Теорія Ймовір. та Матем. Статист.
Вип. 98, 2018

Theor. Probability and Math. Statist.
No. 98, 2019, Pages 243–254
https://doi.org/10.1090/tpms/1074
Article electronically published on August 19, 2019

SOME INEQUALITIES FOR THE RISK FUNCTION IN THE TIME AND SPACE NONHOMOGENEOUS CRAMÉR–LUNDBERG RISK MODEL

UDC 519.21

M. V. KARTASHOV AND V. V. GOLOMOZYĬ

Abstract. We consider a generalized Cramér–Lundberg risk model with a time nonhomogeneous Poisson process of insurance claims where the intensity of premiums depends on the current insurance capital. Some explicit bounds are found for the exponential normalized uniform distance between

(a) the risk function in a time nonhomogeneous model and that in a time homogeneous but space nonhomogeneous model,
(b) the risk functions in a time homogeneous model or in a space nonhomogeneous model and that in a time and space homogeneous model.

It is assumed that the intensity of premiums approaches a constant as the current insurance capital increases.

1. Introduction

The stability of distributions and asymptotic behavior of the distributions of Markov times are studied in the monograph [8] for general Markov chains under rather wide assumptions concerning the mixing properties. A number of applications of these results are also given in [8].

The foundations of the theory of stability of stochastic models are laid by V. M. Zolotarev in [3]. Important results of the theory of stability are obtained by Meyn and Tweedie in the monograph [7].

Considering some further applications, it is desirable to extend the classical results of the Cramér–Lundberg theory to the class of nonhomogeneous risk processes. For example, the price of a vehicle insurance policy may depend on the seasonal factor which is explained by the difference between intensities of road accidents during different seasons.

The asymptotic behavior of the risk function is studied in [11] for a time continuous generalized Cramér–Lundberg risk model where the risk function is space nonhomogeneous but time homogeneous. The space nonhomogeneity of the model considered in [11] is explained by the variation of the premium intensity.

The current paper is a continuation of [11]. Here we compare the risk functions in either a time homogeneous, or nonhomogeneous, or space nonhomogeneous model and that in a homogeneous model. More precisely, we obtain explicit bounds for the difference between the risk functions in these models. The definition of the risk function (or, which

2010 *Mathematics Subject Classification.* Primary 91B30; Secondary 60J25.

Key words and phrases. Time and space nonhomogeneous Markov processes, ruin probability, Cramér–Lundberg risk model.

M. V. Kartashov passed away on January 1, 2018. See the tribute to him by the Theory of Probability and Mathematical Statistics Editorial Board in this issue.

©2019 American Mathematical Society

is the same, of the ruin probability function depending on the insurance capital) is given in the monograph [5].

With applications in mind, we obtain explicit exponential bounds for the difference between the corresponding risk functions in generalized models being space as well as time nonhomogeneous by using the results of the papers [9, 10, 12].

Sections 2–7 of the current paper contain the statements of the results, while the proofs are given in Section 8.

2. A time and space nonhomogeneous Cramér–Lundberg model

Let $(c(x),\ x \in \mathbb{R})$ be a positive Borel function such that $1/c(x)$ is locally integrable and

$$c(x) - c = O\big(\exp(-\gamma x)\big), \qquad x \to \infty, \tag{1}$$

for some constants $c > 0$ and $\gamma > 0$

Consider a time nonhomogeneous Markov chain in $\mathbb{R}$ that describes the dynamic behavior of the capital of an insurance company. We assume that the chain is a right continuous solution of the stochastic differential equation

$$dX(t) = c\big(X(t)\big)dt - d\widetilde{Z}(t), \qquad t \ge s, \quad X(s) = x. \tag{2}$$

The existence and uniqueness of a solution of equation (2) are proved in Lemma 1 below. The function $c(x)$ in equation (2) means the premium rate for the current capital x, and the process $\widetilde{Z}(t)$ with independent increments,

$$\widetilde{Z}(t) - \widetilde{Z}(s) = \sum_{\widetilde{\nu}(s) < n \le \widetilde{\nu}(t)} \xi_n, \tag{3}$$

describes the payments according to insurance claims on the interval $(s,t]$; the stochastic process $\widetilde{\nu}(t)$ is defined below.

The random variables $(\xi_n, n \ge 1)$ in (3) are nonnegative, independent, and identically distributed. We treat the variables $(\xi_n, n \ge 1)$ as payments due to insurance claims. We assume that $(\xi_n, n \ge 1)$ are such that

$$\mathsf{P}(\xi_n < x) = F(x), \qquad G(x) = 1 - F(x), \qquad m = \mathsf{E}\xi_1,$$

and Cramér's condition holds; that is, their moment generating function is finite for some $\alpha > 0$,

$$\widehat{f}(\alpha) = \mathsf{E}\exp(\alpha\xi_1) < \infty. \tag{4}$$

Also let

$$\widehat{G}(\alpha) = \left(\widehat{f}(\alpha) - 1\right)/\alpha = \int_0^\infty \exp(\alpha x) G(x)\,dx.$$

The stochastic process $(\widetilde{\nu}(t), t \ge 0)$ in (3) that describes the number of insurance claims on $(0,t]$ has independent increments, does not depend on (ξ_n), and is a nonhomogeneous Poisson process whose structural measure

$$\widetilde{\Lambda}(s,t) = \int_s^t \widetilde{\lambda}(u)\,du$$

is such that the intensity $\widetilde{\lambda}(u)$ is a Borel nonnegative locally integrable function for which $\widetilde{\Lambda}(s,\infty) = \infty$. Thus

$$\mathsf{P}\big(\widetilde{\nu}(t) - \widetilde{\nu}(s) = n\big) = \exp\left(-\widetilde{\Lambda}(s,t)\right)\left(\widetilde{\Lambda}(s,t)\right)^n/n!, \qquad n \in \mathbb{Z}_+. \tag{5}$$

We define the first jump moment after a moment s for the process $\widetilde{\nu}(t)$ by

$$\theta_s = \inf\big(t > s \colon \widetilde{\nu}(t) > \widetilde{\nu}(s)\big). \tag{6}$$

The density of θ_s is given by

$$\mathsf{P}_{sx}(\theta_s \in du) = \widetilde{\lambda}(u) \exp\bigl(-\widetilde{\Lambda}(s,u)\bigr)\, du, \qquad u \geq s.$$

It is well known in the theory of stochastic processes with independent increments (see [1, 2]) that the infinitesimal operator $A(t)$ (or, in other words, the characteristic Dynkin operator [2, II.5]) of the process $(X(t))$ is given by

$$A(t)g(x) = c(x)g'(x) + \widetilde{\lambda}(t)[g * F(x) - g(x)], \qquad x \in \mathbb{R}, \tag{7}$$

for $g \in C_1(\mathbb{R}) \cap C_b(\mathbb{R})$ (or for $g \in \mathfrak{D}(A(t))$, where $\mathfrak{D}(A(t))$ is the range of the linear operator $A(t)$).

The convolution here is defined by

$$g * F(x) = \int_0^\infty g(x-y)\, dF(y).$$

The convolution for a function f is defined as usual, namely

$$f * g(x) = \int_0^\infty f(x-y)g(y)\, dy.$$

Now we define continuous increasing functions

$$D(x) = \int_0^x \frac{1}{c(y)}\, dy, \qquad C(y) = \sup(x\colon D(x) < y) = D^{(-1)}(y). \tag{8}$$

Both $D(x)$ and $C(y)$ are isomorphisms on $\mathbb{R}_+$ in view of the assumptions imposed on the function $c(x)$.

Also let

$$H(x,t) = C(D(x) + t) - x, \qquad x, t \geq 0. \tag{9}$$

Note that H defined by (9) is a nonnegative and nondecreasing function with respect to t and $H(x,0) = 0$.

Lemma 1. *If $X(s) = x \geq 0$ up to the moment θ_s in* (6), *then*

$$X(t) = x + H(x, t-s), \qquad s \leq t < \theta_s,$$

$$X(\theta_s) = x + H(x, \theta_s - s) - \xi_{\widetilde{\nu}(s)+1}. \tag{10}$$

The symbols P_{sx} and E_{sx} denote the conditional probability and conditional expectation given $X(s) = x$ in the rest of the paper.

3. Ruin moment and stopped process

We study the asymptotic behavior of the ruin moment

$$\zeta_{sx} = \inf(t > s\colon X(t) < 0), \quad \text{where } X(s) = x. \tag{11}$$

Consider the terminating process

$$X_t = X(t) 1_{t<\zeta_{sx}} - \infty 1_{t \geq \zeta_{sx}}, \qquad t \geq s,$$

assuming values in $\mathbb{R}_+ \cup \{-\infty\}$. Then X_t is the Markov subprocess for $X(t)$ that corresponds to the multiplicative functional $\mu_s^t = 1_{\{\zeta_{sx} > t\}}$ [2, Chapter 1.5].

Since $X(t)$ is right continuous, the definition of the subprocess X_t and infinitesimal operator (the trajectories of the processes $X(t)$ and X_t coincide up to a positive Markov moment ζ_{sx} for all $x \geq 0$) implies that the infinitesimal operator A_t of the process X_t is of the form

$$A_t g(x) = [A(t)g](x) 1_{x\geq 0}, \qquad x \in \mathbb{R}, \\ A_t g(-\infty) = 0, \tag{12}$$

for functions g such that $g(x) = 0$ for $x < 0$.

For $B \in \mathfrak{B}(\mathbb{R}_+)$, $t \geq s$, we define the transition probability

$$P_{st}(x, B) = \mathsf{P}_{sx}(X_t \in B) = \mathsf{P}_{sx}(X(t) \in B, \zeta_{sx} > t). \tag{13}$$

For $x \geq 0$, we consider the survival probability and ruin probability (risk function) for the infinite horizon, namely

$$\widetilde{p}_s(x) = P_{s\infty}(x, \mathbb{R}_+) = P_{s\infty} 1(x) \tag{14}$$

and

$$\widetilde{q}_s(x) = 1 - \widetilde{p}_s(x) = P_{sx}(\zeta_{sx} < \infty), \tag{15}$$

where

$$P_{s\infty} 1(x) = \lim_{t\to\infty} P_{st} 1(x) = \lim_{t\to\infty} \mathsf{P}_{sx}(\zeta_{sx} > t) = \mathsf{P}_{sx}(\zeta_{sx} = \infty).$$

Here and in what follows the symbol $1(x)$ denotes the unit function on $\mathbb{R}_+$. We extend these functions for negative arguments by letting $\widetilde{p}_s(x) = 0$ and $\widetilde{q}_s(x) = 0$ for $x < 0$.

4. The Cramér bound in the time homogeneous model

Let $\lambda > 0$ be a constant, and let $(\nu(t))$ be a time homogeneous Poisson process with a constant intensity $\lambda(t) = \lambda$. Thus $\Lambda(s,t) = \lambda(t-s)$ in equality (5).

Consider a homogeneous stochastic risk process (Y_t) similar to (X_t) with the same intensity of premiums $(c(x))$ and with the same payments (ξ_n) but with $(\nu(t))$ instead of the nonhomogeneous process $(\widetilde{\nu}(t))$. The infinitesimal operator of the process (Y_t) is given by

$$Ag(x) = \big(c(x)g'(x) + \lambda[g * F(x) - g(x)]\big) 1_{x\geq 0}, \qquad x \in \mathbb{R}_+, \tag{16}$$

for functions g such that $g(x) = 0$ for $x < 0$.

Denote by

$$p(x) = p_s(x) \quad \text{and} \quad q(x) = q_s(x) \tag{17}$$

the survival probability and ruin probability for the process (Y_t), respectively, according to (14) and (15). It is obvious that both probabilities do not depend on s.

It is proved in [11] that the nonnegative derivative

$$r(x) = -q'(x), \qquad q(x) = \int_x^\infty r(y)\, dy, \tag{18}$$

exists for almost all x (see Lemma 3 and equality (30) in [11]).

For functions g on $\mathbb{R}_+$ and $\alpha \geq 0$, we introduce the notation

$$\|g\|_\alpha = \sup_{x\geq 0} \exp(\alpha x) |g(x)|, \qquad \widehat{g}(\alpha) = \int_0^\infty \exp(\alpha x) g(x)\, dx, \tag{19}$$

$$I_s(g) = \int_s^\infty |g(x)|\, dx, \qquad g^{\pm}(x) = \max(0, \pm g(x)).$$

Assume that the constant c in (1) satisfies the balance condition

$$\lambda m < c. \tag{20}$$

Thus conditions (4) and (20) imply that the Cramér index is positive,

$$\beta \equiv \sup\Big(\alpha \ge 0\colon \lambda\widehat{G}(\alpha) < c\Big) > 0, \tag{21}$$

since $\widehat{G}(0) = m$.

For $\alpha \ge 0$, we consider the condition

$$\Delta(\alpha) \equiv \inf_{x\ge 0}\left(c(x) - \lambda\int_0^x \exp(\alpha y)G(y)\,dy\right)\Big/\lambda\widehat{f}(\alpha) > 0. \tag{22}$$

Remark 1. Relation (22) holds for small $\alpha \ge 0$ if and only if $\Delta(0) > 0$. If the intensity $c(x)$ is nonincreasing, then (22) holds for $\alpha < \beta$.

For the proof below, we note that the function $\Delta(\alpha)$ is continuous at zero in view of assumption (4) and that it is nonincreasing. In addition, $\Delta(0) = c/\lambda - m$. Therefore, if $\Delta(0) > 0$, then condition (20) holds and $\Delta(\alpha) > 0$ for all $\alpha \in [0, \beta)$.

Theorem 1.

(a) *Let conditions* (1) *and* (20) *hold, and let*

$$0 \le \alpha < \min(\gamma, \beta).$$

Then

$$\|q\|_\alpha \le \widehat{r}(\alpha) \le \Big(\big\|(c(\cdot) - c)^-\big\|_\alpha + \lambda\widehat{G}(\alpha)\Big)\Big/\Big(c - \lambda\widehat{G}(\alpha)\Big). \tag{23}$$

(b) *If the denominator of the right-hand side of* (24) *is positive and* $0 \le \alpha < \beta$, *then*

$$\|q\|_\alpha \le \widehat{r}(\alpha) \le \lambda\widehat{G}(\alpha)\Big/\Big(c - \lambda\widehat{G}(\alpha) - \big\|(c(\cdot) - c)^-\big\|_0\Big). \tag{24}$$

(c) *If* (22) *holds, then, given* $\alpha > 0$,

$$\|q\|_\alpha \le \|r\|_\alpha/\alpha \le 1/\alpha\Delta(\alpha). \tag{25}$$

5. A comparison with the time homogeneous Cramér–Lundberg model

By a comparison of two models, we understand the obtaining of explicit bounds for the difference between the corresponding risk functions.

Assume that a nonhomogeneous process is a solution of (2) and satisfies the following condition:

$$\sup_{x\ge 0}\widehat{f}(\alpha)\int_s^\infty \widetilde{\lambda}(u)\exp\Big(-\widetilde{\Lambda}(s,u)\Big)\exp\big(-\alpha H(x, u-s)\big)\,du \le \rho_\alpha(s) < 1 \tag{26}$$

for a fixed number $s \ge 0$ and some constants $\alpha > 0$ and $\rho_\alpha(s) < 1$.

Remark 2. Inequality (26) holds for sufficiently small $\alpha > 0$ if and only if

$$\inf_{x\ge 0}\int_s^\infty \widetilde{\lambda}(u)\exp\Big(-\widetilde{\Lambda}(s,u)\Big)H(x, u-s)\,du > m. \tag{27}$$

Indeed, the Taylor expansion on the left-hand side of inequality (26) is given by

$$\big(1 + m\alpha + o(\alpha)\big)\big(1 - \alpha B(x) + o(\alpha)\big) = 1 - \big(B(x) - m\big)\alpha + o(\alpha), \qquad \alpha \to 0,$$

for every fixed x where $B(x)$ is the integral on the left-hand side of (27). All terms $o(\cdot)$ are uniform in x (do not depend on x), since H is positive almost everywhere. Consider the least upper bound with respect to x in the latter relation and choose a sufficiently small positive α (this α does not necessarily coincide with the constant in (26) denoted by the same symbol α). Then the equivalence of two conditions mentioned above follows, since the difference $B(x) - m$ is separated from zero.

Remark 3. Condition (27) is equivalent to the balance condition (20) for a homogeneous model with $\widetilde{\lambda}(u) = \lambda$ and $c(x) = c$.

Indeed, $H(x,t) = ct$ in this case, and the left-hand side of (27) equals c/λ.

Corollary 1. *If the function $c(x)$ is nondecreasing (nonincreasing), then $H(x,t)$ is nondecreasing (nonincreasing) with respect to the argument x.*

If $c(\cdot)$ is nondecreasing, then condition (27) *is equivalent to*

$$\int_s^\infty \widetilde{\lambda}(u) \exp\left(-\widetilde{\Lambda}(s,u)\right) C(u-s)\, du > m. \tag{28}$$

If $c(\cdot)$ is nonincreasing, then condition (27) *is equivalent to*

$$\int_s^\infty \widetilde{\lambda}(u) \exp\left(-\widetilde{\Lambda}(s,u)\right) \left(C(u-s) - K(u-s)\right) du > m, \tag{29}$$

where

$$K(t) = \int_0^\infty \left(1 - c(y + H(y,t))/c(y)\right) dy.$$

Theorem 2. *Let condition* (26) *hold with $\alpha \ge 0$, and let one of the assumptions* (a), (b), *or* (c) *of Theorem* 1 *hold. Then*

$$\|\widetilde{q}_s - q\|_\alpha \le I_s\left(\widetilde{\lambda}(\cdot) - \lambda\right) \widehat{f}(\alpha) \max\left(1, \|q\|_\alpha\right) / \left(1 - \rho_\alpha(s)\right), \tag{30}$$

where the norm $\|q\|_\alpha$ on the right-hand side is bounded as in the corresponding part of Theorem 1.

In each case, the risk probability is such that

$$\widetilde{q}_s(x) \le \exp(-\alpha x)(\|\widetilde{q}_s - q\|_\alpha + \|q\|_\alpha). \tag{31}$$

6. A comparison with the classical Cramér–Lundberg model

Consider the stochastic risk process $(\overline{Y}_t)$ constructed similarly to (Y_t) and with the same payments (ξ_n). The difference between $(\overline{Y}_t)$ and (Y_t) is that the intensity of premiums $(c(x))$ is nonhomogeneous in the case of $(\overline{Y}_t)$. The infinitesimal operator of this process is given by

$$\overline{A}g(x) = \left(cg'(x) + \lambda[g * F(x) - g(x)]\right) 1_{x \ge 0}, \qquad x \in \mathbb{R}_+. \tag{32}$$

We denote the corresponding functions in (17) and (18) by

$$\overline{p}(x), \qquad \overline{q}(x), \quad \text{and} \quad \overline{r}(x).$$

Explicit bounds are known for these functions (see [5]).

Theorem 3. *Let conditions* (1) *and* (20) *hold, and let $0 \le \alpha < \min(\gamma, \beta)$. Then*

$$\|q - \overline{q}\|_\alpha \le \left(\|c(\cdot) - c\|_\alpha + \lambda \widehat{G}(\alpha) \min\left(\|c(\cdot) - c\|_0 / c, 1\right)\right) / \left(c - \lambda \widehat{G}(\alpha)\right). \tag{33}$$

Further,

$$\lambda m / \left(c + \|c(\cdot) - c\|_0\right) \le q(0).$$

Moreover, if the additional condition $\|c(\cdot) - c\|_0 < c$ holds, then

$$q(0) \le \lambda m / (c - \|c(\cdot) - c\|_0). \tag{34}$$

7. Example: A two-valued intensity of premiums

Consider an example of intensity of premiums that depends on the insurance capital of a company and such that

$$c(x) = b1_{x<z} + c1_{x\geq z} \tag{35}$$

for some b, c, and $z > 0$.

We assume that condition (1) is satisfied for every $\gamma > 0$.

The basic functions of the model are given by

$$D(x) = (x/b)1_{x\leq z} + \big(z/b + (x-z)/c\big)1_{x>z},$$
$$C(y) = (by)1_{y\leq z/b} + \big(z + c(y - z/b)\big)1_{y>z/b},$$
$$H(x,t) = (ct)1_{x>z} + (bt)1_{0\leq t\leq (z-x)/b} + \big(ct + (z-x)(1-c/b)\big)1_{x\leq z, t>(z-x)/b}. \tag{36}$$

The first two equalities are derived from (8), while the third one is easy to obtain from (10) and equality (2) for $t < \zeta_{sx}$.

Condition (22) holds for some $\alpha > 0$ if and only if balance condition (20) holds and if

$$b > \lambda \int_0^z G(y)\, dy.$$

This result is an obvious corollary of Remark 1 and (35).

Corollary 2. *Let assumption* (35) *hold.*

(a) *For* $0 \leq \alpha < \beta$,

$$\|q\|_\alpha \leq \left((b-c)^- e^{\alpha z} + \lambda\widehat{G}(\alpha)\right) / \left(c - \lambda\widehat{G}(\alpha)\right).$$

(b) *For* $0 \leq \alpha < \beta$, *if the denominator on the right-hand side is positive, then*

$$\|q\|_\alpha \leq \lambda\widehat{G}(\alpha) / \left(c - \lambda\widehat{G}(\alpha) - (b-c)^-\right).$$

(c) *Let* $0 < \alpha < \beta$, *and let*

$$\overline{\Delta}(\alpha) = \min\left(c - \lambda\widehat{G}(\alpha), b - \lambda\int_0^z \exp(\alpha y) G(y)\, dy\right) \Big/ \lambda\widehat{f}(\alpha) > 0.$$

Then

$$\|q\|_\alpha \leq 1/\alpha\overline{\Delta}(\alpha).$$

Note that the intensity in (35) is always monotone.

If the intensity is nondecreasing in x and $b \leq c$, then we derive from (28) and (36) and an explicit expression for the density θ_s that condition (27) is equivalent to

$$\mathsf{E}_s\left[b(\theta_s - s)1_{b(\theta_s - s)<z} + \big(c(\theta_s - s) + z(1-b/c)\big)1_{b(\theta_s-s)\geq z}\right] > m.$$

Analogously, if the intensity is nonincreasing in x and $b \geq c$ we prove that condition (27) is equivalent to

$$\mathsf{E}_s[c(\theta_s - s)] > m.$$

Corollary 3. *Assume that one of the conditions* (a), (b), *or* (c) *of Corollary* 2 *holds.*

(a1) *If* $b \leq c$ *and*

$$\overline{\rho}_\alpha(s) \equiv \widehat{f}(\alpha) \int_s^\infty \widetilde{\lambda}(u) \exp\left(-\widetilde{\Lambda}(s,u)\right) \exp\big(-\alpha H(0, u-s)\big)\, du < 1,$$

where $H(0,t) = (bt)1_{t<z/b} + \big(ct + z(1-c/b)\big)1_{t\geq z/b}$, *then*

$$\|\widetilde{q}_s - q\|_\alpha \leq I_s\left(\widetilde{\lambda}(\cdot) - \lambda\right) \widehat{f}(\alpha) \max\left(1, \|q\|_\alpha\right) / \big(1 - \overline{\rho}_\alpha(s)\big). \tag{37}$$

Here $\|q\|_\alpha$ *is bounded as in the corresponding part of Corollary* 2.

(b1) *Let $b \geq c$, and let*

$$\overline{\rho}_\alpha(s) \equiv \widehat{f}(\alpha) \int_s^\infty \widetilde{\lambda}(u) \exp\left(-\widetilde{\Lambda}(s,u)\right) \exp\bigl(-\alpha H(\infty, u-s)\bigr) du < 1,$$

where $H(\infty, t) = ct$. Then inequality (37) *holds.*

Corollary 4. *Let condition* (20) *hold, and let $0 \leq \alpha < \beta$. Then*

$$\|q - \overline{q}\|_\alpha \leq \left(|c-b| \exp(\alpha z) + \min(|b/c - 1|, 1)\lambda \widehat{G}(\alpha)\right) / \left(c - \lambda \widehat{G}(\alpha)\right).$$

Further,

$$\lambda m/(c + |c-b|) \leq q(0).$$

In addition, if $b < 2c$, then

$$q(0) \leq \lambda m/(c - |c-b|).$$

Note that all inequalities of Corollary 4 become equalities if $b = c$.

8. Proofs

Proof of Lemma 1. We derive from (2) and (6) that $dX(t) = c(X(t))\,dt$ for $s \leq t < \theta_s$, whence

$$t - s = \int_s^t dX(u)/c(X(u)) = D(X(t)) - D(x)$$

by (8). Thus (8) implies the first equality in (10). The second equality obviously follows from the first one and in view of the right continuity of the process $X(t)$. □

The following result allows one to study the asymptotic behavior of the derivative of the risk function.

Lemma 2 ([11, Lemma 4])**.** *The function $r(x)$ in* (18) *is a solution of the equation*

$$c(x)r(x) = \lambda \int_0^x r(y) G(x-y)\,dy + \lambda\bigl(1 - q(0)\bigr) G(x), \qquad x \geq 0, \tag{38}$$

for almost all x. Moreover,

$$\int_0^\infty c(x) r(x)\,dx = \lambda m. \tag{39}$$

The following result describes the properties of solutions of an equation of type (38).

Lemma 3. *Let $k(t,s)$ be a bounded Borel nonnegative function. Let $y(t)$ be a bounded Borel function such that $y(t) \geq 0$. Then the Volterra equation*

$$x(t) = y(t) + \int_0^t k(t,s) x(s)\,ds, \qquad t \geq 0, \tag{40}$$

has a unique solution in the class of measurable locally bounded functions. Moreover, the solution is nonnegative, that is, $x(t) \geq 0$ for all t.

Proof. The solution is the sum of the Neyman series of the method of successive approximation. In addition, the solution is nonnegative, since the powers of nonnegative operators are nonnegative operators (see [13, Lemma 3] or [4, Chapter 3]). Note that $y_1(t) \geq y_2(t)$ implies $x_1(t) \geq x_2(t)$ for all t if $x_i(t)$ are solutions of equation (40) with $y_i(t)$ on the right-hand side. □

Proof of Theorem 1(a). The inequality

$$\|q\|_\alpha = \sup_{x\geq 0} \exp(\alpha x) \int_x^\infty r(y)\,dy \leq \widehat{r}(\alpha)$$

is obvious by definition (18). Now we rewrite (38) as follows:

$$cr(x) = (c - c(x))r(x) + \lambda r * G(x) + \lambda p(0)G(x). \tag{41}$$

Multiplying both sides of equality (41) by $\exp(\alpha x)$ and then integrating over $[0,\infty)$ we obtain

$$\begin{aligned} c\widehat{r}(\alpha) &\leq \left\|(c - c(\cdot))^+\right\|_\alpha \int_0^\infty r(y)\,dy + \lambda\widehat{r}(\alpha)\widehat{G}(\alpha) + \lambda p(0)\widehat{G}(\alpha) \\ &\leq \left\|(c(\cdot) - c)^-\right\|_\alpha + \lambda\widehat{r}(\alpha)\widehat{G}(\alpha) + \lambda\widehat{G}(\alpha) \end{aligned}$$

in view of equalities (18) and inequalities $p(0) \leq 1$ and $q(0) \leq 1$. This completes the proof of (23). ☐

Proof of Theorem 1(b). Similarly, we obtain from equation (41) that

$$c\widehat{r}(\alpha) \leq \left\|(c - c(\cdot))^+\right\|_0 \widehat{r}(\alpha) + \lambda\widehat{r}(\alpha)\widehat{G}(\alpha) + \lambda\widehat{G}(\alpha),$$

whence (24) follows. ☐

Proof Theorem 1(c). The inequality $\|q\|_\alpha \leq \|r\|_\alpha/\alpha$ is obvious in view of (18).

Now we obtain from (38) that the function $x(t) = -r(t) + K\exp(-\alpha t)$ is a solution of equation (40) with the kernel $k(s,t) = \lambda G(t-s)$ and right-hand side

$$y(t) = -\lambda p(0)G(t) + K\exp(-\alpha t)\left(c(t) - \lambda\int_0^t \exp(\alpha s)G(s)\,ds\right).$$

Then the inequality $x(t) \geq 0$ (in other words, the inequality $\|r\|_\alpha \leq K$) follows from the inequality $y(t) \geq 0$, $t \geq 0$, by Lemma 3. Since $p(0) \leq 1$, the latter inequality holds by condition (22) with $K = 1/\Delta(\alpha)$. Hence $\|r\|_\alpha \leq 1/\Delta(\alpha)$ and (25) follows. ☐

Proof of Corollary 1. For almost all x, we derive from (9) that

$$\partial/\partial x H(x,t) = C'(D(x)+t)/c(x) - 1,$$

whence

$$\partial/\partial x H(x,t) = c(x + H(x,t))/c(x) - 1 \tag{42}$$

for almost all x in view of (8) and $C'(D(x))/c(x) = 1$. This proves the first statement of Corollary 1, since $H(x,t) \geq 0$.

If the intensity $c(x)$ is nondecreasing with respect to x, then the minimal value of the integral on the left-hand side of (27) is attained at $x = 0$. It remains to take into account that $H(0,t) = C(t)$.

Otherwise, if the intensity $c(x)$ is nonincreasing with respect to x, the minimal value of the integral on the left-hand side of (27) is attained at $x = \infty$ and one should use the equality

$$H(\infty,t) = C(t) - K(t)$$

that follows after the integration of equality (42) over $[0,x]$ and by taking into account the property $H(0,t) = C(t)$. ☐

Proof of Theorem 2. Consider a nonhomogeneous stochastic process (X_t) with the infinitesimal operator defined by (12), transient probability P_{st} in (13), and the basic homogeneous process (Y_t) with the infinitesimal operator given by (16). These operators are quasi-homogeneous (see [12, p. 3]), since the difference operator in equality (AD) of [12] is of the form

$$D_s g(x) \equiv (A_s - A)g(x) = \left(\widetilde{\lambda}(s) - \lambda\right)\left(g * F(x) - g(x)\right)1_{x\ge 0} \tag{43}$$

according to (12) and (16). In addition, this operator is bounded with respect to the norm $\|\cdot\|_\alpha$.

Taking into account that $\widetilde{q}_s - q = -(\widetilde{p}_s - p)$ we apply inequality (17) of Theorem 2 in [12] for the norm $\|\cdot\|_\alpha$,

$$\|\widetilde{q}_s - q\|_\alpha \le \int_s^\infty \|P_{su}\|_\alpha \|D_u p\|_\alpha \, du, \tag{44}$$

where the kernel of the operator P_{su} is defined by (13).

According to (43) with $u \ge s$,

$$\begin{aligned} \int_s^\infty \|D_u p\|_\alpha \, du &\le I_s\left(\widetilde{\lambda}(\cdot) - \lambda\right) \sup_{u\ge 0} \exp(\alpha u)\left|(p * F(u) - p(u))\right| \\ &= I_s\left(\widetilde{\lambda}(\cdot) - \lambda\right) \sup_{u\ge 0} \exp(\alpha u)\left|(-G(u) - q * F(u) + q(u))\right| \\ &\le I_s\left(\widetilde{\lambda}(\cdot) - \lambda\right) \max\big(\max(1, \|q\|_\alpha)\widehat{f}(\alpha), \|q\|_\alpha\big) \\ &= I_s\left(\widetilde{\lambda}(\cdot) - \lambda\right) \widehat{f}(\alpha) \max(1, \|q\|_\alpha). \end{aligned} \tag{45}$$

Then we consider the functions

$$\varphi_{st}(x) = \mathsf{E}_{sx} \exp(-\alpha X_t + \alpha x) \quad \text{and} \quad \varphi_s = \sup_{t\ge s,\, x\ge 0} \varphi_{st}(x). \tag{46}$$

The definition of the operator norm and kernel P_{st} in (13) implies that

$$\|P_{st}\|_\alpha = \sup_{x\ge 0} \varphi_{st}(x) \le \varphi_s. \tag{47}$$

The Markov property of the process (X_t) applied to the Markov moment θ_s in (6) and Lemma 1 imply

$$\begin{aligned} \varphi_{st}(x) &= \mathsf{E}_{sx} \exp(-\alpha H(x, t-s)) 1_{\theta_s > t} \\ &\quad + \mathsf{E}_{sx} \exp(-\alpha J_s + \alpha x) \mathsf{E}_{\theta_s J_s} \exp(-\alpha X_t + \alpha J_s) 1_{\theta_s \le t} \\ &= \mathsf{E}_{sx} \exp(-\alpha H(x, t-s)) 1_{\theta_s > t} \\ &\quad + \mathsf{E}_{sx} \exp(-\alpha J_s + \alpha x) \varphi_{\theta_s, t}(J_s) 1_{\theta_s \le t} \\ &= \int_t^\infty \widetilde{\lambda}(u) \exp\left(-\widetilde{\Lambda}(s,u)\right) \exp(-\alpha H(x, u-s)) \, du \\ &\quad + \int_s^t \widetilde{\lambda}(u) \exp\left(-\widetilde{\Lambda}(s,u)\right) du \exp(-\alpha H(x, u-s)) \\ &\quad\quad \times \int_0^\infty \exp(\alpha y) \varphi_{ut}(H(x, u-s) + x - y) \, dF(y) \end{aligned} \tag{48}$$

in view of $J_s \equiv X_{\theta_s} = x + H(x, \theta_s - s) - \xi_{\nu(s)+1}$.

Note that the upper bound φ_u for $\varphi_{ut}(\cdot)$ does not depend on x. Changing $\varphi_{ut}(\cdot)$ by its upper bound φ_u in (46) and considering the least upper bound for $t \ge s$ and $x \ge 0$

in (48) we obtain

$$\varphi_s \le 1 + \widehat{f}(\alpha) \sup_{t\ge s, x\ge 0} \int_s^t \widetilde{\lambda}(u) \exp\left(-\widetilde{\Lambda}(s,u)\right) \exp(-\alpha H(x, u-s))\varphi_u \, du$$

by equality (5).

The kernels on the right-hand side are contraction mappings by condition (26), and their norm does not exceed $\rho_\alpha(s)$. Since φ_s is monotone, we take the least upper bound in (46) and get the inequality

$$\|P_{st}\|_\alpha \le \varphi_s \le 1/(1-\rho_\alpha(s)) \tag{49}$$

according to definition (26).

Finally, substituting (49) into (44) and using (45) we prove inequality (30).

Inequality (31) follows from the triangle inequality for the norm (19). □

Proof of Theorem 3. We apply equality (38) to the processes (Y_t) and $(\overline{Y}_t)$ and subtract the first equality from the second one. Then we get

$$(r(x) - \overline{r}(x))c = r(x)(c - c(x)) + \lambda(r - \overline{r}) * G + \lambda(\overline{q}(0) - q(0))G(x). \tag{50}$$

Multiplying both sides of (50) by $\exp(\alpha x)$ and integrating over $\mathbb{R}_+$ we deduce that

$$\begin{aligned} c\int_0^\infty & \exp(\alpha x)\,|r(x) - \overline{r}(x)|\,dx \\ &\le \|c - c(\cdot)\|_\alpha \int_0^\infty r(x)\,dx + \widehat{G}(\alpha)\int_0^\infty \exp(\alpha x)\,|r(x) - \overline{r}(x)|\,dx \\ &\quad + \lambda\widehat{G}(\alpha)\,|\overline{q}(0) - q(0)|\,. \end{aligned} \tag{51}$$

Since $q(0) \le 1$, we obtain

$$\begin{aligned} \int_0^\infty & \exp(\alpha x)\,|r(x) - \overline{r}(x)|\,dx \\ &\le \left(\|c - c(\cdot)\|_\alpha + \lambda\widehat{G}(\alpha)\,|\overline{q}(0) - q(0)|\right) / \left(c - \lambda\widehat{G}(\alpha)\right). \end{aligned} \tag{52}$$

Considering (39) we conclude that

$$\begin{aligned} \|q - \overline{q}\|_\alpha &\le \int_0^\infty \exp(\alpha x)\,|r(x) - \overline{r}(x)|\,dx \\ &\le \left(\|c - c(\cdot)\|_\alpha + \lambda\widehat{G}(\alpha)\,|\overline{q}(0) - q(0)|\right) / \left(c - \lambda\widehat{G}(\alpha)\right) \end{aligned} \tag{53}$$

in view of (18).

Next we write equation (39) separately for the processes (Y_t) and $(\overline{Y}_t)$ and obtain two equalities. Subtracting the first equality from the second one we get

$$c\int_0^\infty (r(x) - \overline{r}(x))\,dx = \int_0^\infty r(x)(c - c(x))\,dx,$$

whence

$$\begin{aligned} c\,|q(0) - \overline{q}(0)| = c\left|\int_0^\infty (r(x) - \overline{r}(x))\,dx\right| &= \left|\int_0^\infty r(x)(c - c(x))\,dx\right| \\ &\le \|c - c(\cdot)\|_0 \int_0^\infty r(x)\,dx = \|c - c(\cdot)\|_0\, q(0). \end{aligned} \tag{54}$$

Since $\overline{q}(0) = \lambda m/c$ (see [5]), we prove inequality (34).

On the other hand, $c\,|q(0) - \overline{q}(0)| \le c$. Substituting two latter inequalities into (53) we complete the proof. □

Proof of Corollary 2. Statements (a), (b), and (c) are straightforward consequences of the corresponding statements of Theorem 1 in view of (35) and (36). One should also use the inequality $\Delta(\alpha) \geq \overline{\Delta}(\alpha)$ in the proof of statement (c). $\square$

Proof of Corollary 3. The function $H(x, t)$ is monotone with respect to the argument x according to Corollary 1. Now Corollary 3 follows from equality (36). $\square$

Proof of Corollary 4. The proof follows directly from inequality (33) and equality (35). $\square$

Bibliography

[1] E. B. Dynkin, *Markov Processes*, Academic Press/Springer, New York/Berlin, v. 1 and v. 2, 1966. MR0242252

[2] I. I. Gikhman and A. V. Skorokhod, *Theory of Stochastic Processes II*, Springer-Verlag, Berlin–Heidelberg, 1983. MR0375463

[3] V. M. Zolotarev, *Modern Theory of Summation of Random Variables*, "Nauka", Moscow, 1986; English transl., VSP, Utrecht, 1997. MR1640024

[4] V. M. Shurenkov, *Ergodic Markov Processes*, "Nauka", Moscow, 1989. (Russian) MR1087782

[5] J. Grandell, *Aspects of Risk Theory*, Springer Verlag, New York, 1991. MR1084370

[6] L. N. Bowers, H. U. Gerber, J. C. Hickman, D. A. Jones, and C. J. Nesbitt, *Actuarial Mathematics*, The Society of Actuaries, 1997. MR2345807

[7] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, Springer-Verlag, New York, 1993. MR1287609

[8] N. V. Kartashov, *Strong Stable Markov Chains*, VSP/TBiMC, Utrecht/Kiev, 1996. MR1451375

[9] N. V. Kartashov, *The stability of almost homogeneous in time Markov semigroups of operators*, Teor. Imovirnost. Matem. Statyst. **71** (2003), 105–113; English transl. in Theory Probab. Math. Statist. **71** (2004), 119–128. MR2144325

[10] N. V. Kartashov, *The ergodicity and stability of quasi-homogeneous Markov semigroups of operators*, Teor. Imovirnost. Matem. Statyst. **72** (2005), 54–62; English transl. in Theory Probab. Math. Statist. **72** (2006), 59–68. MR2168136

[11] N. V. Kartashov and O. M. Stroev, *Lundberg approximation for the risk function in an almost homogeneous environment*, Teor. Imovirnost. Matem. Statyst. **73** (2005), 63–71; English transl. in Theory Probab. Math. Statist. **73** (2006), 71–79. MR2213842

[12] N. V. Kartashov, *The stability of transient quasi-homogeneous Markov semigroups and an estimate of the ruin probability*, Teor. Imovirnost. Matem. Statyst. **75** (2006), 36–44; English transl. in Theory Probab. Math. Statist. **75** (2007), 41–50. MR2321179

[13] N. V. Kartashov, *Boundedness, limits, and stability of solutions of a perturbation of a non-homogeneous renewal equation on a semiaxis*, Teor. Imovirnost. Matem. Statyst. **81** (2009), 65–75; English transl. in Theory Probab. Math. Statist. **81** (2010), 71–83. MR2667311

Department of Probability Theory, Statistics, and Actuarial Mathematics, Faculty for Mechanics and Mathematics, Kyiv Taras Shevchenko National University, Volodymyrs'ka Street, 64/13, Kyiv 01601, Ukraine

Email address: nkartashov@skif.com.ua

Department of Probability Theory, Statistics, and Actuarial Mathematics, Faculty for Mechanics and Mathematics, Kyiv Taras Shevchenko National University, Volodymyrs'ka Street, 64/13, Kyiv 01601, Ukraine

Email address: vitaliy.golomoziy@gmail.com

Received 02/NOV/2017

Translated by N. N. SEMENOV