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ON THE LACK OF MEMORY FOR DISTRIBUTIONS OF OVERSHOOT FUNCTIONALS IN THE CASE OF UPPER ALMOST SEMICONTINUOUS PROCESSES DEFINED ON A MARKOV CHAIN

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Abstract. We study the question on whether or not the property of lack of memory that is valid for the geometrical and exponential distributions remains valid for hitting functionals in the case of upper almost semicontinuous processes defined on a Markov chain. If a boundary is attainable and the state of the environment is known at the moment when this boundary is attained, then the lack of memory holds only for an overshoot over a boundary $x \geq 0$ and the distribution of the overshoot does not depend on the overshoot moment as well as on x. The distribution of an undershoot for a boundary x is determined via the distribution of the undershoot for a zero boundary. A similar property is proved for a jump crossing a boundary x.

1. Introduction

Consider a two-dimensional Markov process $\{\xi(t), J(t)\}$ whose second component is a nonreducible ergodic Markov chain with the set of states $\{1, \dots, m\}$, infinitesimal matrix $\mathbf{Q}$, and with the initial stationary distribution $\boldsymbol{\pi}$. We also assume that, given the second component, the first component is a Lévy process starting from zero. For all integer numbers $k, r = 1, \dots, m$, real numbers $x \in \mathbb{R}$, and Borel sets A of $\mathbb{R}$,

$$\begin{aligned}&\mathsf{P}\left\{\xi\left(s+t\right) \in A, J\left(s+t\right) = r \mid \xi(s) = x, J(s) = k\right\} \\ &\quad = \mathsf{P}\left\{\xi(t) \in A - x, J(t) = r \mid J(0) = k\right\}.\end{aligned}$$

The processes of this kind are called Markov additive processes and can serve as a basic model in risk theory and queuing theory. This model allows one to study the dependence of processes on a state of the "outer" environment (see, for example, [1, Chapter VII], [2, Chapter 3], and [3, Chapter 8]).

A number of problems can be reduced to the study of distributions of the following hitting functionals:

$$\begin{aligned}&\tau^+(x) = \inf\left\{t > 0 \colon \xi(t) > x\right\}, \qquad \gamma_1(x) = \gamma^+(x) = \xi\left(\tau^+(x)\right) - x, \\ &\gamma_2(x) = x - \xi\left(\tau^+(x) - 0\right), \qquad \gamma_3(x) = \gamma_1(x) + \gamma_2(x), \qquad x \geq 0.\end{aligned} \tag{1}$$

If the process $\xi(t)$ is upper semicontinuous (that is, its trajectories do not have positive jumps) and if a boundary x is attainable, then the distributions of functionals (1) are quite easy to evaluate, since $\gamma_{1,2,3}(x) = 0$ almost surely. In the case of positive jumps

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with phase-type distributions, one can obtain a matrix exponential expression for generating functions of hitting functionals by using the method of fluid flow models (see, for example, [1, IX.5]).

Following this method, Breuer [4] obtained a matrix exponential expression for the distribution of an overshoot. In [5], he generalized the result of [6] concerning the distribution of a time-spatial position of an overshoot as well as undershoot for a perturbed Poisson process defined on a Markov chain and with phase-type distributed jumps. Breuer and Badescu [7] applied this result to find the so-called Gerber–Shiu measure that gives the expected discounted penalty function (see, for example, [8, Chapter 1]). In addition, the joint distribution of hitting functionals is important in studies of two boundary functionals that can be used for pricing the perpetual American put option (see, for example, [9]).

If we restrict our consideration to the joint distribution of the overshoot and first hitting time for a positive boundary, then the problem can be solved (almost) without assumptions imposed on the distribution of negative jumps (the same conclusion follows explicitly from the second factorization identity of [10, Chapter I, Theorem 3.1]). In contrast, if the problem requires one to determine the distribution of an undershoot, as well, then the method of fluid flow models necessarily assumes that the distributions of both positive and negative jumps are of the phase-type.

Following the factorization method, some expressions for the joint generating function of hitting functionals are obtained in [10, Chapter I, Theorem 3.3] for positive boundaries in terms of integral transformations of distributions of extreme values $\xi^{\pm}(t) = \sup(\inf)_{0\le u\le t}\xi(u)$ and their complements to the process, $\bar{\xi}(t) = \xi(t) - \xi^{+}(t)$ and $\check{\xi}(t) = \xi(t) - \xi^{-}(t)$, stopped at an exponentially distributed moment. It is worthwhile mentioning that the derivation of such transformations also is a rather complicated problem. However, if a positive boundary is crossed by jumps whose distribution possesses the property of lack of memory (in other words, if $\xi(t)$ is an upper almost semicontinuous process), the problem simplifies somehow [11, 12]. Recall that the property of lack of memory for geometrical (exponential) random variable $\eta_c > 0$ means

$$\mathsf{P}\{\eta_c - x > y \mid \eta_c > x\} = \mathsf{P}\{\eta_c > y\} = \begin{cases} c^y, & y \in \mathbb{N},\ 0 < c < 1; \\ e^{-cy}, & y > 0,\ c > 0. \end{cases}$$

An importance of this property for positive jumps is mentioned in [13] for the scalar case (that is, for $m = 1$). In particular, Proposition 2.1 of [13] for perturbed Poisson processes with exponentially distributed jumps proves the conditional independence of the overshoot and hitting time for a boundary if the overshoot is positive. The independence of $\gamma_1(x)$ and $\tau^{+}(x)$ is studied in [14, Chapter 5] for the case where a positive boundary is crossed by exponentially distributed jumps. The question on whether or not $\gamma_{2,3}(x)$ and $\tau^{+}(x)$ are dependent is not considered in [13] and [14].

Note that the independence of $\gamma_k(x)$ and $\tau^{+}(x)$ is important when determining the Gerber–Shiu measure, while the dependence between $\gamma_2(x)$ and $\tau^{+}(x)$ appears when analyzing the properties of this measure.

The distributions of hitting functionals of a positive boundary is studied in [15, 16] for the case of Poisson processes defined on a Markov chain under the assumption that negative jumps are geometrically or exponentially distributed. No assumption on the distribution of positive jumps is in fact imposed [15, 16].

Analogous functionals for dual processes are studied in the current paper. In particular, we study the independence as well as dependence between $\gamma_{1,2,3}(x)$ and $\tau^{+}(x)$. We also find a relation between the generating functions of the pairs $\{\gamma_k(x), \tau^{+}(x)\}$, $k = 2, 3$, and joint generating function of $\{\gamma_2(0), \tau^{+}(0)\}$ that can be expressed in terms of the distribution of negative values of $\xi(t)$. The result we obtain below allows one to restrict the consideration to the generating function of $\{\gamma_2(0), \tau^+(0)\}$ when studying the hitting functionals for the processes described above. Moreover, various representations for a solution of the cumulant equation depending on the sign of $\mathsf{E}\,\xi(1)$ are obtained in the case of a scalar integer valued process. These representations can be used for determining the limit behavior of a solution of the cumulant equation.

For the case of real values, we consider Poisson processes defined on a Markov chain under the assumption that a positive boundary is crossed by exponentially distributed jumps. For the integer values, the jumps are assumed to be geometrically distributed. In both cases, the underlying processes are called upper almost semicontinuous. The case of integer values is considered separately, since the distribution is discrete.

2. Integer valued processes

First we consider upper almost semicontinuous integer valued Poisson processes $\xi(t)$ defined on a Markov chain $J(t)$. Let $\|p_{kr}\|_{k,r=1}^m$ be the matrix of transition probabilities of the embedded chain for $J(t)$ and let $\{n_k, k = 1, \dots, m\}$ be the transition rates. Denote by $S_k^{(1)}(t)$ and $S_k^{(2)}(t)$ jointly independent compound Poisson processes with the jump rates $\lambda_{1k} > 0$ and $\lambda_{2k} > 0$, respectively. The jumps $S_k^{(1)}(t)$ are distributed according to the geometric law with a parameter $0 < c_k < 1$, while the distribution of jumps $S_k^{(2)}(t)$ is $f_k(x)$, $x \in \mathbb{N}$, $k = 1, \dots, m$. The process $\xi(t)$ starts from the origin. If $J(t) = k$, then its increments are determined by the process $S_k^{(1)}(t) - S_k^{(2)}(t)$. At the moment when $J(t)$ jumps from the state k to the state r, the jump of the process $\xi(t)$ is negative and its value is equal to χ_{kr}. The evolution of the process defined in this way is determined by the matrix valued generating function and cumulant

$$\mathsf{E}\, z^{\xi(t)} = \left\| \mathsf{E}\left[z^{\xi(t)}, J(t) = j \mid J(0) = i \right] \right\|_{k,r=1}^m = \exp\{t\mathbf{K}(z)\},$$

$$\mathbf{K}(z) = \sum_{x \neq 0} (z^x - 1)\, \mathbf{\Pi}_0(x) + \mathbf{Q}, \tag{2}$$

where $\mathbf{\Pi}_0(x) = \|\delta_{kr}\lambda_{2k} f_k(-x) + n_k p_{kr} \mathsf{P}\{\chi_{kr} = x\}\|$, if $x < 0$. In addition,

$$\mathbf{\Pi}_0(x) = \mathbf{\Lambda}_1 (\mathbf{I} - \mathbf{C})\, \mathbf{C}^{x-1}$$

for $x \in \mathbb{N}$ with $\mathbf{C} = \|\delta_{kr} c_k\|$ and $\mathbf{\Lambda}_1 = \|\delta_{kr}\lambda_{1k}\|$ (see [11] for more detail).

Denote by θ_s a random variable being independent of $\xi(t)$ and $J(t)$ and such that $\mathsf{P}\{\theta_s > t\} = e^{-st}$, $s, t > 0$. With the help of θ_s, we obtain the Laplace–Carson transform, in particular, $\mathbf{P}_s \overset{\text{def}}{=} \mathsf{E}\, e^{\theta_s \mathbf{Q}} = s(s\mathbf{I} - \mathbf{Q})^{-1}$. The generating functions of the extreme values and their complements are denoted by

$$\mathbf{g}_\pm(s, z) = \mathsf{E}\, z^{\xi^\pm(\theta_s)} = \sum_{x \in \mathbb{Z}_\pm} z^x \mathbf{p}_x^\pm(s),$$

$$\mathbf{g}^-(s, z) = \mathsf{E}\, z^{\bar{\xi}(\theta_s)} = \sum_{x \in \mathbb{Z}_+} z^{-x} \check{\mathbf{p}}_{-x}^-(s), \qquad \mathbf{g}^+(s, z) = \mathsf{E}\, z^{\check{\xi}(\theta_s)} = \sum_{x \in \mathbb{Z}_+} z^x \check{\mathbf{p}}_x^+(s),$$

where $\mathbf{p}_x^\pm(s) = \mathbf{P}\{\xi^\pm(\theta_s) = x\}$, $\check{\mathbf{p}}_{-x}^-(s) = \mathbf{P}\left\{\bar{\xi}(\theta_s) = -x\right\}$, and $\check{\mathbf{p}}_x^+(s) = \mathbf{P}\left\{\check{\xi}(\theta_s) = x\right\}$. The basic factorization identity is written in terms of these generating functions, namely

$$\mathbf{g}(s, z) \overset{\text{def}}{=} \mathsf{E}\, z^{\xi(\theta_s)} = \begin{cases} \mathbf{g}_+(s, z)\, \mathbf{P}_s^{-1} \mathbf{g}^-(s, z), \\ \mathbf{g}_-(s, z)\, \mathbf{P}_s^{-1} \mathbf{g}^+(s, z), \quad |z| = 1. \end{cases} \tag{3}$$

Put $\mathbf{p}_+(s) = \mathbf{P}\{\xi^+(\theta_s) = 0\}$ and $\mathbf{q}_+(s) = \mathbf{P}_s - \mathbf{p}_+(s)$ and consider the matrix

$$\mathbf{Z}_s = \left(\mathbf{q}_+(s)\mathbf{P}_s^{-1} + \mathbf{p}_+(s)\mathbf{P}_s^{-1}\mathbf{C}\right)^{-1}.$$

It is proved in [11] that

$$\begin{aligned} \mathbf{g}_+(s,z) &= (\mathbf{I} - \mathbf{C}z)\left(\mathbf{I} - \mathbf{Z}_s^{-1}z\right)\mathbf{p}_+(s), \\ \mathbf{p}_k^+(s) &= (\mathbf{I} - \mathbf{C}\mathbf{Z}_s)\,\mathbf{Z}_s^{-k}\mathbf{p}_+(s), \qquad k \geq 1, \\ \mathbf{P}\left\{\xi^+(\theta_s) > x\right\}\mathbf{P}_s^{-1} &= \left(\mathbf{Z}_s^{-1} - \mathbf{C}\right)\mathbf{Z}_s^{-x}(\mathbf{I} - \mathbf{C})^{-1}. \end{aligned} \tag{4}$$

According to [15, Lemma 2.1], the moment generating function of the joint generating function of hitting functionals is expressed in terms of the components written in the first line in the main factorization identity (3). Analogously, the moment generating function of the generating functions of functionals related to the crossing of a “lower” boundary $x \leq 0$ is expressed in terms of the components written in the second line in the main factorization identity (3). Since positive jumps are geometrically distributed in the case of upper almost semicontinuous processes $\xi(t)$, the convolutions

$$\begin{gathered} \mathbf{W}_k(s,x,u) \stackrel{\text{def}}{=} \sum_{y\geq 0} \check{\mathbf{p}}_{-y}^-(s)\mathbf{A}_k(x+y,u), \qquad k = 1,2,3, \\ \mathbf{A}_1(x,u) = \sum_{k\geq x+1} u^{k-x}\mathbf{\Pi}_0(k), \qquad \mathbf{A}_2(x,u) = u^x \sum_{k\geq x+1}\mathbf{\Pi}_0(k), \\ \mathbf{A}_3(x,u) = \sum_{k\geq x+1} u^k \mathbf{\Pi}_0(k), \end{gathered} \tag{5}$$

are simplified and reduced to the power-type. The generating functions of the pairs $\{\tau^+(x), \gamma_k(x)\}$, $k = 1,2,3$, are determined from relations that are reduced to the following form:

$$\begin{aligned} \mathbf{V}_k(s,x,u) &\stackrel{\text{def}}{=} \mathsf{E}\left[e^{-s\tau^+(x)}u^{\gamma_k(x)}, \tau^+(x) < \infty\right] \\ &= \sum_{y=0}^{x} s^{-1}\mathbf{p}_{x-y}^+(s)\mathbf{W}_k(s,y,u), \qquad x \geq 0,\ k = 1,2,3. \end{aligned} \tag{6}$$

A matrix analogue of the second factorization identity (see [17, Theorem 7.3]) is more convenient when analyzing the generating functions of $\{\tau^+(x), \gamma_1(x)\}$. Averaging with respect to ν_ε such that $\mathsf{P}\{\nu_\varepsilon = k\} = (1-\varepsilon)\,\varepsilon^{k-1}$, $k \in \mathbb{N}$, this matrix analogue is expressed in terms of the generating function of $\xi^+(\theta_s)$ as follows:

$$\mathsf{E}\left[e^{-s\tau^+(\nu_\varepsilon)}u^{\gamma_1(\nu_\varepsilon)}, \tau^+(\nu_\varepsilon) < \infty\right] = \frac{(1-\varepsilon)\,u}{u-\varepsilon}\left(\mathbf{I} - \mathbf{g}_+(s,\varepsilon)\,\mathbf{g}_+^{-1}(s,u)\right). \tag{7}$$

Theorem 1. *Let $\xi(t)$ be an integer valued upper almost semicontinuous process with cumulant* (2). *Then the generating function of $\{\tau^+(x), \gamma_1(x)\}$ is such that*

$$\mathbf{V}_1(s,x,u) = \left(\mathbf{Z}_s^{-1} - \mathbf{C}\right)\mathbf{Z}_s^{-x}(\mathbf{I} - \mathbf{C})^{-1}u\,(\mathbf{I} - \mathbf{C})\,(\mathbf{I} - u\mathbf{C})^{-1}. \tag{8}$$

The generating functions of $\{\tau^+(x), \gamma_2(x)\}$ *and* $\{\tau^+(0), \gamma_2(0)\}$ *are related to each other as follows:*

$$\begin{aligned} \mathbf{V}_2(s,0,u) &= s^{-1}\mathbf{p}_+(s)\mathbf{W}_2(s,0,u) = s^{-1}\mathbf{p}_+(s)\,\mathsf{E}\,(u\mathbf{C})^{|\bar\xi(\theta_s)|}\,\mathbf{\Lambda}_1, \\ \mathbf{W}_2(s,x,u) &= \mathsf{E}\,(u\mathbf{C})^{|\bar\xi(\theta_s)|}\,(u\mathbf{C})^x\,\mathbf{\Lambda}_1, \qquad x\ge 0, \\ \mathbf{V}_2(s,x,u) &= \mathbf{p}_+^{-1}(s)\sum_{y=0}^{x}\mathbf{p}_y^+(s)\mathbf{V}_2(s,0,u)\,(u\mathbf{C})^{x-y}, \qquad x\ge 0, \\ \mathsf{E}\left(\mathbf{C}^{|\bar\xi(\theta_s)|}\right) &= \mathbf{P}_s\mathbf{p}_+^{-1}(s)\,\mathsf{E}\left[\mathbf{C}^{|\xi(\theta_s)|}, \xi(\theta_s)\le 0\right] \\ &= (\mathbf{I}-\mathbf{C})\left(\mathbf{I}-\mathbf{Z}_s^{-1}\right)\mathsf{E}\left[\mathbf{C}^{|\xi(\theta_s)|}, \xi(\theta_s)\le 0\right], \\ \mathbf{V}_2(s,0,1) &= s^{-1}\,\mathsf{E}\left[\mathbf{C}^{|\xi(\theta_s)|}, \xi(\theta_s)\le 0\right]\mathbf{\Lambda}_1. \end{aligned} \tag{9}$$

Further, the generating functions $\mathbf{V}_3(s,0,u)$ *and* $\mathbf{V}_3(s,x,u)$ *are such that*

$$\begin{aligned} \mathbf{V}_3(s,0,u) &= s^{-1}\mathbf{p}_+(s)\mathbf{W}_3(s,0,u) \\ &= s^{-1}\mathbf{p}_+(s)\,\mathsf{E}\,(u\mathbf{C})^{|\bar\xi(\theta_s)|}\,\mathbf{\Lambda}_1 u\,(\mathbf{I}-\mathbf{C})\,(\mathbf{I}-u\mathbf{C})^{-1}, \\ \mathbf{W}_3(s,x,u) &= \mathsf{E}\,(u\mathbf{C})^{|\bar\xi(\theta_s)|}\,u\,(\mathbf{I}-\mathbf{C})\,(\mathbf{I}-\mathbf{C}u)^{-1}\,(u\mathbf{C})^x\,\mathbf{\Lambda}_1, \qquad x\ge 0, \\ \mathbf{V}_3(s,x,u) &= \mathbf{p}_+^{-1}(s)\sum_{y=0}^{x}\mathbf{p}_y^+(s)\mathbf{V}_3(s,0,u)\,(u\mathbf{C})^{x-y}, \qquad x\ge 0. \end{aligned} \tag{10}$$

Let $m_1^0 = \boldsymbol{\pi}^\top\mathbf{K}'(1)\mathbf{e} > 0$, *where* $\mathbf{e}$ *is the vector column whose components all equal one. If the distribution of* $\bar\xi$ *is nondegenerate, then relation*(26) *of* [11] *implies that*

$$\mathsf{E}\left(\mathbf{C}^{|\bar\xi|}\right) = \frac{1}{\mu_*^+}\,(\mathbf{I}-\mathbf{C})\,\mathbf{\Pi}_*\int_0^\infty\sum_{x\le 0}\mathbf{C}^{|x|}\mathsf{P}\left\{\xi(t)=x\right\}dt, \tag{11}$$

where $\mathbf{\Pi}_*$ *is the matrix with equal rows defined by the stationary distribution of the chain* $\{J(\tau^+(z-1)), z\ge 0\}$ *and where* μ_*^+ *is the expectation of* $\tau^+(0)$ *averaged with respect to this stationary distribution* (*see* [11, p. 1040] *for more detail*).

Proof. In general,

$$\begin{aligned} \frac{\Delta\mathbf{g}_+(s,u)}{\Delta u} &= \frac{1}{u-\varepsilon}\left[\mathbf{g}_+(s,u)-\mathbf{g}_+(s,\varepsilon)\right] = \sum_{k\ge 0}\frac{u^k-\varepsilon^k}{u-\varepsilon}\mathbf{p}_k^+(s) \\ &= \sum_{k\ge 1}\left(\sum_{r=0}^{k-1}\left(\frac{\varepsilon}{u}\right)^r u^{k-1}\right)\mathbf{p}_k^+(s) = \sum_{r\ge 0}\left(\frac{\varepsilon}{u}\right)^r\sum_{k\ge r+1}u^{k-1}\mathbf{p}_k^+(s). \end{aligned} \tag{12}$$

Substituting $\mathbf{p}_k^+(s)$ from (4) to (12) gives us

$$\begin{aligned} \frac{\Delta\mathbf{g}_+(s,u)}{\Delta u} &= \sum_{r\ge 0}\left(\frac{\varepsilon}{u}\right)^r(\mathbf{I}-\mathbf{C}\mathbf{Z}_s)\sum_{k\ge r+1}\mathbf{Z}_s^{-1}u^{k-1}\mathbf{p}_+(s) \\ &= \left(\mathbf{Z}_s^{-1}-\mathbf{C}\right)\left(\mathbf{I}-\varepsilon\mathbf{Z}_s^{-1}\right)\left(\mathbf{I}-\mathbf{Z}_s^{-1}u\right)^{-1}\mathbf{p}_+(s). \end{aligned}$$

Hence

$$\frac{\mathbf{g}_+(s,u)-\mathbf{g}_+(s,\varepsilon)}{u-\varepsilon} = \left(\mathbf{Z}_s^{-1}-\mathbf{C}\right)\left(\mathbf{I}-\varepsilon\mathbf{Z}_s^{-1}\right)\left(\mathbf{I}-\mathbf{Z}_s^{-1}u\right)^{-1}\mathbf{p}_+(s). \tag{13}$$

Multiplying (13) by $u\,(1-\varepsilon)$ and by $\mathbf{g}_+^{-1}\,(s,u)$ from the right, the second factorization identity (7) yields

$$
\begin{aligned}
(14) \qquad & \mathsf{E}\left[e^{-s\tau^+(\nu_\varepsilon)}u^{\gamma_1(\nu_\varepsilon)}, \tau^+\,(\nu_\varepsilon)<\infty\right] \\
& \qquad = u\,(1-\varepsilon)\left(\mathbf{Z}_s^{-1}-\mathbf{C}\right)\left(\mathbf{I}-\varepsilon\mathbf{Z}_s^{-1}\right)^{-1}(\mathbf{I}-\mathbf{C}u)^{-1}.
\end{aligned}
$$

Then we obtain (8) by inverting (14) with respect to ε.

Expressions for the generating functions of the hitting functionals in the case of zero boundary in relations (9) and (10) are derived from (6) with $x=0$. Substituting

$$
\mathbf{\Pi}_0\,(k)=\mathbf{\Lambda}_1\,(\mathbf{I}-\mathbf{C})\,\mathbf{C}^{k-1}, \qquad k\in\mathbb{N},
$$

to (5) for $k=2,3$ we obtain power representations for $\mathbf{W}_k\,(s,u,x)$. Taking into account the expressions for $\mathbf{V}_2(s,0,u)$ and $\mathbf{V}_3(s,0,u)$ we derive the last equalities in (9) and in (10) from (6). Two last relations in (9) follow from a matrix generalization of equality (7.29) in [17] with $z=\mathbf{C}^{-1}$ (also see [11]). Indeed,

$$
\begin{aligned}
\mathbf{g}^-\,(s,z) &= \mathbf{P}_s\mathbf{p}_+^{-1}(s)\Bigg(\mathsf{E}\left[z^{\xi(\theta_s)},\xi\,(\theta_s)\le 0\right] \\
&\qquad + \left(\mathbf{Z}_s^{-1}-\mathbf{C}\right)z\,(\mathbf{I}-\mathbf{C}z)^{-1}\,\mathsf{E}\left[\mathbf{C}^{|\xi(\theta_s)|}-z^{\xi(\theta_s)},\xi\,(\theta_s)<0\right]\Bigg) \\
&= \mathbf{P}_s\mathbf{p}_+^{-1}(s)\int_0^\infty se^{-st}\sum_{x\le 0}\left(z^x\mathbf{I}+\left(\mathbf{Z}_s^{-1}-\mathbf{C}\right)z\,(\mathbf{I}-\mathbf{C}z)^{-1}\left(\mathbf{C}^{|x|}-z^x\mathbf{I}\right)\right) \\
&\qquad \times \mathsf{P}\left\{\xi(t)=x\right\}\,dt.
\end{aligned}
$$

Now equality (11) follows from the first relation in (9) by passing to the limit as $s\to 0$ and $u\to 1$. $\square$

Let $\boldsymbol{\eta}_{\mathbf{C}}=\|\delta_{kr}\eta_{c_k}\|_{k,r=1}^m$ and $\boldsymbol{\eta}_{\mathbf{C}}^0=\|\delta_{kr}\eta_{c_k}^0\|_{k,r=1}^m$, where η_{c_k} and $\eta_{c_k}^0$ are geometrically distributed random variables with parameter c_k and whose minimal values are 1 and 0, respectively. Then

$$
\mathsf{E}\,u^{\boldsymbol{\eta}_{\mathbf{C}}}=\|\delta_{kr}\,\mathsf{E}\,u^{\eta_{c_k}}\|=\left\|\delta_{kr}\frac{u\,(1-c_k)}{1-uc_k}\right\|=u\,(\mathbf{I}-\mathbf{C})\,(\mathbf{I}-u\mathbf{C})^{-1}
$$

and $\mathsf{E}\,u^{\boldsymbol{\eta}_{\mathbf{C}}^0}=\left\|\delta_{kr}\,\mathsf{E}\,u^{\eta_{c_k}^0}\right\|=(\mathbf{I}-\mathbf{C})\,(\mathbf{I}-u\mathbf{C})^{-1}$. Equality (8) can be rewritten in the following form:

$$
\mathbf{V}_1(s,x,u)=\mathbf{T}^+(s,x)\,\mathsf{E}\,u^{\boldsymbol{\eta}_{\mathbf{C}}}
$$

where

$$
\mathbf{T}^+(s,x)=\mathsf{E}\left[e^{-s\tau^+(x)},\tau^+(x)<\infty\right].
$$

More precisely,

$$
\begin{aligned}
&\mathsf{E}\left[e^{-s\tau^+(x)}u^{\gamma_1(x)},\tau^+(x)<\infty, J\left(\tau^+(x)\right)=j\mid J(0)=i\right] \\
&\qquad = \mathsf{E}\left[e^{-s\tau^+(x)},\tau^+(x)<\infty, J\left(\tau^+(x)\right)=j\mid J(0)=i\right]\mathsf{E}\,u^{\eta_{c_j}}
\end{aligned}
$$

for $i,j=1,\dots,m$ or

$$
\begin{aligned}
&\mathsf{E}\left[e^{-s\tau^+(x)}u^{\gamma_1(x)}\mid \tau^+(x)<\infty, J\left(\tau^+(x)\right)=j\right] \\
&\qquad = \mathsf{E}\left[e^{-s\tau^+(x)}\mid \tau^+(x)<\infty, J\left(\tau^+(x)\right)=j\right]\mathsf{E}\,u^{\eta_{c_j}}.
\end{aligned}
$$

Now we conclude that $\tau^+(x)$ and $\gamma_1(x)$ are independent given

$$
\tau^+(x)<\infty \quad \text{and} \quad J\left(\tau^+(x)\right)=j.
$$

Moreover, $\gamma_1(x)$ is geometrically distributed given the same condition. This means that lack of memory property holds for $\gamma_1(x)$ if $\tau^+(x)<\infty$ and $J(\tau^+(x))=j$.

Equalities (9) and (10) imply that the generating functions $\mathbf{V}_2(s,0,u)$ and $\mathbf{V}_3(s,0,u)$ are such that

$$\mathbf{V}_3(s,0,u)=\mathbf{V}_2(s,0,u)\,\mathsf{E}\,u^{\boldsymbol{\eta}_{\mathbf{C}}}.$$

The generating functions $\mathbf{V}_2(s,x,u)$ and $\mathbf{V}_3(s,x,u)$ depend on $\mathbf{V}_{2,3}(s,0,u)$ and on x as well as on the convolution of the distribution of $\xi^+(\theta_s)$ with the distribution of $\boldsymbol{\eta}^0_{u\mathbf{C}}$.

In the scalar case, only one state exists for the chain $J(t)$ and thus Theorem 1 can be rewritten in a somewhat different form in view of the commutative property of the factors.

Corollary 1. *If $\xi(t)$ is a scalar integer valued upper almost semicontinuous process, then*

$$V_1(s,x,u)=T^+(s,x)\,\mathsf{E}\,u^{\gamma_+(x)},$$

$$T^+(s,x)=q_+(s)z_s^{-x},\qquad \mathsf{E}\,u^{\gamma_+(x)}=u\,(1-c)\,(1-uc)^{-1},\quad 0<c<1, \tag{15}$$

$$z_s^{-1}=q_+(s)+p_+(s)c.$$

If $m_1^0=\mathsf{E}\,\xi(1)\ge 0$, then $z_s\to 1$ as $s\to 0$ and thus z_s^{-1} reduces to

$$z_s^{-1}=1+\lambda_1\,(1-c)\int_0^\infty\left(1-e^{-st}\right)\sum_{x\le 0}c^{|x|}\mathsf{P}\left\{\xi(t)=x\right\}\,dt. \tag{16}$$

For $m_1^0<0$, we have $z_0^{-1}<1$ and

$$z_s^{-1}=z_0^{-1}+\lambda_1\,(1-c)\int_0^\infty\left(1-e^{-st}\right)\sum_{x\le 0}c^{|x|}\mathsf{P}\left\{\xi(t)=x\right\}\,dt. \tag{17}$$

For $k=2$,

$$V_2\,(s,0,1)=q_+(s)=\lambda_1\int_0^\infty\sum_{x\le 0}c^{|x|}e^{-sx}\mathsf{P}\left\{\xi(t)=x\right\}\,dt.$$

If $m_1^0<0$, then $p_+=\left(1-z_0^{-1}\right)(1-c)^{-1}$, $q_+=q_+(0)=\left(z_0^{-1}-c\right)(1-c)^{-1}$,

$$q_+(s)=q_+(0)+\lambda_1\int_0^\infty\left(e^{-st}-1\right)\sum_{x\le 0}c^{|x|}\mathsf{P}\left\{\xi(t)=x\right\}\,dt. \tag{18}$$

If $m_1^0\ge 0$, then $q_+(s)\to 1$ as $s\to 0$ and

$$q_+(s)=1+\lambda_1\int_0^\infty\left(e^{-st}-1\right)\sum_{x\le 0}c^{|x|}\mathsf{P}\left\{\xi(t)=x\right\}\,dt. \tag{19}$$

The joint generating function of $\{\tau^+(x),\gamma_2(x)\}$ is such that

$$V_2(s,0,u)=s^{-1}p_+(s)W_2(s,0,u)=\lambda_1 s^{-1}p_+(s)E\,(uc)^{\left|\xi^-(\theta_s)\right|},$$

$$W_2(s,x,u)=\lambda_1\,\mathsf{E}\,(uc)^{\left|\xi^-(\theta_s)\right|}\,(uc)^x=\lambda_1 g_-\left(s,(uc)^{-1}\right)\mathsf{P}\left\{\eta_{cu}>x\right\},$$

$$V_2\,(s,0,1)=q_+(s)=s^{-1}\lambda_1\,\mathsf{E}\left[c^{|\xi(\theta_s)|},\xi\,(\theta_s)\le 0\right],$$

$$\mathsf{E}\left[c^{|\xi(\theta_s)|},\xi\,(\theta_s)\le 0\right]=\frac{\lambda_1}{s}q_+(s)=\frac{\lambda_1}{s}\,\frac{z_s^{-1}-c}{1-c}, \tag{20}$$

$$V_2(s,x,u)=p_+^{-1}(s)V_2(s,0,u)\sum_{y=0}^{x}p_y^+(s)\,(uc)^{x-y}$$

$$=V_2(s,0,u)\left((uc)^x-\frac{(1-cz_s)\left((uc)^x-z_s^{-x}\right)}{1-ucz_s}\right),\qquad x\ge 0.$$

Note that (20) *for* $u = 1$ *implies*

$$T^{+}(s,x) = T^{+}(s,0)\, z_s^{-x}, \qquad x \ge 0, \qquad T^{+}(s,0) = q_{+}(s).$$

For $k = 3$,

$$\begin{aligned} V_3(s,0,u) &= s^{-1} p_{+}(s) W_3(s,0,u) \\ &= \lambda_1 s^{-1} p_{+}(s)\, \mathsf{E}\,(uc)^{\left|\xi^{-}(\theta_s)\right|}\, \mathsf{E}\, u^{\eta_c} = V_2(s,0,u)\, \mathsf{E}\, u^{\eta_c}, \end{aligned}$$

$$W_3(s,x,u) = \lambda_1\, \mathsf{E}\,(uc)^{\left|\xi^{-}(\theta_s)\right|} u\,(1-c)\,(1-cu)^{-1}\,(uc)^{x}, \qquad x \ge 0, \tag{21}$$

$$\begin{aligned} V_3(s,x,u) &= p_{+}^{-1}(s) V_3(s,0,u) \sum_{y=0}^{x} p_y^{+}(s)\,(uc)^{x-y} \\ &= V_3(s,0,u)\left((uc)^{x} - \frac{(1-cz_s)\left((uc)^{x} - z_s^{-x}\right)}{1-ucz_s}\right), \qquad x \ge 0, \end{aligned}$$

$$\mathsf{E}\, z^{\xi^{-}(\theta_s)} = p_{+}^{-1}(s) \sum_{x\le 0}\left(z^{x} + \frac{\left(z_s^{-1}-c\right)z}{1-cz}\left(c^{|x|} - z^{x}\right)\right) \mathsf{P}\left\{\xi\left(\theta_s\right) = x\right\}.$$

According to [17, Theorem 7.5], *if* $m_1^0 > 0$, *then the distribution of* ξ^{-} *is nondegenerate and*

$$p_{+}'(0) = \mathsf{E}\,\tau^{+}(0) = \frac{z'(0)}{1-c}, \qquad z'(0) = \frac{1}{m_1^0},$$

whence

$$\mathsf{E}\, c^{\left|\xi^{-}\right|} = m_1^0\,(1-c)\int_0^{\infty} \sum_{x\le 0} c^{|x|}\, \mathsf{P}\left\{\xi(t) = x\right\}\, dt. \tag{22}$$

It is worth mentioning that $\gamma_1(x) = \gamma_3(x) = 1$ and $\gamma_2(x) = 0$, $x \ge 0$ for integer valued upper semicontinuous Poisson processes. Note also that analogous results (in particular, on the independence of $\tau^{-}(x) = \inf\{t > 0\colon \xi(t) < x\}$, $x \le 0$, and $\gamma^{-}(x) = \xi\left(\tau^{-}(x)\right) - x$) hold for lower almost semicontinuous processes.

Example 1. Let $m = 1$ and let the cumulant of $\xi(t)$ be of the following form:

$$K(z) = \lambda_1\,(z-1)\,(1-cz)^{-1} + \lambda_2\,(z-1)\,(z-b)^{-1}, \qquad 0 < b,\ c < 1.$$

In other words, negative jumps are geometrically distributed with parameter b. According to relation (7.12) of [17],

$$\mathsf{E}\, u^{\left|\xi^{-}(\theta_s)\right|} = p_{-}(s)\frac{u+b}{u+z_1(s)}, \qquad b < z_1(s) = q_{-}(s) + b p_{-}(s) < 1,$$

since the process is lower almost semicontinuous. Then the first equality in (20) implies that

$$V_2(s,0,u) = \lambda_1 s^{-1} p_{+}(s) p_{-}(s)\frac{uc+b}{uc+z_1(s)},$$

whence $q_{+}(s) = V_2(s,0,1) = \lambda_1 s^{-1} p_{+}(s) p_{-}(s)\frac{c+b}{c+z_1(s)}$, and thus

$$V_2(s,0,u) = q_{+}(s)\frac{c+z_1(s)}{c+b}\,\frac{uc+b}{uc+z_1(s)}.$$

Substituting this equality to the fifth equality of (20) we obtain an expression for $V_2(s,x,u)$. The corresponding expression for $V_2(s,x,u)$ shows that, even in this simple case, the joint generating function $\{\tau^{+}(x), \gamma_2(x)\}$ is not equal to the product of generating functions $\mathsf{E}\left[e^{-s\tau^{+}(x)}, \tau^{+}(x) < \infty\right]$ and $\mathsf{E}\, u^{\gamma_2(x)}$.

3. Real valued processes

Let $\xi(t)$ be an upper almost semicontinuous Poisson process with a drift defined on the Markov chain $J(t)$ (see [12, 16] for more detail) and with the cumulant

$$\begin{aligned}\mathbf{\Psi}(\alpha) &= \left\| \frac{1}{t} \ln \mathsf{E}\left[e^{\imath\alpha\xi(t)}, J(t) = r \mid J(0) = k \right] \right\| \\ &= \imath\alpha\mathbf{A} + \mathbf{\Lambda}_1 \imath\alpha \left(\mathbf{C} - \imath\alpha\mathbf{I}\right)^{-1} + \int_{-\infty}^{0} \left(e^{\imath\alpha} - 1\right) d\mathbf{K}_0(x) + \mathbf{Q},\end{aligned} \tag{23}$$

where

$$\mathbf{K}_0(x) = \|\delta_{kr}\lambda_{2k}F_k\left(-x\right) + p_{kr}n_k \mathsf{P}\left\{\chi_{kr} < x\right\}\|, \qquad x \le 0,$$

$F_k(x)$ is the distribution function of jumps of the process $S_{2k}(t)$, $\mathbf{A} = \|\delta_{kr}a_k\|$, $a_k \le 0$. We keep notation (1) for hitting functionals and the main factorization identity is given by

$$\mathbf{\Phi}\left(s,\alpha\right) \overset{\text{def}}{=} \mathsf{E}\, e^{\imath\alpha\xi(\theta_s)} = \begin{cases} \mathbf{\Phi}_+\left(s,\alpha\right)\mathbf{P}_s^{-1}\mathbf{\Phi}^-\left(s,\alpha\right), \\ \mathbf{\Phi}_-\left(s,\alpha\right)\mathbf{P}_s^{-1}\mathbf{\Phi}^+\left(s,\alpha\right), \end{cases} \tag{24}$$

where $\mathbf{\Phi}_\pm\left(s,\alpha\right) = \mathsf{E}\, e^{\imath\alpha\xi^\pm(\theta_s)}$, $\mathbf{\Phi}^+\left(s,\alpha\right) = \mathsf{E}\, e^{\imath\alpha\check{\xi}(\theta_s)}$, and $\mathbf{\Phi}^-\left(s,\alpha\right) = \mathsf{E}\, e^{\imath\alpha\bar{\xi}(\theta_s)}$.

Then the counterparts of equalities (4) are given by

$$\begin{aligned}\mathbf{p}_+^*(s) = \mathbf{p}_+(s)\mathbf{P}_s^{-1} &= \left(\mathbf{I} + s^{-1}\int_{-\infty}^{0} \mathsf{P}\left\{\bar{\xi}\left(\theta_s\right) \in dy\right\} e^{\mathbf{C}y}\mathbf{\Lambda}_1\right)^{-1}, \\ \mathsf{E}\, e^{\imath\alpha\xi^+(\theta_s)} &= \left(\mathbf{C} - \imath\alpha\mathbf{I}\right)\left(\mathbf{p}_+^*(s)\mathbf{C} - \imath\alpha\mathbf{I}\right)^{-1}\mathbf{p}_+(s), \\ \mathsf{P}\left\{\xi^+\left(\theta_s\right) > x\right\} &= \mathbf{T}^+(s,x)\mathbf{P}_s = \mathbf{q}_+(s)\mathbf{P}_s^{-1}e^{-\mathbf{C}\mathbf{p}_+^*(s)x}\mathbf{P}_s.\end{aligned} \tag{25}$$

Note that $\mathsf{E}\, e^{\imath\alpha\xi^+(\theta_s)}$ can be rewritten in terms of the matrix $\mathbf{R}_+(s) = \mathbf{C}\mathbf{p}_+^*(s)$ that like $\mathbf{Z}_s$ in the integer valued case generalizes the scalar root of the Lundberg equation. In particular, using stochastic representations given in the proof of Theorem 3.2 in [10, p. 62] we conclude that

$$\begin{aligned}&\left(s\mathbf{I} + \mathbf{\Lambda} + \mathbf{N}\right)\left(\mathbf{I} - \mathbf{p}_+^*(s)\right) \\ &\quad = \mathbf{A}\left(\mathbf{I} - \mathbf{p}_+^*(s)\right)\mathbf{R}_+(s) + \mathbf{\Lambda}_1 + \int_{-\infty}^{0} d\mathbf{K}_0\left(z\right)\left(\mathbf{I} - \mathbf{p}_+^*(s)\right)e^{\mathbf{R}_+(s)z},\end{aligned}$$

which also is a generalization of the Lundberg equation.

Note that the generating functions of "upper" functionals $\{\tau^+(x), \gamma_k(x)\}$ related to the crossing of an "upper" boundary $x \ge 0$ are determined as in the integer valued case by components of the first equality in the main factorization identity (see equality (3.21) of Theorem 3.3 in [10, p. 70] for $\mathbf{B} = 0$ and $\mathbf{A} \le 0$).

Theorem 2. *The generating functions*

$$\mathbf{V}_k(s,x,u) \overset{\text{def}}{=} \mathsf{E}\left[e^{-s\tau^+(x) - u\gamma_k(x)}, \tau^+(x) < \infty\right], \quad k = 1,2,3,$$

in the case of a real valued upper almost semicontinuous process defined on a Markov chain are given by

$$\mathbf{V}_1(s,x,u) = \left(\mathbf{I} - \mathbf{p}_+^*(s)\right)e^{-\mathbf{C}\mathbf{p}_+^*(s)x}\mathbf{C}\left(\mathbf{C} + u\mathbf{I}\right)^{-1}, \tag{26}$$

$$\begin{aligned}\mathbf{V}_2(s,x,u) &= \mathbf{V}_2(s,0,u)e^{-(u\mathbf{I}+\mathbf{C})x} \\ &\quad + \left(\mathbf{I} - \mathbf{p}_+^*(s)\right)\int_0^x e^{-\mathbf{C}\mathbf{p}_+^*(s)y}\mathbf{C}\mathbf{V}_2(s,0,u)e^{-(u\mathbf{I}+\mathbf{C})(x-y)}\, dy,\end{aligned} \tag{27}$$

where

$$\mathbf{V}_2(s,0,u) = s^{-1}\mathbf{p}_+^*(s)\int_{-\infty}^0 d\mathbf{P}^-(s,z)\,e^{(u\mathbf{I}+\mathbf{C})z}\boldsymbol{\Lambda}_1 = s^{-1}\mathbf{p}_+^*(s)\,\mathsf{E}\,e^{\bar{\xi}(\theta_s)(u\mathbf{I}+\mathbf{C})}\boldsymbol{\Lambda}_1,$$

$$\mathbf{V}_3(s,x,u) = \mathbf{V}_2(s,x,u)\mathbf{C}\,(u\mathbf{I}+\mathbf{C})^{-1}. \tag{28}$$

The joint generating function for $\{\tau^+(0),\gamma_2(0)\}$ *can also be represented as follows:*

$$\begin{aligned}\mathbf{V}_2(s,0,u) = s^{-1}\Bigg(&\mathsf{E}\left[e^{\xi(\theta_s)(u\mathbf{I}+\mathbf{C})},\xi(\theta_s)\le 0\right]\\ &-\left(\mathbf{I}-\mathbf{p}_+^*(s)\right)\mathbf{C}\int_{-\infty}^0 \mathsf{E}\left[e^{\mathbf{C}(\xi(\theta_s)-y)},\xi(\theta_s)\le y\right]\\ &\times e^{(u\mathbf{I}+\mathbf{C})y}\,dy\Bigg)\boldsymbol{\Lambda}_1,\end{aligned} \tag{29}$$

whence

$$\begin{aligned}\mathbf{V}_2(s,0,0) = {}& s^{-1}\,\mathsf{E}\left[e^{\xi(\theta_s)\mathbf{C}},\xi(\theta_s)\le 0\right]\boldsymbol{\Lambda}_1\\ &- s^{-1}\left(\mathbf{I}-\mathbf{p}_+^*(s)\right)\mathbf{C}\int_{-\infty}^0 \mathsf{E}\left[e^{\mathbf{C}(\xi(\theta_s)-y)},\xi(\theta_s)\le y\right]e^{\mathbf{C}y}\,dy\boldsymbol{\Lambda}_1.\end{aligned} \tag{30}$$

Proof. According to Corollary 3.4 of [10, p. 72], we have

$$\mathbf{V}_k(s,x,u) = \int_0^x s^{-1}\mathsf{P}\left\{\xi^+(\theta_s)\in dy\right\}\mathsf{P}_s^{-1}\int_{-\infty}^0 \mathsf{P}\left\{\bar{\xi}(\theta_s)\in dz\right\}\mathbf{W}_k(x-y-z,u) \tag{31}$$

for $k=1,2,3$ and for an arbitrary Lévy process with a bounded variation and $\mathbf{A}\le 0$, where

$$\mathbf{W}_1(x,u) = \int_x^\infty e^{u(x-z)}\,d\mathbf{K}_0(z), \qquad \mathbf{W}_2(x,u) = \int_x^\infty e^{-ux}\,d\mathbf{K}_0(z),$$
$$\mathbf{W}_3(x,u) = \int_x^\infty e^{-uz}\,d\mathbf{K}_0(z).$$

The power form of $\mathbf{K}_0(x)$ for $x\ge 0$ implies for an almost semicontinuous process that

$$\mathbf{W}_1(x,u) = e^{-\mathbf{C}x}\boldsymbol{\Lambda}_1\mathbf{C}\,(u\mathbf{I}+\mathbf{C})^{-1}, \qquad \mathbf{W}_2(x,u) = e^{-(u\mathbf{I}+\mathbf{C})x}\boldsymbol{\Lambda}_1,$$
$$\mathbf{W}_3(x,u) = \mathbf{W}_2(x,u)\mathbf{C}\,(u\mathbf{I}+\mathbf{C})^{-1}.$$

Since $s^{-1}\,\mathsf{E}\,e^{\bar{\xi}(\theta_s)\mathbf{C}}\boldsymbol{\Lambda}_1 = \left(\mathbf{p}_+^*(s)\right)^{-1}-\mathbf{I}$, equality (31) implies the following relation:

$$\begin{aligned}\mathbf{V}_1(s,x,u) = {}&\left(\mathbf{I}-\mathbf{p}_+^*(s)\right)e^{-\mathbf{C}x}\mathbf{C}\,(u\mathbf{I}+\mathbf{C})^{-1}\\ &+\left(\mathbf{I}-\mathbf{p}_+^*(s)\right)\int_0^x e^{-\mathbf{C}\mathbf{p}_+^*(s)y}\left(\mathbf{C}-\mathbf{C}\mathbf{p}_+^*(s)\right)e^{-\mathbf{C}(x-y)}\,dy\mathbf{C}\,(u\mathbf{I}+\mathbf{C})^{-1},\end{aligned}$$

whence (26) follows in view of $\frac{d}{dx}\left(e^{\mathbf{D}_1x}e^{\mathbf{D}_2x}\right) = e^{\mathbf{D}_1x}\left(\mathbf{D}_1+\mathbf{D}_2\right)e^{\mathbf{D}_2x}$ for constant matrices $\mathbf{D}_1$ and $\mathbf{D}_2$ of appropriate sizes. Substituting $\mathbf{W}_k(x,u)$ into (31) we derive relations (27) and (28) for $\mathbf{V}_k(s,x,u)$, $k=2,3$.

The matrix counterpart of equality (3.101) in [17, p. 120] or equality (3.97) of [14, p. 85],

$$\begin{aligned}\mathbf{P}^-(s,x) = \left(\mathbf{p}_+^*(s)\right)^{-1}\Bigg[&\mathsf{P}\left\{\xi(\theta_s)<x\right\}\\ &-\left(\mathbf{I}-\mathbf{p}_+^*(s)\right)\mathbf{C}\int_{-\infty}^0 e^{\mathbf{C}y}\mathsf{P}\left\{\xi(\theta_s)<x+y\right\}dy\Bigg],\\ &x\le 0,\end{aligned} \tag{32}$$

allows us to rewrite (28) as follows:

$$\mathbf{V}_2(s,0,u) = s^{-1}\Bigg(\mathsf{E}\left[e^{(u\mathbf{I}+\mathbf{C})\xi(\theta_s)}, \xi(\theta_s) \le 0\right] - \big(\mathbf{I} - \mathbf{p}_-^*(s)\big) \times \mathbf{C}\int_{-\infty}^{0}\int_{-\infty}^{0} e^{\mathbf{C}y}\frac{\partial}{\partial x}\left(\mathsf{P}\left\{\xi(\theta_s) < x+y\right\}\right)e^{(u\mathbf{I}+\mathbf{C})x}\,dy\,dx\Bigg)\mathbf{\Lambda}_1,$$

whence we conclude that (30) holds by changing the variables and interchanging the order of integration. Substituting $u=0$ we prove (29). □

Note that all relations in (28) for $\mathbf{V}_k(s,x,u)$, $k=2,3$, are of the same type in the real valued case. The difference between relations for $\mathbf{V}_2(s,x,u)$ and $\mathbf{V}_3(s,x,u)$ in the integer valued case is explained by the property that $\gamma_2(x)$ attains zero values with a positive probability (in contrast to $\gamma_{1,3}(x)$).

If $\boldsymbol{\eta}_{\mathbf{C}}$ is a diagonal matrix with exponentially distributed diagonal entries η_{c_j}, $j=1,\dots,m$, then $\mathsf{E}\,e^{-u\boldsymbol{\eta}_{\mathbf{C}}} = \mathbf{C}\left(\mathbf{C}+u\mathbf{I}\right)^{-1}$. This allows us to rewrite equalities (26) and (28) in a simpler form,

$$\mathbf{V}_1(s,x) = \mathbf{T}^+(s,x)\,\mathsf{E}\,e^{-u\boldsymbol{\eta}_{\mathbf{C}}}, \qquad \mathbf{V}_3(s,x,u) = \mathbf{V}_2(s,x,u)\,\mathsf{E}\,e^{-u\boldsymbol{\eta}_{\mathbf{C}}}.$$

These expressions yield results analogous to those in the integer valued case concerning the conditional independence of $\gamma_1(x)$ and $\tau^+(x)$. Moreover, applying Corollary 3.4 of [10, p. 72] in the same way as in the proof of (28) we obtain a representation for the joint generating function

$$\mathbf{V}(s,x,u,v) \stackrel{\text{def}}{=} \mathsf{E}\left[e^{-s\tau^+(x)-u\gamma_1(x)-v\gamma_2(x)}, \tau^+(x)<\infty\right]$$

as the product of $\mathbf{V}_2(s,x,v)$ and $\mathsf{E}\,e^{-u\boldsymbol{\eta}_{\mathbf{C}}}$. This implies that $\gamma_1(x)$ is conditionally independent of both $\tau^+(x)$ and $\gamma_2(x)$.

Corollary 2. *In the scalar case and for an integer valued upper almost semicontinuous process,*

$$V_1(s,x,u) = q_+(s)e^{-cp_+(s)x}c\left(c+u\right)^{-1} = q_+(s)e^{-cp_+(s)x}\,\mathsf{E}\,e^{-u\gamma_1(x)},$$

$$V_2(s,x,u) = \frac{ue^{-(u+c)x} + cq_+(s)e^{-cp_+(s)x}}{u+cq_+(s)}V_2(s,0,u),$$

where $V_2(s,0,u) = s^{-1}p_+(s)\,\mathsf{E}\,e^{\xi^-(\theta_s)(u+c)}\lambda_1$ *and* $V_3(s,x,u) = V_2(s,x,u)\,\mathsf{E}\,e^{-u\gamma_1(x)}$. *The generating function for* $\{\tau^+(0),\gamma_2(0)\}$ *can be represented as follows:*

$$V_2(s,0,u) = \frac{\lambda_1}{s}\Bigg(\mathsf{E}\left[e^{(u+c)\xi(\theta_s)}, \xi(\theta_s)\le 0\right] + \frac{c}{u}q_+(s)\,\mathsf{E}\left[e^{c\xi(\theta_s)}\left(e^{u\xi(\theta_s)}-1\right), \xi(\theta_s)\le 0\right]\Bigg). \tag{33}$$

Note that the above representation for $V_1(s,x,u)$ is compatible with some other results (see, for example, [17, Corollary 3.4]).

Example 2. Let $m=1$ and let $\Psi(\alpha) = \imath\alpha a + \imath\alpha\big(\lambda_1\left(c-\imath\alpha\right)^{-1} - \lambda_2\left(b+\imath\alpha\right)^{-1}\big)$, where $a<0$ and $b,c,\lambda_1,\lambda_2>0$. In other words, $\xi(t)$ is a compound Poisson process with a negative drift and exponentially distributed negative and positive jumps. This process can be viewed as an aggregate loss process (see [1, p. 1]) for which the claims are exponentially distributed and with exponentially distributed income amounts. Applying [13, Theorem 3.1], we obtain

$$\mathsf{E}\, e^{\imath\alpha\xi^-(\theta_s)} = \frac{r_1(s)r_2(s)}{b}\frac{b+\imath\alpha}{(r_1(s)+\imath\alpha)(r_2(s)+\imath\alpha)},$$

where $-r_1(s)$, $-r_2(s)$ are negative roots of the cumulant equation $\Psi(-\imath\alpha) = s$ (note that $r_1(s) < b < r_2(s)$). Since $cp_+(s)$ also is a root of this equation and, as a result, $-cp_+(s)r_1(s)r_2(s) = scb$, we get

$$V_2(s,0,u) = s^{-1}p_+(s)\,\mathsf{E}\, e^{\xi^-(\theta_s)(u+c)}\lambda_1 = \lambda_1\frac{u+c+b}{(r_1(s)+u+c)\,(r_2(s)+u+c)}.$$

Then Corollary 2 implies that

$$V_2(s,x,u) = \lambda_1\frac{u+c+b}{(r_1(s)+u+c)\,(r_2(s)+u+c)}\frac{ue^{-(u+c)x}+cq_+(s)e^{-cp_+(s)x}}{u+cq_+(s)}.$$

This means, as in Example 1, that the joint generating function $\{\gamma_2(x),\tau^+(x)\}$ cannot be represented as the product of the corresponding marginal generating functions.

Putting

$$B_1(s) = \lambda_1\frac{b-r_1(s)}{r_2(s)-r_1(s)}, \qquad B_2(s) = \lambda_1\frac{r_2(s)-b}{r_2(s)-r_1(s)},$$

and inverting $V_2(s,x,u)$ with respect to u we obtain

$$\begin{aligned}
&\mathsf{E}\left[e^{-s\tau^+(x)}, \gamma_2(x)\in dz, \tau^+(x)<\infty\right]\\
&\quad= \sum_{i=1}^{2}\Bigg(e^{r_i(s)x-(c+r_i(s))z}\left(c+r_i(s)-cq_+(s)e^{-(cp_+(s)+r_i(s))(x-z)}\right)\mathbb{1}_{\{z>x\}}\\
&\qquad\qquad + cq_+(s)\left(e^{-cq_+(s)z}-e^{(c+r_i(s))z}\right)\Bigg)\\
&\qquad\times\frac{B_i(s)}{cp_+(s)+r_i(s)},
\end{aligned}$$

where

$$\mathbb{1}_{\{z>x\}} = \begin{cases}1, & z>x,\\ 0, & z\le x.\end{cases}$$

This result allows us to evaluate the Gerber–Shiu measure, since

$$\begin{aligned}
K^{(s)}(x,dy,dz) &\stackrel{\text{def}}{=} \mathsf{E}\left[e^{-s\tau^+(x)}, \gamma_1(x)\in dy, \gamma_2(x)\in dz, \tau^+(x)<\infty\right]\\
&= ce^{-cy}dy\cdot\mathsf{E}\left[e^{-s\tau^+(x)}, \gamma_2(x)\in dz, \tau^+(x)<\infty\right]
\end{aligned}$$

in view of the conditional independence of $\gamma_1(x)$ and $\{\gamma_2(x),\tau^+(x)\}$.

Bibliography

[1] S. Asmussen and H. Albrecher, *Ruin Probabilities*, World Scientific, Singapore, 2010. MR2766220
[2] L. Breuer, *From Markov Jump Processes to Spatial Queues*, Springer, New York, 2003. MR1967164
[3] A. Pacheco, L. Tang, and N. Prabhu, *Markov-Modulated Processes and Semigenerative Phenomena*, World Scientific, Singapore, 2008. MR2484312
[4] L. Breuer, *First passage times for Markov-additive processes with positive jumps of phase type*, J. Appl. Probab. **45** (2008), 779–799. MR2455184
[5] L. Breuer, *A quintuple law for Markov additive processes with phase-type jumps*, J. Appl. Probab. **47** (2010), 441–458. MR2668499
[6] R. A. Doney and A. E. Kyprianou, *Overshoots and undershoots of Lévy processes*, Ann. Appl. Probab. **16** (2006), 91–106. MR2209337

[7] L. Breuer and A. L. Badescu, *A generalized Gerber–Shiu measure for Markov-additive risk processes with phase-type claims and capital injections*, Scand. Actuar. J. **2** (2014), 93–115. MR3177093

[8] A. E. Kyprianou, *Gerber-Shiu Risk Theory*, Springer, New York, 2013. MR3135656

[9] Z. Jiang and M. Pistorius, *On perpetual American put valuation and first-passage in a regime-switching model with jumps*, Finance and Stochastics **12** (2008), 331–355. MR2410841

[10] D. V. Husak, *Boundary Problems for Processes with Independent Increments on Markov Chains and for Semi-Markov Processes*, Institute of Mathematics, National Academy of Science of Ukraine, Kyiv, 1998. (Ukrainian) MR1710395

[11] M. S. Herych and D. V. Husak, *On the moment-generating functions of extrema and their complements for almost semicontinuous integer-valued Poisson processes on Markov chains*, Ukrain. Matem. Zh. **67** (2016), no. 8, 1034–1049; English transl. in Ukr. Math. J. **67** (2016), no. 8, 1164–1182. MR3473711

[12] D. V. Gusak and E. V. Karnaukh, *Matrix factorization identity for almost semi-continuous processes on a Markov chain*, Theor. Stoch. Process. **11(27)** (2005), No 1-2, 40–47. MR2327445

[13] S. G. Kou and H. Wang, *First passage times of a jump diffusion process*, Adv. Appl. Prob. **35** (2003), 504–531. MR1970485

[14] D. Husak, *Boundary Functionals for Lévy Processes and their Applications*, Lambert Academic Publishing, 2014.

[15] M. S. Gerych, *Distributions of overshoots for almost continuous stochastic processes defined on a Markov chain*, Teor. Imovirnost. Matem. Statyst. **94** (2016), 36–49; English transl. in Theor. Probab. Math. Statist. **94** (2017), 37–52. MR3553452

[16] E. V. Karnaukh, *Overshoot functionals for almost semicontinuous processes defined on a Markov chain*, Teor. Imovirnost. Matem. Statyst. **76** (2007), 45–52; English transl. in Theor. Probab. Math. Statist. **76** (2008), 49–57. MR2368739

[17] D. V. Husak, *Boundary Value Problems for Processes with Independent Increments in the Risk Theory*, Institute of Mathematics, National Academy of Science of Ukraine, Kyiv, 2011. (Ukrainian) MR2382816

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