SUNDRY METRIC THEOREMS CONCERNING N LINES IN A PLANE*

BY

FRANK H. LOUD

The point of departure for this paper is furnished by the article of Professor F. MORLEY † in the April number of the Transactions. For the convenience of the reader, the notation of that memoir has been as much as possible followed; but it will be perceived that even where, in the opening sections of the present essay, the consequent resemblance to portions of the former rises to the point of identity of formulæ, the geometric meaning which underlies these is quite distinct, while in the later portions of the article it has been necessary to find forms of statement unlike those suited to the preceding work.

In the last section of the article quoted (p. 184) its author points out that the problems to which that memoir is mainly devoted arise from an initial combination of n lines by pairs, while a grouping by threes, fours, or higher numbers is possible. The present paper is concerned with the case in which the lines are originally grouped in threes, and has for its basic element, analogous to the intersection of two lines as treated in the former article, the center of a circle tangent to the three. I shall briefly indicate a new series of theorems which thus arises from an altered interpretation of formulæ practically identical with those of Professor MORLEY, including analogues to his own theorems upon center-circles, as well as to the chain of propositions associated with the names of STEINER, MIQUEL, KANTOR, and CLIFFORD; I shall then develop a relation by which the new theorems are connected with those of the foregoing case; and finally I shall devote some space to the simpler aspects of the added multiplicity of forms resulting from that character by which the case here treated is chiefly distinguished from the preceding, to wit, the assignment to each line of a definite direction, the reversal of which in any instance, while leaving the original configuration of n lines apparently unaffected, entirely changes the diagram of circles built upon it.

§ 1. *The system of n directed lines in a plane.*

The locus of the centers of circles tangent to two lines, when these are considered as each described indifferently in either of two opposite directions, con-

* Presented to the Society April 28, 1900. Received for publication April 28, 1900.

† *On the Metric Geometry of the Plane n-Line*, Transactions of the American Mathematical Society, vol. 1, no. 2.

sists of the two axes of symmetry, or bisectors of the four angles at the intersection; but when we assign to each line a definite direction to the exclusion of the opposite, and require that a tangent circle shall be so situated that a point moving continuously around the latter shall, at the point of contact with a line, move in the direction in which that line is described, then the locus of the centers of such circles is reduced to a single axis. This is *equidistant* from the given lines, and will be distinguished by that property from its companion axis (to which alone, on the other hand, the name "bisector of the angle" is appropriate), if we consent to regard the distances of a point of the former from the two given lines as agreeing in sign, while those of a point of the latter are opposite to one another. Given three directed lines, the three axes, thus equidistant respectively from the lines of three pairs, meet in a point, the center of the one circle tangent to the lines. Though this may happen to be the circle inscribed in the triangle, or one of the escribed circles, its center is the sole point *equidistant* from the three directed lines. Given four directed lines, it is an elementary theorem—included, as will be seen, under a more general proposition presently to be stated—that the quadrangle formed by the four points each equidistant from three of the lines is inscriptible in a circle.

In expressing algebraically these elementary relations, I follow Professor MORLEY in using circular coördinates * x and y, defined by the equations

$$x = \mathrm{x} + i\mathrm{y},$$

$$y = \mathrm{x} - i\mathrm{y},$$

(where x and y are the ordinary rectangular coördinates) but it becomes convenient in this case to use as the standard equation of the right line a form different from that adapted to the requirements of his article. In place of

$$xt_1 + y = x_1 t_1$$

I employ

$$xt_1 - 2r_1 + \frac{y}{t_1} = 0;$$

so that, while the convention is retained that t_1 shall denote a complex quantity of absolute value 1, the t_1 of this article is t_1^2 in the notation of the other, while r_1 is a real quantity, equal to the length of the perpendicular from the origin to the line.

Then

$$\left(xt_1 - 2r_1 + \frac{y}{t_1}\right) - \left(xt_2 - 2r_2 + \frac{y}{t_2}\right) = 0$$

represents the axis equidistant from two given lines; and elimination between two such equations gives

* SALMON: *Higher Plane Curves*, p. 7.

$$x = \frac{2r_1t_1}{(t_1 - t_2)(t_1 - t_3)} + \frac{2r_2t_2}{(t_2 - t_1)(t_2 - t_3)} + \frac{2r_3t_3}{(t_3 - t_1)(t_3 - t_2)}$$

as the point equidistant from three lines, each determinate in direction.

We have here a sum of fractions closely resembling those formerly denoted * by the symbols a_1, etc. Accordingly, let us put

$$a_\alpha = \sum \frac{2r_1t_1^{\,n-\alpha-1}}{(t_1 - t_2)\cdots(t_1 - t_n)} \qquad (\alpha = 1, 2, \cdots, n),$$

noting, for later use, that the conjugate of a_α is

$$(-1)^{n-1}Ta_{n-\alpha},$$

where

$$T = t_1t_2\cdots t_n,$$

and that a_α for $n = m$ is given in terms of the a's for $n = m + 1$ and t_{m+1} by the formula:

$$a_\alpha - a_{\alpha+1}t_{m+1}.$$

Then the position of the equidistant center of three lines is given by

$$x = a_1 \qquad (n = 3),$$

while the equation:

$$x = a_1 - a_2t \qquad (n = 4),$$

represents the circumcircle of the inscriptible quadrangle defined by four given directed lines. Its center and radius are

$$a_1 \text{ and } |a_2| \qquad (n = 4).$$

It is immediately evident, on writing out the formulæ, that the algebraic argument is here simply a repetition of that used by Professor MORLEY in the article quoted. I therefore omit it altogether, merely stating the theorems into which the equations are translated under the modified code of interpretation.

I. *Let there be given five right lines, each of specified direction. Any four of these determine a quadrangle (viz., the vertices of the latter are points in which the equidistant axes of pairs of lines, taken from the given four, converge by threes) which quadrangle is inscriptible in a circle* C_4, *and the five circumcircles* C_4 *have their centers on one new circle* C_5. *Six given directed lines, taken by fives, lead to six such circles* C_5, *whose centers lie on a new circle* C_6*; and so on indefinitely.*

II. *Given five directed right lines, the five circles* C_4 *of Theorem* I *meet in a point* N_5*; and in general, given n directed right lines, the n circles* C_{n-1} *meet in a point* N_n.

* loc. cit., pp. 99, 100.

III. *Given six directed right lines, the six points* N_5 *of Theorem* II *lie on a circle* O_6; *given seven such lines, the seven circles* O_6 *meet in a point* P_7; *given eight such lines, the eight points* P_7 *lie on a circle* O_8; *and so on in an infinite alternation of points and circles.*

The point P_{2p+1} is given by

$$\begin{vmatrix} a_1 - x & a_2 & \cdots & a_p \\ a_2 & a_3 & \cdots & a_{p+1} \\ \vdots & \vdots & & \vdots \\ a_p & a_{p+1} & \cdots & a_{2p-1} \end{vmatrix} = 0 ,$$

and the equation of the circle O_{2p+2} is

$$\begin{vmatrix} a_1 - x & a_2 & \cdots & a_p \\ a_2 & a_3 & \cdots & a_{p+1} \\ \vdots & \vdots & & \vdots \\ a_p & a_{p+1} & \cdots & a_{2p-1} \end{vmatrix} = t \begin{vmatrix} a_2 & a_3 & \cdots & a_{p+1} \\ a_3 & a_4 & \cdots & a_{p+2} \\ \vdots & \vdots & & \vdots \\ a_{p+1} & a_{p+2} & \cdots & a_{2p} \end{vmatrix} ,$$

but this circle will in the general case be replaced by a right line on condition that

$$\begin{vmatrix} a_3 & a_4 & \cdots & a_{p+1} \\ a_4 & a_5 & \cdots & a_{p+2} \\ \vdots & \vdots & & \vdots \\ a_{p+1} & a_{p+2} & \cdots & a_{2p-1} \end{vmatrix} = 0 .$$

The algebra leading to these theorems may be found in the article of Professor MORLEY in the following sections: for Theorem I, § 2; for II, § 3; for III, § 4.

[The condition that O_6 be a right line, viz.,

$$a_3 = 0 ,$$

is readily developed into the geometric statement that the six given lines must be tangents to a certain curve, parallel to the hypocycloid of fourth class (see its equation in SALMON, *Higher Plane Curves*, § 117, ex. 2), and it is thus shown that what might be inferred from the concluding section of Professor MORLEY'S article to be a *sufficient* condition for such a breaking-up is also a *necessary* one. When the given lines are tangents to such a curve, the equations leading to the foregoing theorems assume a high degree of simplicity, and several subordinate theorems are inferred without difficulty; the results, however, do not seem of sufficient interest to warrant an extended discussion.]

It will be observed that the number of lines, points, or circles named in any one of the foregoing theorems is greater by one than in the corresponding theorem of the preceding article. When we take account of the lines, etc., that are involved in the steps anterior to that which the theorem directly states, these numbers are still further increased.

For instance, that part of Theorem III which forms an analogue to the theorem of MIQUEL would, if stated independently, read as follows:

Given six directed lines, they form fifteen pairs each having an equidistant axis; these axes converge by threes at twenty points, the vertices of fifteen inscriptible quadrangles, whose circumcircles meet by fives in six points, and these points lie on a circle O_6.

§ 2. *Inter-relation between the two series of theorems.*

It is shown in § 5 of the paper above cited that the MIQUEL diagram may be constructed in reverse order, beginning with the circle, on which five points are selected arbitrarily, and ending with the five lines assumed in the usual form of the theorem. This property also has its extension to the present case. To show this, it is necessary to find a set of six lines to which those deduced as above shall be equidistant axes; or, algebraically, to write a self-conjugate equation symmetrical in all but one of six t's, and having a turn for the coefficient of x, in such a form that on the subtraction of another equation, like it save for the particular t in which it is asymmetric, the result shall be the equation (13) of the memoir cited, viz.:

$$x + ys_4 = as_1 + \beta s_2 + as_3 .$$

The terms in x and y of such an equation may be constructed from the description, and it is then found possible to complete the equation. To represent the result, let A stand for the function

$$\frac{i}{t_6}\left(\frac{t_2t_3t_4t_5}{t_1} + \cdots + \frac{t_1t_2t_3t_4}{t_5}\right),$$

A' for the conjugate of the same, and B for the function produced when in the symmetrical expression

$$\tfrac{1}{2}\left(\frac{t_1t_2t_3}{t_4t_5t_6} + \frac{t_1t_2t_4}{t_3t_5t_6} + \cdots + \frac{t_4t_5t_6}{t_1t_2t_3}\right),$$

every term is multiplied either by i or by $1/i$ according as $1/t_6$ or t_6 is already a factor of the term. The desired equation will then be

$$\frac{t_1t_2t_3t_4t_5 i}{t_6}x + \frac{t_6}{t_1t_2t_3t_4t_5 i}y - Aa - B\beta - A'a = 0 .$$

From this subtract a like equation in which t_5 is treated as is t_6 in this; the result divided by $(t_5^2 - t_6^2)i/(t_5 t_6)$ is a form in which it is necessary to replace $1/t_1^2$, $1/t_2^2$, etc., by t_1, t_2, etc., and then the equation (13) above mentioned is obtained. Hence the theorem:

IV. *If a circle on which six points have been selected is made, by the construction quoted, the Miquel circle of six sets of lines, corresponding to the six ways of taking the given points by fives, the fifteen lines thus obtained will be equidistant axes to a new set of six lines.*

This theorem points directly to the inference that when we begin with six given directed lines, it is possible in six different ways to select five from among the fifteen equidistant axes belonging to the given lines in pairs, so that the Miquel circle of these five axes shall be identical with the circle O of the six given lines.

An examination proves that this inference is true, and that the five axes to be selected in any case are those which arise from pairing a given line with each of the others in succession. For let us assume one given line L and n others in a plane with it, these n being, like L, each determinate in direction; there are then n axes, each equidistant from L and one of the given lines. First, let $n = 3$; it is then known from Theorem I, or it can be proved in a variety of ways, that the circumcircle of the three axes contains also the point equidistant from the three lines. Now add a fourth line, and in consequence a fourth equidistant axis; there are four such circumcircles, meeting (as is known) in a point, and also (as shown by STEINER) having their centers on a circle. These four are circumcircles, not only of triangles formed by the axes, but of quadrangles formed by the given lines in conjunction with L. There is a fifth circumcircle, belonging to the quadrangle derived from the four lines without L, but this fifth circle would have been grouped with three of the others had a different line been chosen to play the part of L; and, since three are sufficient to determine both the common point of their circumferences and the circle through their centers, it is seen that the theorems I, II concerning the circumcircles of quadrangles are a necessary consequence of the known theorems on the circumcircles of triangles. When n is increased to 5 and upward, the same considerations apply; whence we are entitled to state the following theorem:

V. *If n lines, each fixed in direction, be given, and their equidistant axes be grouped in sets, those forming one set which are equidistant from a selected line and the other $n - 1$ lines severally (so that, each axis being counted in two sets, the $\frac{1}{2}n(n-1)$ axes form n sets of $n - 1$ axes apiece), then the centercircles of the n sets coincide in one circle C_n', the nodes of the n sets in one point N_n, and (if n be odd) their Clifford points in one point P_n or (if n be even) their Clifford circles in one circle O_n.*

Although this theorem seems to merge the subject-matter of the present article in that of the preceding, rendering a separate examination unnecessary, save as an amplification of the field already explored, this is true only as regards the class of properties thus far discussed. I turn, accordingly, to other aspects which require an independent treatment; namely, those depending upon the effect of reversing the direction in which a given line has been assumed to be described.

§ 3. *Reversal of the direction of the given lines.*

When the second line of the pair:

$$xt_1 - 2r_1 + \frac{y}{t_1} = 0\,, \qquad xt_2 - 2r_2 + \frac{y}{t_2} = 0$$

is reversed in direction, the corresponding parenthesis in the equation of their equidistant axis changes sign from $-$ to $+$, producing

$$\left(xt_1 - 2r_1 + \frac{y}{t_1}\right) + \left(xt_2 - 2r_2 + \frac{y}{t_2}\right) = 0\,;$$

whence the effect of the reversal will be expressed in the resulting formulæ by change of sign of r_2 and t_2. It will be convenient to assume in the case of n lines having one line reversed, that this line is the n-th; and the modified values of the constants, obtained by changing r_n and t_n to $-r_n$ and $-t_n$, will be denoted by the mark $\hat{}$, thus, $\hat{a}_\alpha$. Then the relations of these constants to those appertaining to the original directions of the lines is given by

$$a_\alpha - a_{\alpha+1}t_n = \hat{a}_\alpha + \hat{a}_{\alpha+1}t_n\,;$$

each member of the equation representing the value of a_α for the $n-1$ unreversed lines.

Two circles C_n (see Theorem I) derived on the respective suppositions that the n-th line is not reversed, and that it is reversed, have the equations:

$$x = a_1 - a_2 t\,, \qquad x = \hat{a}_1 + \hat{a}_2 t\,.$$

These coincide at the point which, while the number of lines was only $n-1$, was designated as a_1, now called indifferently

$$a_1 - a_2 t_n \qquad \text{or} \qquad \hat{a}_1 + \hat{a}_2 t_n\,.$$

Their angle at this intersection is that of the strokes extending from their respective centers; hence it is the amplitude of

$$\frac{-a_2 t_n}{\hat{a}_2 t_n} = -\frac{a_2}{\hat{a}_2}\,.$$

To obtain this amplitude separate from the ratio of absolute values, we divide this expression by its conjugate, and take the square root of the result. Now the conjugate of a_2 is $(-1)^{n-1}_{-} Ta_{n-2}$, but that of $\hat{a}_2$ is $(-1)^{n-2}T\hat{a}_{n-2}$; whence that of $-a_2/\hat{a}_2$ is $a_{n-2}/\hat{a}_{n-2}$; and the angle of the circles is the amplitude of

$$\sqrt{\frac{-a_2\hat{a}_{n-2}}{a_{n-2}\hat{a}_2}}.$$

This affords no simple result except in the case $n = 4$, when the amplitude derived is that of i, viz., $\frac{1}{2}\pi$; and we learn that *two such circles cut orthogonally.* Now the circle C_4 is not only the first member of the series of circles C_n, but stands in the same relation to the circles O_n, viz., O_6, O_8, etc., and the inquiry is suggested whether the property of orthogonal section under the given conditions, though not extending beyond the first term of the former series, may not prove to belong to subsequent terms of the series O. This is found to be the fact, as follows:

The equation of O_{2p+2} is:

$$x = \frac{D_2}{D_1} - \frac{D_3}{D_1}t,$$

where D_1, D_2, D_3 are the determinants:

$$D_1 = \begin{vmatrix} a_3 & \cdots a_{p+1} \\ \vdots & \vdots \\ a_{p+1} & \cdots a_{2p-1} \end{vmatrix}, \qquad D_2 = \begin{vmatrix} a_1 \cdots a_p \\ \vdots \quad \vdots \\ a_p \cdots a_{2p-1} \end{vmatrix}, \qquad D_3 = \begin{vmatrix} a_2 & \cdots a_{p+1} \\ \vdots & \vdots \\ a_{p+1} & \cdots a_{2p} \end{vmatrix}.$$

When one line has been reversed, thus changing the signs of r_{2p+2} and t_{2p+2}, the equation of the new circle will be a similar formula which we may write

$$x = \frac{\Delta_2}{\Delta_1} + \frac{\Delta_3}{\Delta_1}t.$$

As in the case of the circles C_n, the two circles O have a common point (here one of the points P_{n-1}) which is fixed by the $n-1$ unreversed lines. The quantity whose amplitude furnishes the angle of section is the product of

$$\frac{-D_3\Delta_1}{D_1\Delta_3}$$

by the ratio of the values of t corresponding, in the two circles severally, to this common point. These values in the former case were alike (viz., t_4); here, though not equal to t_{2p+2}, they are again alike, being the product of t_{2p+2} by quantities dependent on the first $2p+1$ lines only.

To show this it is necessary to rehearse briefly some of the formulæ of § 4 of Professor MORLEY'S paper, and in quoting them a slight modification of their notation will adapt them to the present applications without interfering with reference to the original form. Let s_1, s_2, $\cdots$, s_m represent the sum, sum of products by twos, etc., of m quantities, τ', τ'', etc., upon which no restriction is at first placed save that the absolute value of each is unity. Then the equation

$$x = a_1 - a_2 t \qquad (n=4),$$

here representing a circle C_4, is one of five included in

$$x = a_1 - a_2 s_1 + a_3 s_2 \qquad \binom{n=5}{m=2},$$

in the sense that when one of the two τ's is identified with the constant t_4, the other with the variable t, the latter equation reduces to the preceding. A common point N_5 of these circles is obtained by showing that one τ may still be variable, though x be made constant, if the other τ satisfy two equations into which the one last written is in that case resolved, viz.:

$$\begin{aligned} x &= a_1 - a_2 \tau' \\ 0 &= a_2 - a_3 \tau' \end{aligned} \qquad (n=5).$$

These are in the same way included in

$$\begin{aligned} x &= a_1 - a_2 s_1 + a_3 s_2 \\ 0 &= a_2 - a_3 s_1 + a_4 s_2 \end{aligned} \qquad \binom{n=6}{m=2}.$$

For here, if τ'' be t_6, the equations become identical with the preceding pair, the latter of which shows the value which is thus imposed on τ', viz., a_2/a_3, $(n=5)$. The equation of O_6, viz.,

$$\begin{vmatrix} x - a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = t \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \qquad (n=6)$$

is a direct consequence of the last pair of equations, t being put for $s_2 = \tau' \cdot \tau''$ and therefore consisting, for the point at which this pair falls back into the preceding, of the product of t_6 by a_2/a_3 $(n=5)$.

This process is repeated indefinitely. The equations

$$\begin{aligned} x &= a_1 - a_2 s_1 + a_3 s_2 - a_4 s_3 \\ 0 &= a_2 - a_3 s_1 + a_4 s_2 - a_5 s_3 \end{aligned} \qquad \binom{n=7}{m=3},$$

into which the preceding pair is expanded, will again admit of the location of P_7, by variable τ together with a constant x, if they still hold true when broken up into the following three:

$$\begin{aligned} x &= a_1 - a_2 s_1 + a_3 s_2 \\ 0 &= a_2 - a_3 s_1 + a_4 s_2 \\ 0 &= a_3 - a_4 s_1 + a_5 s_2 \end{aligned} \qquad \binom{n=7}{m=2}$$

The second and third of these fix upon s_2, or $\tau'\tau''$, the value

$$\frac{\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}}{\begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix}} \qquad (n=7);$$

and thus determine that in the equation of O_8, where the variable t is $\tau'\tau''\tau'''$, the value for this variable which will designate the particular point P_7 must be the product of $\tau''' = t_8$ by that value of $\tau'\tau''$ which has just been written, and in which the constants, being all taken for $n = 7$, are independent of t_8. And so in general.

Returning now to the question of the angle at this common point, which was to be found as the amplitude of a quotient, we see that the factor derived from the t's is unity, whence the amplitude depends wholly on the remaining factor already given, viz.:

$$\frac{-D_3\Delta_1}{D_1\Delta_3}.$$

We proceed, as before, to divide by the conjugate. The conjugates of D_1, D_3, Δ_1, Δ_3 are respectively

$$[(-1)^{2p+1}T]^{p-1}D_1, \quad [(-1)^{2p+1}T]^{p}D_3, \quad [(-1)^{2p}T]^{p-1}\Delta_1, \quad [(-1)^{2p}T]^{p}\Delta_3.$$

The net result of the division is -1; whence the derived angle is the amplitude of $\sqrt{-1}$, viz., $\frac{1}{2}\pi$. We have accordingly the following theorem, which forms the basis of all that is herein obtained concerning the reversal of lines.

VI. *If one of $2p+2$ given lines be described successively in opposite directions, the directions of the other $2p+1$ remaining fixed, the two circles O_{2p+2} determined by them will cut each other orthogonally. As respects this property, the circle C_4 (where $p=1$), is to be regarded as belonging to the series O.*

A circle O_{2p+2} determined by $2p+2$ directed lines has $2p+2$ circles orthogonal to it, each obtained by reversing one given line; whence $(p+1)(2p+1)$ circles will be obtained by reversing two lines, etc. The sum of these numbers (the coefficients of the $2p+2$ power of a binomial) is 2^{2p+2}, but because the result of reversing a given number of the lines is the same as that of reversing the remaining lines without the former, this sum is to be halved; whence the effect of regarding all the lines as subject to reversal is to replace the single circle O_{2p+2} by 2^{2p+1} circles. These form two sets of 2^{2p} circles each, characterized by the property that all circles orthogonal to any circle of one set are found in the other set, because any given arrangements of directions can be reached from any other only by an odd or else only by an even number of reversals. For example, when six lines are given, and each is considered as reversible in direction, they form fifteen pairs, each having two equidistant axes; the meeting of each axis with any one of those sixteen others which are derived from line-pairs having one line in common with its own pair would give $(30 \times 16)/2 = 240$ points if all were distinct, but as they coincide in threes, actually eighty points, equidistant from three lines apiece. Each of these points is associated with the thirty-six others having two lines in common with it in $(80 \times 36)/(3! \times 4) = 120$ sets of four, lying upon as many circles C_4. The circles C_n meet by fives in points N_5; and of these, on a like principle of enumeration, there are ninety-six. Finally, those points N_5 lie on thirty-two circles O_6, which form two sets of sixteen, each circle of one set finding all of the six circles orthogonal to it in the set opposite to its own.

§ 4. *Four lines, each susceptible of reversal.*

The simplest case to which the foregoing principles can be applied—that, namely in which $p = 1$, or n, the number of given lines, is four—presents some features peculiar to itself. Here a circle C_4 is drawn through the four points which are equidistant from the given lines taken in threes. There are eight such circles when the lines are reversed; while the number of equidistant points is sixteen, since each belongs to two circles. The circles form two sets of four, every member of either set being orthogonal to every member of the other. It follows immediately that each set is a coaxial system, the pair of real points in which one set meets being inverse points in respect to any circle of the other set. An advantage is obtained if the four lines be regarded as tangents of one parabola, which evidently imposes no restriction upon their generality, but enables us to employ axes of coördinates in a determinate relation to the given lines.

The equation of a tangent to a parabola is very simply obtained from the property that the foot of a perpendicular from the focus lies on the tangent at the vertex. The focus being the origin and the vertex $x = \frac{1}{4}$, this equation is

$$t_1 x - \frac{t_1}{1 + t_1^2} + \frac{y}{t_1} = 0 .$$

The intersections of two such tangents is

$$x = \frac{1}{(1 + t_1^2)(1 + t_2^2)},$$

their equidistant axis is

$$t_1 t_2 x + y = \frac{t_1 t_2 (1 - t_1 t_2)}{(1 + t_1^2)(1 + t_2^2)},$$

and the point equidistant from three of them is

$$x = \frac{1 - t_2 t_3 - t_3 t_1 - t_1 t_2}{(1 + t_1^2)(1 + t_2^2)(1 + t_3^2)} .$$

From the last expression is derived immediately the equation of C_4, which may be written,

$$x = \frac{1}{P}[1 - \sigma_2 + \sigma_4 - (\sigma_1 - \sigma_3)t] ,$$

where

$$\sigma_1 = t_1 + t_2 + t_3 + t_4 , \quad \sigma_2 = \sum t_1 t_2 , \text{ etc.},$$

and

$$P = (1 + t_1^2)(1 + t_2^2)(1 + t_3^2)(1 + t_4^2) .$$

If we consider the expressions for any two opposite vertices of the quadrilateral formed by the given lines, for instance,

$$x = \frac{1}{(1 + t_1^2)(1 + t_2^2)} , \qquad x' = \frac{1}{(1 + t_3^2)(1 + t_4^2)} ,$$

we observe that they satisfy the equation :

$$xx' = \frac{1}{P} .$$

This equation defines a complex involution in the plane, three pairs of which are composed of the vertices just named. Another pair is formed by the points

$$x = \frac{1 - t_2 t_3 - t_3 t_1 - t_1 t_2}{(1 + t_1^2)(1 + t_2^2)(1 + t_3^2)} , \qquad x' = \frac{1}{(1 + t_4^2)(1 - t_2 t_3 - t_3 t_1 - t_1 t_2)} .$$

Of this pair, the former, as has just been noted, is one of the points through which the circle C_4 is passed; but it may now be shown that the latter also lies on that circle. By writing x' in place of x and solving the equation of C_4 for t, we obtain an expression, which will or will not be a value actually assumed by t, according as its absolute value is or is not equal to unity. In writing this expression it is convenient to put $s_1 = t_1 + t_2 + t_3$, $s_2 = t_2 t_3 + t_3 t_1 + t_1 t_2$, $s_3 = t_1 t_2 t_3$, and to retain σ_1, etc., in the sense already stated as expressions involving four variables. We then have

$$\left| \frac{s_1\sigma_1 - s_1\sigma_3 - s_3\sigma_1 + s_3\sigma_3}{\sigma_1 - s_2\sigma_1 - \sigma_3 + s_2\sigma_3} \right| = 1$$

as the condition that the circle C_4 contains the specified point. To verify this equation it is only necessary to replace s_1, s_2, etc., σ_1, σ_2, etc., by their conjugates, s_2/s_3, s_1/s_3, etc., σ_3/σ_4, σ_2/σ_4, etc., and it will be found that the fraction is changed into its reciprocal, which proves the proposition. It has been noted that two of the eight circles C_4 contain the point x; hence both contain x' as well, so that we can in future refer to the latter point as the *second intersection* of these circles. On the other hand, there are four points x on a circle C_4, and to define the partner circle of C_4 in the involution it is necessary to use no more than three points x' ; hence it is apparent that C_4 is its own partner, or a double circle of the involution.

But a double circle either contains the two double points of the involution or else they are inverse in respect to it; whence we come again upon the arrangement of the eight circles C_4 in two coaxial systems, and learn beside that the expression for the two points in which the four circles of one system meet is

$$x = \frac{1}{\sqrt{P}} = \frac{1}{\sqrt{(1 + t_1^2)(1 + t_2^2)(1 + t_3^2)(1 + t_4^2)}}.$$

The center of the involution, midway between these, is the origin, i. e., the focus of the tangent parabola, as is directly evident from the equation :

$$xx' = \frac{1}{P}.$$

By the aid of this equation all lines and circles of the plane are mapped into their partner circles. The twelve equidistant axes meeting by threes in points where the circles C_4 meet by pairs, are represented by as many circles, passing by threes through the second intersections of the same pairs of circles, passing also through the origin, and passing moreover by orthogonal pairs through vertices of the given quadrilateral opposite to those in which their partner axes cross at right angles.

The foregoing results relating to the general quadrilateral may be summed up as follows:

VII. *Given any four lines, there pass through their six intersections twelve lines (axes) each equidistant from two of the original four, and these twelve converge by threes in sixteen points. Through these points pass eight circles, four of which form a coaxial system through two real points, while the other four form a second coaxial system orthogonal to the first. The intersection of the two lines of centers is the focus of the parabola touching the four given lines. The sixteen points above mentioned lie by fours on these circles, each point having two circles through it, and the remaining intersections of the same circles constitute a second set of sixteen points. These lie by fours on twelve other circles, each point having three of these circles through it, while the twelve circles all pass through the focus of the parabola and meet orthogonally in pairs at the six intersections of the given lines.*

The case of four given lines is also susceptible of a different method of treatment leading to another configuration of circles. The proof is embraced in the general discussion in the next section; but the result for four lines only is considered worth stating separately, on account of its simplicity.

VIII. *If to three of any four given lines perpendiculars be drawn at the points where they respectively meet the fourth, the circumcircles of the two homolo-*

*gous triangles thus formed cut orthogonally, and the center of homology is one of the two points of section. The other point of section remains unchanged when a different line of the given four is used as an axis of homology, and is therefore common to all the circles of four orthogonal pairs.**

§ 5. *Any even number of lines, each reversible.*

The theorem VIII just stated for four lines may be generalized, and found to hold good for any higher even number. For this purpose the following lemma will be found serviceable.

LEMMA. *If a rectangle $ABCD$ have a vertex A at a fixed point, which is a $(p-1)$-fold point of a curve Q of p-th order, another vertex B upon the curve Q, a third C upon a fixed right line GH, which cuts Q at a point G and is at right angles with AG, then the locus of the fourth vertex D is a new curve of p-th order, having a $(p-1)$-fold point at A and an ordinary point at G.*

While circular coördinates might be employed in the proof of this proposition, they present no advantage over the ordinary Cartesian coördinates, and the demonstration is accordingly indicated in terms of the latter, designated as x and y. If A be taken as origin, and AG as axis of abscissas, the equation of Q may be written

$$a_0\mathrm{x}^p + pa_1\mathrm{x}^{p-1}\mathrm{y} + \tfrac{1}{2}p(p-1)\,a_2\mathrm{x}^{p-2}\mathrm{y}^2 + \cdots + pa_{p-1}\mathrm{x}\mathrm{y}^{p-1} + a_p\mathrm{y}^p$$
$$-\ [b_0\mathrm{x}^{p-1} + (p-1)b_1\mathrm{x}^{p-2}\mathrm{y} + \cdots + (p-1)b_{p-2}\mathrm{x}\mathrm{y}^{p-2} + b_{p-1}\mathrm{y}^{p-1}] = 0\,.$$

The fixed line GH and the moving line AB have respectively the equations $\mathrm{x} = b_0/a_0$ and $\mathrm{y} = m\mathrm{x}$.

The coördinates of B, the equation of BC, and coördinates of C are now readily formed, and the equation of CD written as parallel to AB. Then the locus of D follows by eliminating m between the last equation and that of AD, viz., $m = -\,\mathrm{x}/\mathrm{y}$.

It is

$$a_0(a_0\mathrm{y}^p - pa_1\mathrm{x}\mathrm{y}^{p-1} + \cdots \pm a_p\mathrm{x}^p)$$
$$+\ b_0\,[pa_1\mathrm{y}^{p-1} - \tfrac{1}{2}p(p-1)a_2\mathrm{x}\mathrm{y}^{p-2} + \cdots \mp a_p\mathrm{x}^{p-1}]$$
$$-\ a_0\mathrm{y}\,[(p-1)b_1\mathrm{y}^{p-2} - \tfrac{1}{2}(p-1)(p-2)\,b_2\mathrm{x}\mathrm{y}^{p-3} + \cdots \pm b_{p-1}\mathrm{x}^{p-2}] = 0\,.$$

The application to the present discussion comes through the theorem proved by CLIFFORD at the end of his memoir, *Synthetic Proof of Miquel's Theorem* (*Mathematical Papers*, p. 38), whence it appears that a curve such as Q is the

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* It can be proved that the four points, which successively become centers of homology when the given lines in turn are used as axes, are also situated upon a ninth circle through the common point of the foregoing eight; but the demonstration of this, not being included in the general discussion of the following section, is omitted for the sake of brevity.

pedal of a p-fold parabola with regard to its focus: so that the lines BC and CD above are tangents to two p-fold parabolas having in common a fixed tangent GH, upon which the moving tangents meet at right angles; and it is seen that two such parabolic curves have a common focus A.

Now let there be given $2p + 2$ lines, from which one line L is arbitrarily selected and its direction assigned. At the point in which L is met by K, any one of the remaining $2p + 1$, let us suppose that an auxiliary directed line is drawn, so that L and this auxiliary shall have K as their equidistant axis, a construction which obviously demands only the doubling of a given angle. Then the circle O_{2p+2} of the set made up from L and the $2p + 1$ auxiliary lines is the same as the Clifford circle of the remaining given lines [Theorem V]. To each of the latter let a perpendicular be drawn at the point at which it meets L. These perpendiculars make up a new set of equidistant axes resulting from the former set by the reversal of the line L; consequently that circle O_{2p+2} which is identical with the Clifford circle of this set meets the former circle O_{2p+2} at right angles [Theorem VI]. The point P_{2p+1} belonging to the auxiliary lines taken by themselves is the same for one circle O as for the other, since these lines are all unreversed; consequently it forms one of the points of section of these two circles. Further, it is proved by CLIFFORD, in the memoir cited, that the Clifford circle of the $2p + 1$ given lines, L omitted, passes through the focus of the p-fold parabola which touches all the given lines. For the same reason, the Clifford circle of the $2p + 1$ perpendiculars passes through the focus of a second p-fold parabola, touching the perpendiculars and L. But these two p-fold parabolas, as we have inferred from the lemma, have a common focus. This focus, then, is the second point of section of the two circles. (It cannot be identical with the first, since its position depends upon L, while that of the former, P_{2p+1}, did not.) If now all the given lines be made to assume successively the part hitherto assigned to L, we have $2p + 2$ pairs of circles O intersecting orthogonally, and one of their points of section, the focus of the p-fold parabola touched by all, is the same for all the pairs.

The foregoing argument applies to all values of p, but it may be remarked that in the case $p = 1$, the curve Q of the lemma becomes the general right line, not passing through A at all; when $p = 2$, Q is a conic and A an ordinary point of it; for higher values A is a multiple point as described.

The result now obtained accordingly includes Theorem VIII as a special case, and (dropping from the statement the auxiliary lines) may be summarized as follows:

IX. *If all but one of an even number of given lines be turned, each through a right angle, about the point at which it meets the one remaining line, and if the Clifford circles be constructed for these rotated lines in both their original position and that to which they are turned, these circles intersect orthogonally.*

By making each line in succession the one upon which the others are turned, as many orthogonal pairs of circles are obtained as there are given lines, and all these circles pass through a common point, which is the focus of the p-fold parabola tangent to all the given lines.

HAVERFORD COLLEGE,
April, 1900.