ON LINEAR CRITERIA FOR THE DETERMINATION OF THE RADIUS OF CONVERGENCE OF A POWER SERIES*

BY

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The most common criteria for the determination of the radius of convergence of a power series $A_0 + A_1x + A_2x^2 + \cdots$ are based upon well known convergence-theorems of CAUCHY and involve expressions containing at most two coefficients of the series. According to one criterion the radius of convergence is the limit of $|A_n^{-1/n}|$ or, in case no limit exists, the reciprocal of the greatest affix of any point of condensation of the sequence $|A_n^{1/n}|$, $(n = 1, 2, 3, \cdots)$. CAUCHY'S ratio test, on the other hand, gives the radius of convergence only when $|A_n/A_{n-1}|$ approaches a limit, the reciprocal of which is then the desired radius.† But though the first criterion is perfectly general and of the highest value for theoretical investigations, it nevertheless has little superiority as a practical test for the determination of the radius of convergence of a given series. For when $|A_n/A_{n-1}|$ approaches a limit, the two criteria are of equal import, and the ratio-test is commonly used, being easier of application. On the other hand, in most cases when no limit for the ratio exists, the determination of the limit of $|A_n^{1/n}|$, or of its points of condensation, is extremely difficult,‡ as, for example, when more than two coefficients are connected in the law governing their formation.

The object of this paper is to establish certain criteria for the convergence of a power series when the $(n + 1)$th coefficient A_n is connected with the preceding coefficients by a linear relation which tends to take a limiting form as n increases indefinitely. The criteria obtained include CAUCHY'S ratio test as a special case and may, indeed, be regarded as an extension of that test. They are applicable in many cases in which the simple ratio-test fails. One special class of series to which they apply (see sections 7 and 8) has been previously considered by

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† When the ratio fails to approach a limiting value, only a minor limit to the radius of convergence is obtained from the sequence $|A_{n-1}/A_n|$.

‡ An exception worthy of note is a series in which A_n does not approach a limiting value but has a finite upper limit. The radius of convergence is then equal to unity.

HADAMARD* and includes every series defining a function which has only polar singularities on the circle of convergence of the series. Another application, which will be made in a subsequent paper, is to the subject of linear differential equations.

The linear criteria can also be extended to power series in two or more variables. The last two theorems of the present paper give the extension for the case of two variables, and may be of some interest, inasmuch as criteria for the convergence of power series in two or more variables seem as yet to be very rare. †

1. *Simple proof of convergence.*

Consider a series:

$$S = A_0 + A_1 x + A_2 x^2 + \cdots, \tag{1}$$

in which after some fixed term every $p + 1$ consecutive coefficients are connected by a linear relation:

$$A_n = b_1^{(n)} A_{n-1} + b_2^{(n)} A_{n-2} + \cdots + b_p^{(n)} A_{n-p}. \tag{2}$$

If the coefficients $b_i^{(n)}$ in this relation, as n indefinitely increases, converge toward fixed limits b_i, the series must converge for values of x other than zero. A simple proof of this is as follows. According to the hypothesis of the existence of a limiting form for (2), if we arbitrarily prescribe a small positive quantity ϵ, an integer m can be found such that for $n \geqq m$ the moduli of the p differences $b_i^{(n)} - b_i$ will all be less than ϵ. Denote now with b' the greatest of the quantities $|b_i| + \epsilon$ and with A' the greatest of the moduli $|A_{m-p}|$, $|A_{m-p+1}|$, $\cdots$, $|A_{m-1}|$. Then since for $n \geqq m$ every $|b_i^{(n)}| < b'$, it follows first from (2), by putting $n = m$, that

$$|A_m| < pb'A'.$$

Hence if c is the larger of the two quantities pb' and 1, none of the p moduli from $|A_{m-p+1}|$ to $|A_m|$ inclusive can exceed cA'. Placing next $n = m + 1$ in (2), we find that

$$|A_{m+1}| < pb'cA' \leqq c^2 A'$$

and, continuing in the same manner, we have in general

$$|A_{m+i}| < c^{i+1} A'.$$

Therefore after the first m terms of (1) the moduli of the successive terms will be less than the corresponding terms of

$$A'cx^m + A'c^2x^{m+1} + A'c^3x^{m+2} + \cdots.$$

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*Liouville's Journal, ser. 4, vol. 8.

† In a dissertation published in volume 34 of the Mathematische Annalen J. HORN has given criteria of an entirely different character for the convergence of certain series which he terms hypergeometric series in two or more variables.

The latter series obviously converges for $|x| < 1/c$. It follows that S must converge within a circle of radius $1/c$ having its center at the origin.

2. *On the formation of the coefficients.*

For many purposes a simple proof such as this of the convergence of S gives all that is desired, although only a minor limit for the radius of convergence has been found. The proof, for example, suffices as an existence-theorem in certain simple cases in the theory of linear differential equations. To determine, however, what the radius of convergence is, another line of proof must be adopted. Consider as a preliminary the power series $a_0 + a_1x + a_2x^2 + \cdots$ for

$$\frac{1 + c_1x + \cdots + c_qx^q}{1 - b_1x - \cdots - b_px^p},$$

which may be obtained by the method of indeterminate coefficients. Beginning with the $(q+2)$th term of the series, each coefficient is obtained from the preceding coefficients by the recurring relation:

$$a_n = b_1a_{n-1} + b_2a_{n-2} + \cdots + b_pa_{n-p}\,. \tag{3}$$

Conversely, if, after some fixed point in a series, the coefficients are connected by the relation (3), the series is the expansion of some rational fraction $G(x)/B_p(x)$ whose denominator $B_p(x)$ is the polynomial $1 - b_1x - \cdots - b_px^p$. A particular case which we shall have occasion to use is the series for $1/B_p(x)$, the first few coefficients of which are obtained from the equations:

$$\begin{aligned} a_0 &= 1\,, \\ a_1 &= b_1a_0\,, \\ a_2 &= b_1a_1 + b_2a_0\,, \\ a_3 &= b_1a_2 + b_2a_1 + b_3a_0\,, \quad \text{etc.}; \end{aligned} \tag{4}$$

the law for the formation of the coefficients may then be said to hold from the outset. In any case, if the fraction has been expressed in its lowest terms, the radius of convergence of the series is equal to the distance from the origin to the nearest root (or roots) of $B_p(x) = 0$.

Let us now return to our series S. The characteristic relation (2) may be written in the form:

$$A_n - (b_1 + \epsilon_1^{(n)})A_{n-1} - (b_2 + \epsilon_2^{(n)})A_{n-2} - \cdots - (b_p + \epsilon_p^{(n)})A_{n-p} = 0\,; \tag{5}$$

or

$$A_n = b_1A_{n-1} + b_2A_{n-2} + \cdots + b_pA_{n-p} + \epsilon_n\,, \tag{6}$$

in which

$$\epsilon_n = \epsilon_1^{(n)}A_{n-1} + \epsilon_2^{(n)}A_{n-2} + \cdots + \epsilon_p^{(n)}A_{n-p}\,. \tag{7}$$

Beginning with the $(m+1)$th term of the series, let each coefficient be expressed in terms of the p coefficients A_{m-p}, A_{m-p+1}, $\cdots$, A_{m-1} and of the quantities ϵ_m, ϵ_{m+1}, $\cdots$. Let also the part of each coefficient which is independent of the latter quantities be written first, then in turn the parts containing ϵ_m, ϵ_{m+1}, $\cdots$. Thus, for example,

$$\begin{aligned} A_m &= (\quad) + \epsilon_m , \\ A_{m+1} &= (\quad) + b_1\epsilon_m + \epsilon_{m+1} , \\ A_{m+2} &= (\quad) + b_1(b_1\epsilon_m) + b_2\epsilon_m + b_1\epsilon_{m+1} + \epsilon_{m+2} , \end{aligned}$$

in which the parts independent of the ϵ_i are indicated by parentheses. If now, after the mth term, we should neglect all the quantities ϵ_i, $(i \geqq m)$, the series would be the expansion of some rational fraction $G(x)/B_p(x)$. As regards ϵ_m, it will be observed that it enters explicitly into the successive coefficients of S in the combinations:

$$\epsilon_m , \quad b_1\epsilon_m , \quad b_1(b_1\epsilon_m) + b_2\epsilon_m , \quad b_1[b_1(b_1\epsilon_m) + b_2\epsilon_m] + b_2[b_1\epsilon_m] + b_3 , \ \cdots ,$$

that is to say, in accordance with the law expressed in (4). If, therefore, the parts of the successive terms of S into which ϵ_m explicitly enters should be collected, they would give a power series $P(x)$ for $\epsilon_m x^m/B_p(x)$. The portion of the series depending upon any subsequent ϵ_n gives in like manner the expansion of $\epsilon_n x^n/B_p(x)$ into a power series. We conclude therefore that *the coefficient of any power of x in the series S can be obtained by summing the coefficients of the same power of x in the various series which represent the respective members of the function :*

$$F(x) = \frac{G(x)}{B_p(x)} + \sum_{n=m}^{\infty} \frac{\epsilon_n x^n}{B_p(x)} . \tag{8}$$

The question whether it is permissible to rearrange S so as to put it into the form (8), or, in other words, whether $F(x)$ and S are identical, will be considered later (§ 5). For the present it suffices that the coefficients of S can be obtained from (8) in the manner stated.

3. *Fundamental proof of convergence.*

We now proceed, with the aid of (8), to prove anew the convergence of S. According to our fundamental hypothesis the relation (5) tends with increasing n to take the form (3). If, therefore, any positive quantity ϵ, however small, be prescribed, an integer m can be found such that for $n \geqq m$ every $|\epsilon_i^{(n)}|$ will be less than ϵ. To obtain a major limit to $|A_n|$ we shall have recourse to the following well known theorem : If a series

$$\sum_{i=0}^{\infty} c_i x^i$$

converges absolutely for $|x| = \xi$, then $|c_n| < \gamma\xi^{-n}$ where γ is some fixed number independent of n. Let ξ designate here any positive quantity smaller than the radius of convergence of the series for $G(x)/B_p(x)$ and for $1/B_p(x)$, and let γ be taken sufficiently large that the inequality shall apply simultaneously to both series. Since the first m coefficients of S are also coefficients of the series for $G(x)/B_p(x)$, we have

$$|A_{m-i}| < \frac{\gamma}{\xi^{m-i}}, \qquad (i=1,\ 2,\ \cdots,\ m).$$

It follows then from (7), by putting $n = m$, that

$$|\epsilon_m| < \epsilon\gamma\left(\frac{1}{\xi^{m-1}} + \frac{1}{\xi^{m-2}} + \cdots + \frac{1}{\xi^{m-p}}\right) = \frac{X}{\xi^m},$$

where

$$X = \epsilon\gamma(\xi + \xi^2 + \cdots + \xi^p).$$

Hence $|A_m|$, which is the modulus of the coefficient of x^m in the series for

$$\frac{G(x)}{B_p(x)} + \frac{\epsilon_m x^m}{B_p(x)},$$

is less than

$$\frac{\gamma}{\xi^m} + \gamma\,|\epsilon_m| = \frac{\gamma(1+X)}{\xi^m}.$$

Since also $|A_{m-i}| < \gamma(1+X)\xi^{-(m-i)}$, we next obtain by placing $n = m+1$ in (7)

$$(9) \qquad |\epsilon_{m+1}| < \epsilon\gamma(1+X)\left(\frac{1}{\xi^m} + \frac{1}{\xi^{m-1}} + \cdots + \frac{1}{\xi^{m-p+1}}\right) = \frac{X(1+X)}{\xi^{m+1}}.$$

From this it follows that A_{m+1}, or the coefficient of x^{m+1} in the series for

$$\frac{G(x)}{B_p(x)} + \frac{\epsilon_m x^m}{B_p(x)} + \frac{\epsilon_{m+1}x^{m+1}}{B_p(x)},$$

has a modulus smaller than

$$(10) \qquad \gamma\frac{(1+X)}{\xi^{m+1}} + \gamma\frac{X(1+X)}{\xi^{m+1}} = \gamma\frac{(1+X)^2}{\xi^{m+1}}.$$

To obtain a major limit for $|\epsilon_{m+2}|$ and $|A_{m+2}|$ we have only to replace m by $m+1$ and γ by $\gamma(1+X)$ in (9) and (10). Proceeding in this manner we obtain

$$(11) \qquad |\epsilon_{m+q}| < \frac{X(1+X)^q}{\xi^{m+q}}, \qquad |A_{m+q}| < \frac{\gamma(1+X)^{q+1}}{\xi^{m+q}}.$$

From the last inequality it is evident that S must converge within the circle of convergence of

$$(12)\qquad 1 + \frac{1+X}{\xi}x + \frac{(1+X)^2}{\xi^2}x^2 + \cdots,$$

the radius of which is $\xi/(1+X)$. But by taking m sufficiently large the value of ϵ and therefore of X may be made smaller than any assigned quantity, while ξ denotes any positive value smaller than the radius of convergence of the series for $1/B_p(x)$. The series S must therefore converge within a circle C whose radius is the distance from the origin to the nearest root (or roots, if several have equal moduli) of $B_p(x) = 0$. The result thus reached may be recapitulated as follows:

I. *Let a series $A_0 + A_1x + A_2x^2 + \cdots$ be given in which, after some fixed term, every $p+1$ consecutive coefficients are connected by a linear relation:*

$$A_n = (b_1 + \epsilon_1^{(n)})A_{n-1} + (b_2 + \epsilon_2^{(n)})A_{n-2} + \cdots + (b_p + \epsilon_p^{(n)})A_{n-p}.$$

If for any arbitrarily assigned positive value ϵ, an integer m can be found such that for $n \geqq m$ every $|\epsilon_i^{(n)}|$ will be less than ϵ, the series will converge within a circle C whose center is the origin and whose radius is the distance from the origin to the nearest root (or roots) of the polynomial:

$$1 - b_1x - b_2x^2 - \cdots - b_px^p.$$

A special case of this theorem is that in which $p = 1$. The root of the polynomial, $1/b_1$, is then also the limit of A_{n-1}/A_n, and we have CAUCHY's ratio test for the determination of the circle of convergence.

4. *The linear relation with the smallest number of terms.*

A necessary consequence of the equation (5) is the existence of similar relations containing a greater number of terms. For if in (5) we replace n by $n+1$ and subtract the original equation multiplied with a constant a, we obtain

$$A_{n+1} - (b_1 + a + e_1^{(n+1)})A_n - (b_2 - ab_1 + e_2^{(n+1)})A_{n-1} - \cdots - (-ab_p + e_{p+1}^{(n+1)})A_{n-p} = 0$$

in which the $e_i^{(n+1)}$ denote certain infinitesimals whose moduli may be made smaller than any assigned quantity by sufficiently increasing n. But this is exactly the form for a linear relation corresponding to the polynomial $(1-ax)B_p(x)$. We conclude therefore that *for every polynomial which contains $B_p(x)$ as a factor there exists a linear relation between the coefficients of S whose limiting form corresponds to this polynomial.*

Suppose now that there is a linear relation of the same character as (5) but containing a smaller number of terms, say

$$(13)\qquad A_r - (b'_1 + e_1^{(r)})A_{r-1} - \cdots - (b'_{p'} + e_{p'}^{(r)})A_{r-p'} = 0, \qquad (p' < p).$$

Then by taking successively

$$r = n + p' - p, \quad n + p' - p + 1, \cdots, n$$

in this equation and subtracting the results multiplied into appropriate constants from (5), the last $p - p' + 1$ terms of (5) can be removed, and a relation is thereby obtained which connects only p' coefficients of S. This relation may then be combined with (13) so as still further to reduce the number of coefficients linearly connected, and so on. It will be observed that the process is exactly parallel to that by which the highest common factor of the polynomials $B_p(x)$ and $B_{p'}(x) = 1 - b'_1 x - \cdots - b'_{p'} x^{p'}$ is derived. Since also the coefficients of x^i and of A_{n-i} at the outset of the two processes differ only by infinitesimals, it is clear that the same holds true after the first subtractions. It follows then by mathematical induction that they will differ also by infinitesimals at the close of the parallel processes. *If therefore there coexist two linear relations between the coefficients of S whose limiting forms correspond to two polynomials $B_p(x)$ and $B_{p'}(x)$, there must also be a linear relation whose limiting form corresponds to the highest common factor of these polynomials.*

There can not, however, be two linear relations which correspond to polynomials without a common polynomial factor. For the last subtraction in the process of finding their highest common factor would give a constant as a remainder. Consequently, to the last minuend and subtrahend there would correspond two linear relations whose limiting forms would differ only in a single coefficient, which is clearly impossible. By combining the two results last italicized we conclude finally that *there is a single characteristic linear relation between the coefficients of S which contains a smaller number of coefficients than any similar expression.*

5. *Fundamental expression for the function defined by the series.*

We now return to the question whether the series S can be rearranged so as to take the form (8). A sufficient condition for such a rearrangement is contained in the following theorem of WEIERSTRASS:* Given a series:

$$F = \sum_{q=0}^{\infty} u_q,$$

of which the general term u_q is a power series:

$$u_q = \sum_{n=0}^{\infty} a_{qn} x^n;$$

if the separate series u_q and the collective series F converge within a circle (R) and if also the latter series converges uniformly along every concentric circle

* BIERMANN, *Theorie der analytischen Functionen,* § 30, p. 146, or HARKNESS AND MORLEY, *Analytic Functions,* § 81.

(R_1) of radius $R_1 < R$, then within the circle (R) the collective series can be expressed as a power series:

$$F = \sum_{n=0}^{\infty} A_n x^n,$$

in which the general coefficient A_n is the sum of the corresponding coefficients a_{qn} of the component series u_q.

In the collective series which we condensed in (8) u_{q+1} is the power series for $\epsilon_{q+m} x^{q+m}/B_p(x)$, which converges within the circle C. Within this circle the collective series $F(x)$ evidently converges in the same manner as $\sum_{n=m}^{\infty} \epsilon_n x^n$. But the latter, being a power series, converges uniformly within any circle concentric with its circle of convergence and having a smaller radius. We infer therefore the equivalence of our two expressions under the following condition:

If $\sum_{n=m}^{\infty} \epsilon_n x^n$ *converges within a circle whose radius is equal to or less than the radius of* C, *the series* S *will within this circle be identical with* $F(x)$.

The inequalities (11) show that $\sum_{n=m}^{\infty} \epsilon_n x^n$ and S have a like connection with (12), and the former series, as well as S, must accordingly converge within the circle C. Hence *upon the hypothesis of a limiting form for our linear relation the function* $F(x)$ *and the series* S *are identical within the circle* C.

6. *The radius of convergence.*

One important inference can now be drawn from the form of (8). If $B_p(x)$ *is taken to denote the polynomial which corresponds to the linear relation containing the minimum number of coefficients of* S, *the roots of* $B_p(x)$ *will be, in general, singular points of the function defined by* S. *Hence the distance* ρ *from the origin to the nearest root (or roots) of* $B_p(x) = 0$ *is in general the radius of convergence.*

Attention should, however, be called to certain *exceptional cases*, in which the radius of convergence is greater than this distance. This may occur if one or more of the roots of $B_p(x)$ are zeros of the analytic function defined by $G(x) + \sum_{n=m}^{\infty} \epsilon_n x^n$. Such a root, at least if simple, is then no longer a singularity. That such exceptional cases do arise may be seen by multiplying together

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

and

$$\frac{1}{x-1} = \frac{1}{x} + \frac{1}{x^2} + \cdots.$$

The resulting LAURENT'S series is one in which for all values of n, positive or negative,

$$A_{n+1} = \left(1 - \frac{1}{n+1}\right) A_n + \frac{A_{n-1}}{n+1}.$$

Confining our attention to the part of the series which consists of the positive powers, we have a series which converges over the entire plane. Yet it is characterized by a limiting linear relation which corresponds to the polynomial $1 - x$.

7. *Series defining functions whose only singularities upon the circle of convergence are poles.*

From inspection of (8) it is evident that all the singularities of the function defined by our series S which lie upon its circle of convergence will be poles when $\sum_{n=m}^{\infty} \epsilon_n x^n$ has a radius of convergence greater than that of S. We shall show, conversely, that *if a series defines a function whose only singularities upon the circle of convergence are poles, the series is characterized by a limiting linear relation,* $\sum_{n=m}^{\infty} \epsilon_n x^n$ *having a radius of convergence greater than the distance from the origin to the nearest root of the polynomial which corresponds to the linear relation.*

To demonstrate the existence of a limiting linear relation, let the function be expressed as the sum of a rational fraction whose poles lie upon the circle of convergence and of a series $C_0 + C_1 x + C_2 x^2 + \cdots$ which converges beyond this circle. We will denote the denominator of the rational fraction by $B_p(x)$. Decompose the rational fractions into simple fractions $D_i(1 + x a_i^{-1})^{-r_i}$ and expand each simple fraction into a series in ascending powers of x. The $(n+1)$th coefficient will be

$$D_{in} = (-1)^n \frac{r_i(r_i+1)\cdots(r_i+n-1)D_i}{n!\, a_i^n}. \tag{14}$$

If now $r_i > r_j$, we obtain from (14)

$$\frac{|D_{in}|}{|D_{jn}|} \leqq \frac{|D_i|}{|D_j|} \cdot \frac{r_i + n - 1}{r_j}.$$

When, therefore, n is indefinitely increased, D_{jn} becomes negligible in comparison with D_{in}. We will accordingly resolve our rational fraction into two parts, the first of which is the sum of all simple fractions in which r_i has the maximum value, while the other part comprises the remaining fractions.

Let the expansions in series for the two parts be $\sum C'_n x^n$ and $\sum C''_n x^n$. If the former part consists of k simple fractions, we have

$$C'_n = D_{1n} + D_{2n} + \cdots + D_{kn}, \tag{15}$$

a like equation holding for C''_n. As n increases C''_n becomes infinitesimal in comparison with each of the k components of C'_n. The same is true of $C''_{n-i}(i = 1, 2, \cdots, k)$, inasmuch as the ratio between D_{jn} and $D_{j,\, n-i}$, any two corresponding components of C''_n and C''_{n-i}, approaches a finite limit. We will show also that the same quantities become infinitesimal not simply in comparison with the components of the corresponding coefficients C'_{n-i} but also in comparison with at least one of these coefficients.

If possible, suppose the contrary to hold, and consider the system of simultaneous equations obtained by substituting $n-1, n-2, \cdots, n-k$ for n in (15). When n is indefinitely increased, the left hand member of each equation should become an infinitesimal part of any term on the other side of the equation. Introducing into the system the values of the D_{in} furnished by (14) we have

$$\delta_i = a_1^i(a_1^{-n}D_1) + a_2^i(a_2^{-n}D_2) + \cdots + a_k^i(a_k^{-n}D_k), \qquad (i=1, 2, \cdots k),$$

in which the δ_i designate the negligible left hand members. Now the determinant of the coefficients of the quantities $a_i^{-n}D_i$ becomes, after the removal of the factor $a_1a_2\cdots a_k$, a well known expression for the product of the differences of the a_i. Since this product does not vanish, the system of equations can be satisfied only by supposing that each of the quantities $a_i^{-n}D_i$ is an infinitesimal part of itself—that is, only if $D_1 = D_2 = \cdots = D_k = 0$. Our initial hypothesis was therefore untenable. Some one (or more) of the coefficients $C'_{n-i}(i = 1, 2, \cdots, k)$ must remain of the same order of magnitude as $D_{1,n-i}$, and in comparison with it the k quantities C''_{n-i} may be neglected.

A glance at (14) shows that this same coefficient is at least as great as $\gamma|a_1|^{-n}$, in which γ designates an appropriate finite number independent of n. Since the radius of convergence of $C_0 + C_1x + C_2x^2 + \cdots$ is greater than $|a_1|$, we have also $|C_n| < \gamma'|a'|^{-n}$, in which $|a'| > |a_1|$. It follows that the k coefficients C_{n-i} are likewise infinitesimal in comparison with the above coefficient.

The existence of a limiting linear relation between the coefficients of the series representing our function can now be quickly established. For if $B_{p'}(x)$ denotes the product of the denominators of the k simple fractions whose exponents had the greatest value, we have

$$C'_n = b'_1C'_{n-1} + b'_2C'_{n-2} + \cdots + b'_{p'}C'_{n-p'}.$$

By virtue of this relation the general coefficient $C_n + C'_n + C''_n$ in the series for the function—call it A_n—can be thrown into the form:

$$A_n = b'_1A_{n-1} + b'_2A_{n-2} + \cdots + b'_{p'}A_{n-p'} - (b'_1C''_{n-1} + b'_2C''_{n-2} + \cdots + b'_{p'}C''_{n-p'})$$
$$- (b'_1C_{n-1} + b'_2C_{n-2} + \cdots + b'_{p'}C_{n-p'}) + C''_n + C_n.$$

But, by the preceding paragraphs, when n increases, C_n, C''_n, and the two parentheses must become an infinitesimal part of some of the preceding terms. There is therefore a linear relation which corresponds to the polynomial $B_{p'}(x)$. The same must also be true of $B_p(x)$ since it contains $B_{p'}(x)$ as a factor.

It has thus been shown that the existence of a limiting linear relation for which $\sum_{m=n}^{\infty}\epsilon_n x^n$ has a greater radius of convergence than S is a necessary as well as a sufficient condition that all the singularities on the circle of convergence should be poles. The positions of the poles and their orders are determined by the polynomial $B_p(x)$ which corresponds to the lowest linear relation of this character. It is, however, to be emphasized that *unless all the poles are of the first order, this will not be the limiting linear relation containing the smallest number of terms. The latter is dependent only upon the poles of highest order.*

8. *Hadamard's conditions for polar singularities upon the circle of convergence.*

The necessary and sufficient conditions that there should be p poles upon the circle of convergence of a given series $A_0 + A_1 x + A_2 x^2 + \cdots$, and no other singularities, have been previously studied* by HADAMARD and were thrown by him into the following very interesting form:

If ρ is the radius of convergence of the series and

$$\Delta_{n,l} = \begin{vmatrix} A_n & A_{n+1} & \cdots & A_{n+l} \\ A_{n+1} & A_{n+2} & \cdots & A_{n+l+1} \\ \cdots & \cdots & \cdots & \cdots \\ A_{n+l} & A_{n+l+1} & \cdots & A_{n+2l} \end{vmatrix},$$

then the greatest affix of a point of condensation of the sequence

$$|\Delta_{n,l}|^{1/n} \qquad (n=1, 2, \cdots)$$

must be equal to

$$\text{I)} \qquad 1/\rho^{l+1} \quad (l<p),$$

$$\text{II)} \qquad 1/\rho^{p}\rho' \quad (l=p),$$

in which ρ' is a number greater than ρ.

The second of these conditions can easily be shown to be a necessary one with the aid of our linear relation. For this purpose subtract from the last column of $\Delta_{n,p}$ the preceding column multiplied into b_1, the second preceding column multiplied into b_2 and so on. The constituents of the last column then become the quantities ϵ_{n+p+i} $(i = 0, 1, \cdots, p)$. They must therefore be less than $\gamma'/(\rho')^{n+p+i}$ while each constituent of any preceding column is less than γ/ρ^i. The modulus of $\Delta_{n,p}$ must accordingly be less than $\Gamma/\rho^{np}(\rho')^p$, in which Γ denotes an appropriate number independent of n. The desired conclusion now immediately follows. Moreover $|\Delta_{n,l}|$ is less than $\Gamma/\rho^{(n+l)}$ for any value of l.

* HADAMARD, Liouville's Journal, ser. 4, vol. 8, p. 119.

The greatest affix of a point of condensation of the above sequence can therefore never exceed $1/\rho^{l+1}$.

To demonstrate that the conditions I and II together are sufficient, HADAMARD proves successively the following three points:

1) When n increases $|\Delta_{n,p-1}|^{1/n}$ converges toward ρ^{-p}.

2) Simultaneously the solutions of the system of equations

$$(16) \qquad A_{n+p+i} + x_1^{(n)} A_{n+p+i-1} + \cdots + x_p^{(n)} A_{n+i} = 0 \qquad (i = 0, 1 \cdots, p-1)$$

approach definite limits,* these limits being the negatives of the coefficients b_i of $B_p(x)$.

$$3) \qquad \epsilon_{n+p} \equiv A_{n+p} - b_1 A_{n+p-1} - \cdots - b_n A_n < \left(\frac{1+\epsilon}{\rho'} \right)^n,$$

in which ϵ decreases indefinitely as n increases.

The last point shows that the singularities can be removed by multiplying the series with $B_p(x)$, and thus that they are poles.

HADAMARD has no occasion to explicitly note the existence of a limiting linear relation of the character here considered. This is, however, implied in 2).

As regards the significance of the above three points, the first is used by HADAMARD as a lemma by which to establish the second under the given conditions. Logically, however, the second always carries with it the first, though the converse does not seem to be true. If, in fact, $|x_p^{(n)}|$ or $|\Delta_{n+1,p-1}|/|\Delta_{n,p-1}|$ approaches the limit $1/\rho^p$, we have

$$\frac{(1-\epsilon)^j}{\rho^{pj}} < \frac{|\Delta_{n+j,p-1}|}{|\Delta_{n,p-1}|} < \frac{(1+\epsilon)^j}{\rho^{pj}}.$$

Hence the limit of $|\Delta_{n+j,p-1}|^{1/n+j}$ for $j = \infty$ is $1/\rho^p$.—The third point carries with it the two preceding.

It should be expressly noted that *no one of the three points is necessary that there should be a limiting relation.* The second, however, suffices, and this fact suggests one method of determining the existence of a linear relation for a given series, namely, by an examination of the determinant ratios which are the solution of (16) to see whether for any value of p they converge toward definite limits.

9. *The linear relation with an infinite number of terms.*

The theory developed in the preceding sections can be extended in a restricted form to series whose coefficients satisfy a linear relation which tends to take the form:

$$(17) \qquad A_n = b_1 A_{n-1} + b_2 A_{n-2} + b_3 A_{n-3} + \cdots,$$

* HADAMARD'S notation has been varied to bring it into accord with that used here.

the number of terms in the relation increasing indefinitely with n. In place of $B_p(x)$ an infinite series $B(x) = 1 - b_1x - b_2x^2 \cdots$ must then be introduced. Let, for convenience, the linear relation be written thus:

$$A_n = (b_1A_{n-1} + b_2A_{n-2} + \cdots + b_nA_0) + \epsilon_n ,$$

in which

$$\epsilon_n = \epsilon_1^{(n)}c_1A_{n-1} + \epsilon_2^{(n)}c_2A_{n-2} + \cdots + \epsilon_n^{(n)}c_nA_0 .$$

Suppose now that the quantities $\epsilon_i^{(n)}$ and c_i satisfy the following conditions:

(1) For any arbitrarily assigned positive quantity ϵ an integer m can be found such that for $n \geqq m$ every $|\epsilon_i^{(n)}|$ is less than ϵ.

(2) The series $\sum_{i=1}^{\infty} c_ix^i$ converges at least within the circle of convergence of

$$P(x) \equiv \frac{1}{B(x)} .$$

Should every ϵ_n $(n \geqq m)$ be neglected, the coefficients after the mth term would be formed in accordance with the law (17), and the series would be the expansion of some fraction $G(x)/B(x)$, in which $G(x)$ denotes a polynomial. By collecting, as in § 2, the parts of the successive terms of the series which explicitly contain ϵ_n $(n = m, m+1, \cdots)$, we obtain the function,

$$\frac{G(x)}{B(x)} + \frac{\sum_{n=m}^{\infty} \epsilon_n x^n}{B(x)}$$

in place of (8). Let γ and ξ have meanings similar to those in which they were before employed. Then

$$|A_0| < \gamma, \quad |A_1| < \gamma\xi^{-1}, \quad \cdots, \quad |A_{m-1}| < \gamma\xi^{-(m-1)},$$

and

$$|\epsilon_m| < \epsilon\gamma\left(\frac{|c_1|}{\xi^{m-1}} + \frac{|c_2|}{\xi^{m-2}} + \cdots + \frac{|c_{m-1}|}{\xi} + |c_m|\right) = \frac{\epsilon\gamma X'}{\xi^m} ,$$

where

$$X' = |c_1|\,\xi + |c_2|\,\xi^2 + \cdots + |c_m|\,\xi^m .$$

As ξ lies within the circle of convergence of $\sum_{i=1}^{\infty} c_ix^i$, the sum of the series:

$$|c_1|\,\xi + |c_2|\,\xi^2 + |c_3|\,\xi^3 + \cdots$$

must have a finite value, say V. Then if $X = \epsilon\gamma V$, we have

$$|\epsilon_m| < X\xi^{-m} .$$

With this value of X the argument of § 3 may now be repeated, and our series may thus be proved to converge within a circle having the origin as its center and a radius equal to the distance from the origin to the nearest singular point of $1/B(x)$.

In case condition (2) is not fulfilled, ξ must be taken smaller than the radius of convergence of $\sum_{i=1}^{\infty} c_i x^i$ in order that the above considerations shall apply, and the series $A_0 + A_1 x + A_2 x^2 + \cdots$ is shown merely to converge within this circle. No difficulty will be experienced in extending the same reasoning also to relations of the form:

$$(18) \qquad A_n = \left(b_1 + \sum_{i=1}^{i=k} \epsilon_{1i} c_{1i}\right) A_{n-1} + \left(b_2 + \sum_{i=1}^{i=k} \epsilon_{2i} c_{2i}\right) A_{n-2} + \cdots,$$

the quantity V being replaced by $\sum_{i=1}^{i=k} V_i$. The result which will thus be reached may be summed up as follows:

II. *Let a series* $S = A_0 + A_1 x + A_2 x^2 + \cdots$ *be given such that, after some fixed term, each coefficient is connected with the preceding coefficients by a linear relation* (18) *in which* $b_1, b_2, \cdots$ *are the coefficients of a convergent series* $\sum_{n=1}^{\infty} b_n x^n$. *If*

(1) *for any arbitrarily assigned positive quantity* ϵ *an integer* m *can be found such that every* $|\epsilon_{ni}|$ *for* $n \geqq m$ *will be less than* ϵ, *and*

(2) *the* k *series* $\sum_{n=1}^{\infty} c_{ni} x^n$, $(i = 1, 2, \cdots, k)$, *converge within a circle whose radius is the distance from the origin to the nearest singular point of*

$$\frac{1}{1 - b_1 x - b_2 x^2 - \cdots},$$

then the series S *converges within this circle. In case only condition* (1) *is fulfilled* S *converges within the region of convergence common to the* k *series.*

10. *Series in two or more variables.*

Criteria similar to I and II can be derived for the convergence of power series containing two or more variables. This will be shown here only for the case of two variables, inasmuch as the theory is in every respect parallel for the case of more than two variables. Let

$$B_p(x, y) = 1 - \sum b_{ij} x^i y^j, \qquad (0 < i + j \leqq p).$$

If the reciprocal of this polynomial be expanded into a series

$$(19) \qquad 1 + A_{10} x + A_{01} y + A_{20} x^2 + A_{11} xy + A_{02} y^2 + \cdots,$$

the coefficients of the terms of any given degree are obtained from the coefficients of the terms of lower degree by means of the recurring relation:

$$A_{ij} = \sum b_{qr} A_{i-q,\, j-r}\,, \qquad (0 < q + r \leqq p).^*$$

If in any series such a relation holds after some fixed term, the series is the expansion of some rational function which has $B_p(x, y)$ for its denominator.

Consider now a series S of the form (19) in which every coefficient A_{ij} whose index sum $i + j$ is equal to or exceeds some fixed integer M is derived from the coefficients having a smaller index sum by a relation of the form:

$$A_{ij} = \sum b_{qr} A_{i-q,\, j-r} + \epsilon_{ij}\,, \qquad (0 < q + r \leqq p),$$

where

(20) $$\epsilon_{ij} = \sum \epsilon_{qr}^{(ij)} A_{i-q,\, j-r}\,.$$

Let $G(x, y)/B_p(x, y)$ represent the rational fraction obtained by omitting every ϵ_{ij} for which $i + j$ exceeds some fixed integer M. Then by reasoning similar to that of § 2 it can be shown that the coefficient of any term $x^i y^j$ in S can be obtained by summing the corresponding coefficients in the various power series which represent the separate members of the function:

(21) $$F(x, y) = \frac{G(x, y)}{B_p(x, y)} + \sum_i \sum_j \frac{\epsilon_{ij} x^i y^j}{B_p(x, y)}\,, \qquad (i + j \geqq M).$$

Denote with R and R' the radii of any two circles in the x- and y-planes respectively which together constitute a domain of convergence for the series expanding $G(x, y)/B_p(x, y)$ and $1/B_p(x, y)$. Then if ξ and η are any positive quantities smaller than R and R' respectively, the coefficient of $x^i y^j$ in either series will be less than $\gamma/\xi^i \eta^j$, where γ is some fixed number independent of i and j. We have then

$$|A_{ij}| < \frac{\gamma}{\xi^i \eta^j}, \qquad (i + j < M).$$

Suppose now that for any arbitrarily small ϵ a value can be found for M such that for $i + j \geqq M$, every $|\epsilon_{qr}^{(ij)}|$ will be less than ϵ. It will follow then from (20) that for every combination of values for i and j whose sum is M we have

$$|\epsilon_{ij}| < \frac{X}{\xi^i \eta^j}$$

where

$$X = \epsilon\gamma(\xi + \eta + \xi^2 + \xi\eta + \eta^2 + \cdots + \eta^p)\,,$$

and therefore also

$$|A_{ij}| < \frac{\gamma(1 + X)}{\xi^i \eta^j}\,.$$

Taking for i and j all combinations of values whose sum successively is M, $M + 1$, $M + 2$, $\cdots$, we obtain finally

* No negative subscript is to be admitted either here or elsewhere.

$$|\epsilon_{ij}| < \frac{X(1+X)^{i+j-M}}{\xi^i \eta^j}, \qquad (i+j \geqq M)$$
$$|A_{ij}| < \frac{\gamma(1+X)^{i+j-M+1}}{\xi^i \eta^j}.$$

The last inequalities show that S converges with

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(1+X)^{i+j} x^i y^j}{\xi^i \eta^j}.$$

Now this double series is the expansion for

$$\frac{1}{\left(1 - \frac{x(1+X)}{\xi}\right)\left(1 - \frac{y(1+X)}{\eta}\right)}$$

and converges for

$$|x| < \frac{\xi}{1+X}, \qquad |y| < \frac{\eta}{1+X}.$$

But ϵ and with it X can be made less than any arbitrarily small positive quantity by taking M sufficiently large. We have therefore the following result:

III. *Let a series*

$$S = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} A_{ij} x^i y^j$$

be given in which every coefficient A_{ij} whose index sum $i+j$ exceeds a fixed number is obtained from the coefficients of smaller index sum by a relation of the form

$$A_{ij} = \sum_{q,\,r} (b_{qr} + \epsilon_{qr}^{(ij)})\, A_{i-q,\, j-r}, \qquad (0 < q + r \leqq p).$$

If for any assigned positive value ϵ, however small, an integer M can be found such that for $i+j \geqq M$ every $|\epsilon_{pq}^{(ij)}|$ is less than ϵ, the series will converge at every point within the domain of convergence of $P(x, y) \equiv 1/B_p(x, y)$.

Upon the same hypothesis S can be proved to be identical with the function $F(x, y)$. The proof is based upon a generalization * of the theorem of WEIERSTRASS cited in § 5. The form of (21) shows that if $B_p(x, y)$ *is irreducible, the above mentioned domain is, in general, the domain of convergence of S*, although we have not excluded the possibility of exceptional cases in which there may be a greater domain. If $B_p(x)$ is reducible, a linear relation of lower order may correspond to one of its factors, from which a greater domain of convergence might be argued.

The domain of convergence of the series for $1/B_p(x, y)$ is fixed by the theorem of WEIERSTRASS concerning the reciprocal of a power series. We have

* BIERMANN, loc. cit., pp. 148–150.

only to seek for values (x', y') which satisfy the equation $B_p(x, y) = 0$ and make $|B_p(x, y)| > 0$ for $|x| < |x'|$ and $|y| < |y'|$. For definiteness and brevity the term *radius of convergence* * is sometimes applied to a power series in two variables, being used to designate the maximum quantity ρ such that the series converges if $|x| < \rho$ and $|y| < \rho$. Its value is, in general, determined by $B_p(x, y)$. Thus for the simple case in which $B_p(x, y) = 1 - b_1x - b_2y$, the radius of convergence is obviously $1/(|b_1| + |b_2|)$.

Theorem III permits of extension in the same manner as I, giving the following result:

IV. *Let a series*

$$S = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} A_{ij} x^i y^j$$

be given in which every coefficient having an index sum greater than some fixed number is obtained from the coefficients having a smaller index sum by a relation of the form:

$$A_{ij} = \sum_q \sum_r b_{i-q,\, j-r} A_{i-q,\, j-r} + \epsilon_{qr}^{(ij)}(c_1 m_{ij}^{(1)} + \cdots + c_p m_{ij}^{(p)}) A_{i-q,\, j-r}.$$

If for an arbitrarily assigned positive quantity ϵ an integer M can be found such that for $i + j \geqq M$ every $|\epsilon_{qr}^{(ij)}|$ is less than ϵ, the series will converge within the region of convergence common to the p series $\sum_i \sum_j m_{ij}^{(k)} x^i y^j$, $(k = 1, 2, \cdots, p)$ and to the series for the reciprocal of

$$1 - \sum \sum b_{ij} x^i y^j, \qquad (i + j > 0).$$

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* BIERMANN, loc. cit., p. 138.