MR2130224
[7] Y. Hu, *Analysis on Gaussian Space*, World Scientific, Singapore, 2017.
[8] Y. Hu, J. Huang, D. Nualart, and S. Tindel, *Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency*, Electron. J. Probab. **20** (2015), no. 55. MR3354615
[9] Y. Hu, J. Huang, K. Le, D. Nualart, and S. Tindel, *Stochastic heat equation with rough dependence in space*, Ann. Probab. Probab. **45** (2017), no. 6B, 4561–4616. MR3737918
[10] Y. Hu and D. Nualart, *Stochastic heat equation driven by fractional noise and local time*, Prob. Theory Related Fields **143** (2009), 285–328. MR2449130
[11] Y. Hu, B. Øksendal, and T. Zhang, *Stochastic partial differential equations driven by multiparameter fractional white noise*, Stochastic Processes, Physics and Geometry: New Interplays, II (Leipzig, 1999), CMS Conf. Proc., vol. 29, Amer. Math. Soc., Providence, RI, 2000, pp. 327–337. MR1803426
[12] Y. Hu, B. Øksendal, and T. Zhang, *General fractional multiparameter white noise theory and stochastic partial differential equations*, Comm. Partial Differential Equations **29** (2004), 1–23. MR2038141
[13] Y. Hu and J. A. Yan, *Wick calculus for nonlinear Gaussian functionals*, Acta Math. Appl. Sin. Engl. Ser. **25** (2009), no. 3, 399–414. MR2506982
[14] Yu. G. Kondratiev, *Spaces of entire functions of an infinite number of variables, connected with the rigging of a Fock space*, Spectral Analysis of Differential Operators, Selecta Mathematica Sovietica **10** (1991), no. 2, 165–180. MR1115043
[15] Yu. G. Kondratiev, *Nuclear spaces of entire functions in problems of infinite-dimensional analysis*, Soviet Math. Dokl. **22** (1980), 588–592. MR592501
[16] J. Memin, Y. Mishura, and E. Valkeila, *Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion*, Statist. Probab. Lett. **51** (2001), 197–206. MR1822771
[17] Y. S. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, Lecture Notes in Math., vol. 1929, Springer-Verlag, Berlin, 2008. MR2378138
[18] D. Nualart, *The Malliavin Calculus and Related Topics*, 2nd ed., Probab. Appl. (N. Y.), Springer-Verlag, Berlin, 2006. MR2200233
[19] M. Sanz-Solé and I. Torrecilla, *A fractional Poisson equation: existence, regularity and approximations of the solution*, Stoch. Dyn. **9** (2009), no. 4, 519–548. MR2589036
[20] L. Quer-Sardanyons and S. Tindel, *Pathwise definition of second-order SDEs*, Stochastic Process. Appl. **122** (2012), no. 2, 466–497. MR2868927

Department of Mathematical and Statistical Sciences, University of Alberta at Edmonton, Edmonton, T6G 2G1, Canada
*Email address*: yaozhong@ualberta.ca

Received 1/FEB/2018
Originally published in English