AN APPLICATION OF GROUP THEORY TO HYDRODYNAMICS*

BY

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It has been observed by SOPHUS LIE that the stationary motion of a fluid can serve as a perfect picture of a one-parameter group in three variables. So far as I know, neither he nor any of his followers utilized this fact for the purposes of hydrodynamics. It is the purpose of the present paper to do this. One of the advantages gained for hydrodynamics by this standpoint lies in the general conception. But another advantage is, as is always the case when a class of problems is investigated from a new standpoint, that from the group-theoretical point of view, certain special cases are of exceptional interest, simplicity, and importance, cases which otherwise would appear difficult and unpromising.

§1. *Relation between the steady motion of a fluid aud the theory of one-parameter groups.*

From the definition of the steady flow of a fluid, and LIE'S conception of a one-parameter group, the following theorems will be seen to be true.

1. *Let* x, y, z *be the Cartesian coördinates of any point of the fluid at the time* t, *and let* a, b, c *be the coördinates of the same material point at the time* $t = 0$. *If*

$$(1) \qquad x = \phi(a, b, c; t), \quad y = \psi(a, b, c; t), \quad z = \chi(a, b, c; t)$$

are the equations which describe the steady motion of any point of the fluid, they are also the finite equations of a one-parameter group, the parameter being t.

2. *Conversely, if equations* (1) *are the equations of a one-parameter group, they can also be interpreted as describing the steady flow of a fluid.*

We shall be mainly concerned with the second theorem. Let

$$(2) \qquad \mathrm{K}f = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

be the infinitesimal transformation of a one-parameter group G, where u, v, w are arbitrary functions of x, y, z. The finite equations of the group are found by integrating

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$$(3) \qquad \frac{dx}{dt} = u(x, y, z), \quad \frac{dy}{dt} = v(x, y, z), \quad \frac{dz}{dt} = w(x, y, z),$$

with the condition $x = a$, $y = b$, $z = c$ for $t = 0$, if a, b, c are the variables which are transformed into x, y, z by the operations of the group. Thus u, v, w are the values of the velocity-components at the point x, y, z.

Let X, Y, Z denote the components of the force acting upon the unit of mass, p the pressure, and ρ the density at the point x, y, z. Then EULER'S equations assume the form:

$$(4) \qquad \mathrm{K}u = X - \frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \mathrm{K}v = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \mathrm{K}w = Z - \frac{1}{\rho}\frac{\partial p}{\partial z},$$

and the equation of continuity becomes:

$$(4a) \qquad \mathrm{K}\rho + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0.$$

Moreover p and ρ are connected by an equation:

$$(4b) \qquad p = f(\rho),$$

where the nature of the function $f(\rho)$ depends upon the character of the fluid considered.

Suppose that u, v, w are arbitrarily assigned functions of x, y, z. Then by integrating (3) we get x, y, z as functions of a, b, c, and t. The equations (1), thus found, represent any one-parameter group G in space. We can find the physical conditions corresponding to every group G. First ρ may be found by integrating (4a). The simplest way to do this is to consider ρ as a function of a, b, c and t rather than of x, y, z. For then, since the motion is steady, i. e., since $\partial\rho/\partial t = 0$, $\mathrm{K}\rho$ is simply $d\rho/dt$, and therefore (4a) may be written:

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = -\sigma(a, b, c; t),$$

whence

$$(5) \qquad \rho = \rho_0 e^{-\int_0^t \sigma(a, b, c; t)dt},$$

where ρ_0 denotes the value of ρ for $t = 0$. In the case of steady motion, ρ_0 is not an arbitrary function of a, b, c over any region of space. For if it were, ρ would be the same function of x, y, z that ρ_0 is of a, b, c and therefore ρ would be an arbitrary function of x, y, z which according to (4a) is obviously not the case. The reason for this difficulty is that although we have assumed explicitly that u, v, w are functions of x, y, z alone, we have as yet made no explicit assumption of the same character about ρ, equation (5) being universally true whether the motion be stationary or not.

If S be a surface, cutting all of the lines of flow once and only once, ρ can obviously be assigned arbitrary values for all points upon this surface. Of course S may consist of several separate pieces. When any element of the fluid reaches this surface, its density must then, if the motion is stationary, assume the value corresponding to that point of S. Thus and only thus can the motion be stationary.

After ρ is obtained (4b) gives p, and from (4), X, Y, Z can be found. If u, v, w are real, and ρ_0 is positive, ρ from (5) is positive as it should be. If p should become negative, the fluid would tear.

In the case of an incompressible fluid, these results would be somewhat different: p is not then a function of ρ. We then have

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{6}$$

so that u, v, w must be taken subject to this condition. The surfaces $\rho=$ constant are made up of lines of flow, for the density of an element of an incompressible fluid can not change. From one line of flow to another the density can vary according to an arbitrary law, for (4a) has reduced to $\mathrm{K}\rho=0$, which says only that the surfaces $\rho=$ constant are invariant under all transformations of the group, i. e., that they are made up of lines of flow, which we have already assumed to be the case. Since p is not in general a function of ρ, in this case p can be assumed to be an arbitrary function of x, y, z and then X, Y, Z are determined from (4).

If the fluid is homogeneous, without being incompressible, $\rho=$ constant, and then from (4a), since $\mathrm{K}\rho=0$, we get

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

i. e., although the fluid may not be incompressible, if it is homogeneous and has a steady motion, it moves just as an incompressible fluid of like density would. But if the fluid is not incompressible, p is a certain function of ρ, and since $\rho=$ constant, $p=$ constant all through the fluid. The forces X, Y, Z are again obtained from (4).

The one-parameter group, generated by $\mathrm{K}f$, has two families of invariant surfaces, viz., the ∞^2 surfaces:

$$u_1(x, y, z)=\text{constant}, \quad u_2(x, y, z)=\text{constant},$$

where u_1, u_2 are two independent solutions of $\mathrm{K}f=0$. The intersections of these surfaces are the lines of flow of the fluid.

Like all one-parameter groups, ours can be written in the form:

$$\Omega_i(x, y, z)=\Omega_i(a, b, c)\ (i=1,2), \quad W(x, y, z)=W(a, b, c)+t. \tag{7}$$

Thus all steady fluid motions can be put into this form, and every set of equations of this kind represents a steady fluid motion.

It should be remarked, however, that it is not necessary that the functions Ω_i and W should be represented by the same formula for all portions of space. Thus, we have more generally

$$(8)\qquad \begin{aligned} \Omega_{ki}(x, y, z) &= \Omega_{ki}(a, b, c) && (i=1, 2),\\ W_k(x, y, z) &= W_k(a, b, c) + t && (k=1, 2, 3, \cdots), \end{aligned}$$

the formulæ with index k being valid only for a certain region R_k of space.

Thus while the point x, y, z is in R_1, Ω_{1i} and W_1 are used, when it is in R_2, the next adjoining region, Ω_{2i} and W_2 are employed, etc. Of course, for continuity, the values of x, y, z obtained from the first set of equations must coincide with those found from the second set for all points upon the boundary between R_1 and R_2, i. e., x, y, z must be continuous functions of a, b, c, t. The same is not necessary of u, v, w, i. e., in general the direction of the motion will undergo a discontinuous change on the bounding surface between two such regions. Such discontinuities will actually occur, for instance, if we consider the flow of a fluid in a channel whose walls are portions of analytical surfaces, which meet at angles different from 180°.

These considerations go a great way towards the solution of the general problem of the steady flow of a fluid in a channel bounded by any arbitrary surfaces whatsoever. An infinity of equations of the form (8) can be written down in every case, which fulfil the boundary conditions, and which represent a possible fluid motion. Among all these, it will be necessary to pick out those which correspond to the external forces at work, i. e., which give the values to X, Y, Z belonging to that particular problem.

In most cases the forces X, Y, Z have a potential, so that

$$(9)\qquad X = \frac{\partial V}{\partial x}, \quad Y = \frac{\partial V}{\partial y}, \quad Z = \frac{\partial V}{\partial z}.$$

In all such cases u, v, w cannot be chosen at will, but are subject to the condition, that the values of X, Y, Z computed from (4) can be written in the form (9). If we put

$$(10)\qquad P = \int \frac{dp}{\rho},$$

equations (4) become:

$$(11)\qquad \mathrm{K}u = \frac{\partial(V-P)}{\partial x}, \quad \mathrm{K}v = \frac{\partial(V-P)}{\partial y}, \quad \mathrm{K}w = \frac{\partial(V-P)}{\partial z},$$

and therefore, *if the forces X, Y, Z have a potential,*

(12) $$\mathrm{K}u\,dx + \mathrm{K}v\,dy + \mathrm{K}w\,dz = d\,(V - P)$$

must be a complete differential. This is the condition which u, v, w must verify in this case, or, what amounts to the same thing,

(13) $$\frac{\partial \mathrm{K}v}{\partial z} - \frac{\partial \mathrm{K}w}{\partial y} = 0\,, \quad \frac{\partial \mathrm{K}w}{\partial x} - \frac{\partial \mathrm{K}u}{\partial z} = 0\,, \quad \frac{\partial \mathrm{K}u}{\partial y} - \frac{\partial \mathrm{K}v}{\partial x} = 0\,.$$

§ 2. *The fluid motion expressed by the general projective group.*

We shall confine ourselves to the case in which the forces have a potential and we shall find that the most general ternary projective group, which can express the steady motion of a fluid, if the forces have a potential, is a linear group.

For convenience in this paragraph we shall write x_1, x_2, x_3, u_1, u_2, u_3 instead of x, y, z, u, v, w.

The most general projective infinitesimal transformation is a linear combination of the following fifteen:

$$p_i = \frac{\partial f}{\partial x_i}, \quad x_i p_k = T_{ik}, \quad x_i \sum_{k=1}^{3} x_k p_k = P_i \qquad (i, k = 1, 2, 3).$$

We can write it:

(14) $$\mathrm{K}f = \sum_{i=1}^{3} u_i \frac{\partial f}{\partial x_i},$$

where

(15) $$u_i = \gamma_i + \sum_{k=1}^{3} c_{ki} x_k + x_i \sum_{k=1}^{3} c_k x_k \qquad (i = 1, 2, 3).$$

We must find the necessary and sufficient conditions in order that

$$\sum_{i=1}^{3} \mathrm{K}u_i\,dx_i$$

may be a complete differential.

We obtain

(16) $$v_i = \mathrm{K}u_i = (c_1x_1 + c_2x_2 + c_3x_3)u_i + \sum_{k=1}^{3} (c_{ki} + c_k c_i) u_k \quad (i = 1, 2, 3).$$

But if the forces have a potential, equations (13) of § 1 must be verified, the first of which is

$$\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} = 0.$$

We find

$$\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} = c_2u_3 - c_3u_2 + (c_1x_1 + c_2x_2 + c_3x_3)\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)$$
$$+ \sum_{k=1}^{3} \left((c_{k3} + c_k x_3) \frac{\partial u_k}{\partial x_2} - (c_{k2} + c_k x_2) \frac{\partial u_k}{\partial x_3} \right).$$

Now, from (2), we find

$$\frac{\partial u_i}{\partial x_j} = c_{ji} + c_j x_i + \delta_{ij} \sum_{\lambda=1}^{3} c_\lambda x_\lambda ,$$

where $\delta_{ij} = 0$ for $i \neq j$, and $\delta_{ii} = 1$.

Thus we obtain:

$$\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} = c_2 u_3 - c_3 u_2 + [c_{23} - c_{32} + (c_2 x_3 - c_3 x_2)] \sum_{\lambda=1}^{3} c_\lambda x_\lambda$$

$$+ \sum_{k=1}^{3} [(c_{k3} + c_k x_3)(c_{2k} + c_2 x_k + \delta_{k2} \sum_{\lambda=1}^{3} c_\lambda x_\lambda) - (c_{k2} + c_k x_2)(c_{3k} + c_3 x_k + \delta_{k3} \sum_{\lambda=1}^{3} c_\lambda x_\lambda)] .$$

When this is arranged according to powers of x_1, x_2, x_3, the coefficient of each term must vanish. This gives the following equations, obtained by putting the coefficients of the quadratic terms equal to zero:

$$c_2 c_3 = c_3 c_1 = c_1 c_2 = c_2^2 - c_3^2 = 0 ,$$

which gives at once $c_2 = c_3 = 0$. Using one of the other equations (13) §1, we should find $c_1 = 0$ also.

Equating the other coefficients in $(\partial v_2/\partial x_3) - (\partial v_3/\partial x_2)$ to zero, we get only one other condition, and thus we have proved the theorem:

An arbitrary projective one-parameter group cannot represent the steady motion of a fluid under the influence of forces possessing a potential. To represent such a motion, the group must be a linear group. If

$$\text{(17)} \qquad \mathrm{K}f = \sum_{k=1}^{3} u_i \frac{\partial f}{\partial x_i} ,$$

where

$$\text{(18)} \qquad u_i = \gamma_i + \sum_{k=1}^{3} c_{ki} x_k ,$$

is the infinitesimal transformation of this linear group, the coefficients c_{ki} must verify the three conditions:

$$\text{(19)} \qquad \begin{cases} \sum_{k=1}^{3} (c_{k3} c_{2k} - c_{k2} c_{3k}) = 0 , \\ \sum_{k=1}^{3} (c_{k1} c_{3k} - c_{k3} c_{1k}) = 0 , \\ \sum_{k=1}^{3} (c_{k2} c_{1k} - c_{k1} c_{2k}) = 0 , \end{cases}$$

which necessary conditions are also sufficient.

§ 3. *Fluid motion expressed by the ternary linear group.*

In equations (18) we can put $\gamma_i = 0$ without loss of generality. For the motions in which $\gamma_i \neq 0$ only differ from those for which $\gamma_i = 0$ by a translation.

We assume then that the origin remains fixed during the entire motion, and we will resume the less symmetrical but more convenient notation, x, y, z; u, v, w instead of x_1, x_2, x_3; u_1, u_2, u_3. Let

$$(20)\qquad \begin{cases} u = a_1x + b_1y + c_1z, \\ v = a_2x + b_2y + c_2z, \\ w = a_3x + b_3y + c_3z. \end{cases}$$

The components of the velocity of rotation of any element of the fluid are:

$$\begin{cases} \xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \frac{1}{2}(b_3 - c_2), \\ \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \frac{1}{2}(c_1 - a_3), \\ \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(a_2 - b_1), \end{cases}$$

so that ξ, η, ζ are the same for all points of the fluid. Hence at all points the axis of rotation has the same direction, and the angular velocity of rotation is the same, viz.:

$$\omega = \sqrt{\xi^2 + \eta^2 + \zeta^2}.$$

Let us take our z axis parallel to this direction. Then

$$\xi = \eta = 0, \quad \zeta = \omega,$$

and hence, for this choice of coördinates,

$$b_3 = c_2, \quad c_1 = a_3, \quad a_2 = b_1 + 2\omega,$$

so that we can write more simply

$$(21)\qquad \begin{cases} u = \qquad\quad Ax + (H - \omega)y + Gz, \\ v = (H + \omega)x + \qquad\quad By + Fz, \\ w = \qquad\quad Gx + \qquad\quad Fy + Cz. \end{cases}$$

If the forces have a potential, equations (19) give

$$(A + B)\omega = 0, \quad F\omega = 0, \quad G\omega = 0.$$

Therefore, either $\omega = 0$, or

$$A + B = 0, \quad F = 0, \quad G = 0.$$

If $\omega = 0$, we have a velocity potential

$$\phi = \tfrac{1}{2}(Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy),$$

so that the lines of flow are the orthogonal trajectories of the system of similar surfaces of the second order:

$$\phi = \text{constant}.$$

In the second case (21) reduces to

$$(22)\qquad \begin{cases} u = \phantom{(H+\omega)x -{}} Ax + (H-\omega)y\,, \\ v = (H+\omega)x - Ay\,, \\ w = Cz\,. \end{cases}$$

The motion can be decomposed into a rotation and a motion with a velocity potential. But it is simpler to treat it without so decomposing it. This case again separates into two distinct sub-cases.

In the first sub-case,

$$A^2 + H^2 - \omega \neq 0\,.$$

Then, if

$$(23)\qquad \begin{cases} \rho_1 = +\sqrt{A^2 + H^2 - \omega} = \rho\,, \quad \rho_2 = -\rho_1 = -\rho\,, \\ \lambda_i = -\dfrac{H-\omega}{A-\rho_i} = \dfrac{A+\rho_i}{H+\omega}\,, \end{cases}$$

the finite equations of the group can be written as follows:

$$(24)\qquad x - \lambda_1 y = e^{-\rho t}(a - \lambda_1 b)\,, \quad x - \lambda_2 y = e^{+\rho t}(a - \lambda_2 b)\,, \quad z = ce^{Ct}\,,$$

where a, b, c are the values of x, y, z for $t = 0$.

The surfaces:

$$z^{-\rho}(x - \lambda_1 y)^{-C} = \text{constant}, \qquad z^{\rho}(x - \lambda_2 y)^{-C} = \text{constant},$$

which are real or imaginary cylinders, are therefore invariant. Their real intersections are the lines of flow.

The function

$$(x - \lambda_1 y)(x - \lambda_2 y)$$

is also invariant. Therefore the projections of the lines of flow upon the xy plane are the similar conics:

$$(25)\qquad x^2 - \frac{2A}{H+\omega}xy + \frac{\omega - H}{\omega + H}y^2 = \text{constant},$$

i. e., the lines of flow are situated upon a cylinder of the second order, whose elements are parallel to the z axis.

The physical conditions, corresponding to this motion, are obtained from the considerations of § 1. We have, in particular, from (11),

$$V - p = \tfrac{1}{2}\left[(A^2 + H^2 - \omega)(x^2 + y^2) + c^2z^2\right] + \text{constant}. \tag{26}$$

The motion, here considered, becomes periodic if ρ is purely imaginary, and $c = 0$, as may be seen by solving (24) for x, y, z. The lines of flow are then similar ellipses in parallel planes. It can be shown that this motion is possible for an incompressible homogeneous fluid filling an ellipsoidal space, the particles attracting each other according to Newton's law. This has been studied in detail by DIRICHLET and DEDEKIND.* It can also be easily shown that the analogous result is true for an elliptic cylinder of infinite length, provided that H and ω satisfy the equation:

$$(H^2 - \omega^2)\frac{H^2 + 4}{\pi f \rho} + 4 = 0,$$

where ρ is the density, and f a numerical constant, depending upon the units employed. If a' and b' are the semi-axes of the elliptic section of the cylinder their ratio will be obtained from the equation:

$$\frac{a'}{b'} = \sqrt{\frac{\omega - H}{\omega + H}}.$$

In the second sub-case,

$$\begin{cases} A^2 + H^2 - \omega = 0, \\ \lambda = -\dfrac{H - \omega}{A} = \dfrac{A}{H + \omega}. \end{cases}$$

The finite equations of the group can be written:

$$x - \lambda y = a - \lambda b, \quad \frac{y}{x - \lambda y} = \frac{b}{a - \lambda b} + (H + \omega)t, \quad z = ce^{Ct}.$$

If we chose the plane $x - \lambda y = 0$ as xz-plane, and the plane $\lambda x + y = 0$ as yz-plane, the planes $y =$ constant are invariant, and the lines of flow are the curves:

$$z = \text{constant}\ e^{\frac{C}{2\omega}\frac{x}{y}}, \quad y = \text{constant},$$

i. e., exponential curves in these planes.

§ 4. *Fluid motion expressed by a linearoid group.*

Let us consider next a one-parameter group, generated by

$$\mathrm{K}f = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z},$$

* Crelle's Journal, vol. 58.

where

$$
(27) \qquad \begin{cases} u = \phi_1 x + \phi_2 y + \phi_3 , \\ v = \psi_1 x + \psi_2 y + \psi_3 , \\ w = \chi_1 x + \chi_2 y + \chi_3 , \end{cases}
$$

$\phi_1 , \cdots , \chi_3$ being functions of z only. It will be of the class called linearoid by me in another connection. Then,

$$
(28) \begin{cases} \mathrm{K}u = (\phi_1^2 + \phi_2\psi_1 + \phi_3'\chi_1)x + (\phi_1\phi_2 + \phi_2\psi_2 + \phi_3'\chi_2)y + \phi_1\phi_3 + \phi_2\psi_3 \\ \qquad\qquad + \phi_3'\chi_3 + \phi_1'\chi_1x^2 + (\phi_1'\chi_2 + \phi_2'\chi_1)xy + \phi_2'\chi_2y^2 , \\ \mathrm{K}v = (\psi_1\phi_1 + \psi_2\psi_1 + \psi_3'\chi_1)x + (\psi_1\phi_2 + \psi_2^2 + \psi_3'\chi_2)y + \psi_1\phi_3 + \psi_2\psi_3 \\ \qquad\qquad + \psi_3'\chi_3 + \psi_1'\chi_1x^2 + (\psi_1'\chi_2 + \psi_2'\chi_1)xy + \psi_2'\chi_2y^2 , \\ \mathrm{K}w = (\chi_1\phi_1 + \chi_2\psi_1 + \chi_3'\chi_1)x + (\chi_1\phi_2 + \chi_2\psi_2 + \chi_3'\chi_2)y + \chi_1\phi_3 + \chi_2\psi_3 \\ \qquad\qquad + \chi_3'\chi_3 + \chi_1'\chi_1x^2 + (\chi_1'\chi_2 + \chi_2'\chi_1)xy + \chi_2'\chi_2y_2 , \end{cases}
$$

where the accents denote derivatives with respect to z.

Write this as follows:

$$
(29) \qquad \begin{cases} \mathrm{K}u = \lambda_1x^2 + 2\mu_1xy + \nu_1y^2 + 2\rho_1x + 2\sigma_1y + 2\tau_1 , \\ \mathrm{K}v = \lambda_2x^2 + 2\mu_2xy + \nu_2y^2 + 2\rho_2x + 2\sigma_2y + 2\tau_2 , \\ \mathrm{K}w = \lambda_3x^2 + 2\mu_3xy + \nu_3y^2 + 2\rho_3x + 2\sigma_3y + 2\tau_3 ; \end{cases}
$$

the conditions for the integrability of

$$
\mathrm{K}u \cdot dx + \mathrm{K}v \cdot dy + \mathrm{K}w \cdot dz ,
$$

i. e., for the existence of a potential for the corresponding fluid motion, become:

$$
(30) \qquad \begin{cases} \mu_1 - \lambda_2 = 0 , \quad \nu_1 - \mu_2 = 0 , \quad \sigma_1 - \rho_2 = 0 , \\ \rho_2' - \mu_3 = 0 , \quad \sigma_2' - \nu_3 = 0 , \quad \tau_2' - \sigma_3 = 0 , \\ \rho_1' - \lambda_3 = 0 , \quad \sigma_1' - \mu_3 = 0 , \quad \tau_1' - \rho_3 = 0 , \\ \lambda_1' = \mu_1' = \nu_1' = \lambda_2' = \mu_2' = \nu_2' = 0 . \end{cases}
$$

Hence

$$
(31) \qquad \begin{cases} 2(\mu_1 - \lambda_2) = (\phi_2' - 2\psi_1')\chi_1 + \phi_1'\chi_2 = 0 , \\ 2(\nu_1 - \mu_2) = - \psi_2'\chi_1 + (2\phi_2' - \psi_1')\chi_2 = 0 , \end{cases}
$$

so that we have two cases to distinguish, according as $\chi_1 = \chi_2 = 0$, or the determinant

$$
(32) \qquad (\phi_2' - 2\psi_1')(2\phi_2' - \psi_1') + \phi_1'\psi_2' = 0 .
$$

Let us consider first case I, in which $\chi_1 = \chi_2 = 0$. Then we find from the definition of these quantities,

$$\lambda_1 = \mu_1 = \nu_1 = \lambda_2 = \mu_2 = \nu_2 = \lambda_3 = \mu_3 = \nu_3 = \rho_3 = \sigma_3 = 0,$$

and from (30),

$$\rho_1' = \rho_2' = \sigma_1' = \sigma_2' = \tau_1' = \tau_2' = 0,$$

so that $\rho_1, \rho_2, \sigma_1, \sigma_2, \tau_1, \tau_2$ are constants.

Now since $\chi_1 = \chi_2 = 0$, we have

$$(33) \qquad \begin{cases} 2\rho_1 = \phi_1^2 + \phi_2\psi_1, & 2\rho_2 = \psi_1(\phi_1 + \psi_2), \\ 2\sigma_1 = \phi_2(\phi_1 + \psi_2), & 2\sigma_2 = \phi_2\psi_1 + \psi_2^2. \end{cases}$$

The only further condition is $\sigma_1 = \rho_2$, whence either

$$\phi_1 + \psi_2 = 0, \qquad \text{or} \qquad \phi_2 = \psi_1.$$

We will first assume that $\phi_1 + \psi_2 = 0$, and call this case I*a*. Then we have, in this case,

$$\sigma_1 = \rho_2 = 0, \qquad \phi_1^2 + \phi_2\psi_1 = 2\rho_1 = 2\sigma_2 = 2\sigma,$$

where σ is a constant. Thus in case I*a* the functions χ_1 and χ_2 vanish, and between the other functions we have the relations:

$$(34) \qquad \begin{cases} \phi_1 + \psi_2 = 0, \quad \phi_1^2 + \phi_2\psi_1 = 2\sigma = \text{constant}, \\ \phi_1\phi_3 + \phi_2\psi_3 + \chi_3\phi_3' = 2\tau_1, \\ \psi_1\phi_3 + \psi_2\psi_3 + \chi_3\psi_3' = 2\tau_2, \end{cases}$$

where τ_1, τ_2 are also constants, and where ϕ_1, ϕ_2, χ_3 may be taken as arbitrary functions of z.

In the second case, case I*b*, we have

$$\phi_2 = \psi_1,$$

whence

$$(35) \qquad \begin{cases} 2\rho_2 = 2\sigma_1 = \phi_2(\phi_1 + \psi_2), \\ 2\rho_1 = \phi_2^2 + \phi_2^2, \quad 2\sigma_2 = \phi_2^2 + \psi_2^2, \end{cases}$$

where ϕ_1 and ψ_2 may be taken to be arbitrary functions of z, and the equations for ϕ_3, ψ_3, χ_3 are of the same form as in (34).

If the fluid is incompressible we have further, $\chi_3 =$ constant in case I*a*, and in case I*b*, we find

$$\chi_3 = -\int(\phi_1 + \psi_2)dz + \text{constant},$$

as is seen at once from the condition of incompressibility, viz.:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 .$$

We shall not at present further discuss the fluid motions thus analytically determined. *The group theoretical considerations have enabled us to discover a considerable class of fluid motions, whose entire theory depends only on linear differential equations.*

Let us now carry out the corresponding investigation for case II, in which χ_1 and χ_2 do not both vanish. We have then

$$(32) \qquad (\phi_2' - 2\psi_1')(2\phi_2' - \psi_1') + \phi_1'\psi_2' = 0 ,$$

and further, as before (equations (30)) λ_1, μ_1, ν_1, λ_2, μ_2, ν_2 are constants. By the definition of these quantities, we have

$$(36) \qquad \begin{cases} \phi_1'\chi_1 = \lambda_1, & \phi_2'\chi_1 + \phi_1'\chi_2 = 2\mu_1, \quad \phi_2'\chi_2 = \nu_1 , \\ \psi_1'\chi_1 = \lambda_2, & \psi_2'\chi_1 + \psi_1'\chi_2 = 2\mu_2, \quad \psi_2'\chi_2 = \nu_2 . \end{cases}$$

Therefore

$$(37) \qquad \nu_i\chi_1^2 + \lambda_i\chi_2^2 - 2\mu_i\chi_1\chi_2 = 0 \qquad (i=1, 2),$$

and hence

$$(38) \qquad \lambda_1 : 2\mu_1 : \nu_1 = \lambda_2 : 2\mu_2 : \nu_2 = \alpha : \beta : 1 ,$$

where α and β denote the ratios $\lambda_1 : \nu_1$ and $2\mu_1 : \nu_1$.

Multiplying (32) by $\chi_1^2\chi_2^2$, and using (36) and (38), we obtain a third quadratic for χ_1 and χ_2, viz.:

$$(39) \qquad (\nu_1\chi_1 - \lambda_2\chi_2)^2 = 0 ,$$

so that also

$$\lambda_2^2 : \ 2\lambda_2\nu_1 : \ \nu_1^2 = \alpha : \ \beta : \ 1.$$

Now if we rotate the x axis through an angle θ, the form of u, v, w is not changed, and w becomes

$$\chi_1(x \cos\theta - y \sin\theta) + \chi_2 (x \sin\theta + y \cos\theta) + \chi_3$$
$$= (\chi_1 \cos\theta + \chi_2 \sin\theta)\, x + (-\chi_1 \sin\theta + \chi_2 \cos\theta)\, y + \chi_3 .$$

If we put

$$\tan\theta = \frac{\nu_1}{\lambda_2},$$

according to (39), the coefficient of y in the transformed w will vanish.

We can therefore assume from the beginning, without loss of generality, that $\chi_2 = 0$. Having already treated the case $\chi_1 = \chi_2 = 0$, we will now assume $\chi_1 \neq 0$, $\chi_2 = 0$.

From (36) we have, in this case,

$$\begin{cases} \phi_1' = \dfrac{\lambda_1}{\chi_1}, & \phi_2' = \dfrac{2\mu_1}{\chi_1}, \quad \nu_1 = 0, \\ \psi_1' = \dfrac{\lambda_2}{\chi_1}, & \psi_2' = \dfrac{2\mu_2}{\chi_1}, \quad \nu_2 = 0. \end{cases}$$

Since $\nu_2 = 0$, (30) gives $\mu_2 = 0$. Therefore, if

$$\int \frac{dz}{\chi_1} = \phi,$$

we have, since $\mu_1 = \lambda_2$,

$$(40) \qquad \begin{cases} \phi_1 = \lambda_1 \phi + a, & \phi_2 = 2\lambda_2 \phi + \beta, \\ \psi_1 = \lambda_2 \phi + \gamma, & \psi_2 = \delta, \end{cases}$$

where a, β, γ, δ denote four arbitrary constants.

We find further,

$$\lambda_3 = \chi_1' \chi_1, \qquad \mu_3 = 0, \qquad \nu_3 = 0.$$

Therefore, from (30), we notice that ρ_2, σ_2, σ_1 are constants. Now we have, with the values for ϕ_1, ϕ_2, ψ_1, ψ_2 just found,

$$2\sigma_2 = 2\lambda_2^2 \phi^2 + \lambda_2(\beta + 2\gamma)\phi + \beta\gamma + \delta^2,$$

so that λ_2 must vanish. Then

$$2\sigma_1 = \beta(\lambda_1 \phi + a + \delta),$$

so that λ_1 must also vanish. From the definition of ρ_2, which must be a constant, follows:

$$(41) \qquad \psi_3' = \frac{2\rho_2 - \gamma(a + \delta)}{\chi_1} = \frac{(\beta - \gamma)(a + \delta)}{\chi_1},$$

since $\rho_2 = \sigma_1$. Similarly we find, from $\rho_1' = \lambda_3 = \chi_1' \chi_1$,

$$(42) \qquad \phi_3' = \chi_1 + \frac{\epsilon - a^2 - \beta\gamma}{\chi_1},$$

where ϵ denotes another constant.

Substituting all these values into the only two equations (30) not yet used, viz.: $\tau_2' = \sigma_3$, $\tau_1' = \rho_3$, we find the following conditions for χ_1 and χ_3:

$$
(43)\quad \begin{cases}
(\gamma-\beta)\chi_1 + \dfrac{\gamma(\epsilon - a^2 - \beta\gamma) + \delta(\beta-\gamma)(a+\delta)}{\chi_1} \\
\qquad\qquad + (\beta-\gamma)(a+\delta)\dfrac{d}{dz}\left(\dfrac{\chi_3}{\chi_1}\right) = 0\,, \\
\dfrac{a(\epsilon - a^2 - \beta\gamma) + \beta(\beta-\gamma)(a-\delta)}{\chi_1} + \dfrac{d}{dz}(\chi_1\chi_3) \\
\qquad\qquad + (\epsilon - a^2 - \beta\gamma)\dfrac{d}{dz}\left(\dfrac{\chi_3}{\chi_1}\right) - \chi_1\chi_3' = 0\,.
\end{cases}
$$

The fluid motion, in this case, is therefore given by the equations:

$$
(44)\quad \begin{cases}
u = ax + \beta y + \phi_3\,, \\
v = \gamma x + \delta y + \psi_3\,, \\
w = \chi_1 x \qquad + \chi_3\,,
\end{cases}
$$

where a, β, γ, δ are constants, where χ_1 and χ_3 are obtained as functions of z by integrating (43), and where ϕ_3 and ψ_3 are determined from (41) and (42).

If the fluid is incompressible, we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = a + \delta + \chi_1' x + \chi_3' = 0\,,$$

i. e.,

$$\chi_1 = \eta\,, \quad \chi_3 = -(a+\delta)z + \theta\,,$$

where η and θ are constants. Moreover θ can be put equal to zero, if the xy-plane be appropriately chosen. Substituting these values in (43), we get relations between η and the other constants. But as will be seen from (41) and (42) the group in this case becomes a ternary linear one, all of the coefficients being constants.

Thus Case II of this paragraph gives nothing new for incompressible fluids.

The detailed discussion of the fluid motions here determined, as well as those of the preceding case, will be left for a future occasion.

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