DETERMINATION OF AN ABSTRACT SIMPLE GROUP OF ORDER $2^7 \cdot 3^6 \cdot 5 \cdot 7$ HOLOEDRICALLY ISOMORPHIC WITH A CERTAIN ORTHOGONAL GROUP AND WITH A CERTAIN HYPERABELIAN GROUP*

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1. Among the known simple groups† occur an orthogonal group and a hyperabelian group of the same order $2^7 \cdot 3^6 \cdot 5 \cdot 7$. They are shown to be holoedrically isomorphic in this paper. We first determine in §§ 2–14 an abstract group Γ (§§ 4, 2) simply isomorphic with the orthogonal group. This is accomplished by means of a rectangular array, a direct method of procedure employed by the writer in two recent papers in the Proceedings of the London Mathematical Society (vol. 31, p. 30; vol. 31, p. 351).

§§ 2, 3. *Rectangular array for the orthogonal group $O_{6,3}$ with respect to the subgroup $O_{5,3}$.*

2. The general orthogonal group of senary linear substitutions of modulus 3 having determinant unity has a subgroup H of index two.‡ The group H has as maximal invariant subgroup the group of order two generated by the substitution C which changes the signs of the six indices. The quotient group is a simple group $O_{6,3}$ of order $2^7 \cdot 3^6 \cdot 5 \cdot 7$. The substitutions of H which affect only the indices $\xi_1, \cdots, \xi_5$ form a simple group $O_{5,3}$ of order $2^6 \cdot 3^4 \cdot 5$ and of index $2 \cdot 126$ under H.

In a paper communicated December 28, 1899, to the London Mathematical Society, the writer has shown that $O_{5,3}$ is simply isomorphic with the abstract group G generated by the operators E_1, E_2, E_3, B_1, W subject to the generational relations:

$$(1) \qquad E_1^3 = E_2^2 = E_3^2 = B_1^2 = W^3 = I,$$

$$(2) \qquad (E_1E_2)^3 = (E_2E_3)^3 = (B_1E_1)^3 = (E_1E_3)^2 = (B_1E_2)^2 = (B_1E_3)^2 = I,$$

$$(3) \qquad WE_1 = B_3E_2E_1W, \quad WE_2 = B_3W, \quad WB_1 = B_3E_2W,$$

* Presented to the Society (Chicago) April 14, 1900. Received for publication January 15, 1900.

† See the list of known simple groups in the Bulletin of the American Mathematical Society for July, 1899.

‡ American Journal of Mathematics, vol. 21 (1899), pp. 193–256.

(4) $WB_4 = B_4B_2E_1E_2E_1^2W_1^2$,

(5) $WE_3E_2E_1WE_3 = E_1^2E_2E_3E_2E_1WE_3E_2E_1W$,

where for brevity we have set

(6) $B_2 \equiv E_1B_1E_1^2, \quad B_3 \equiv E_1E_2E_1^2B_1E_1E_2E_1^2, \quad B_4 \equiv E_2E_3B_3E_3E_2.$

From the above relations we derive the following:

(7) $$E_2E_1W = B_3WE_1 = WE_2E_1,$$

(8) $$WE_1^2 = E_1^2WB_1, \quad E_1B_1B_2 = B_1E_1.$$

Indeed, we have

$$WE_1^2 \equiv WE_2E_1E_2E_1E_2 = E_2E_1E_2E_1WE_2 = E_1^2E_2 \cdot B_3W = E_1^2WB_1,$$

$$E_1B_1B_2 = E_1B_1 \cdot E_1B_1E_1^2 = (E_1B_1)^3B_1E_1 = B_1E_1.$$

The isomorphism is defined by the following correspondences:

(9) $E_1 \sim E_1' \equiv (\xi_1\xi_2\xi_3), \quad E_2 \sim E_2' \equiv (\xi_1\xi_2)(\xi_3\xi_4), \quad E_3 \sim E_3' \equiv (\xi_1\xi_2)(\xi_4\xi_5),$

(10) $$W \sim W' \equiv \begin{Bmatrix} \xi_1' = \xi_1 - \xi_2 - \xi_3 - \xi_4 \\ \xi_2' = \xi_1 - \xi_2 + \xi_3 + \xi_4 \\ \xi_3' = \xi_1 + \xi_2 - \xi_3 + \xi_4 \\ \xi_4' = \xi_1 + \xi_2 + \xi_3 - \xi_4 \end{Bmatrix},$$

(11) $B_1 \sim C_1C_2, \quad B_2 \sim C_2C_3, \quad B_3 \sim C_3C_4, \quad B_4 \sim C_4C_5,$

where C_i denotes the orthogonal substitution changing the sign of ξ_i.

3. The group $O_{5,3}$ is extended to the orthogonal group H by the substitution

$$F' \equiv (\xi_1\xi_6)(\xi_2\xi_3).$$

The substitutions of H replace ξ_6 by every one of the $2 \cdot 126 \equiv 3^5 + 3^2$ linear functions *

(f) $$\sum_{i=1}^{6} a_i\xi_i \qquad \left[\sum_{i=1}^{6} a_i^2 \equiv 1 \pmod 3\right].$$

It follows that all the substitutions of H are given by the formula

$$S_iO_{5,3} \qquad (i = 1, 2, \cdots, 2 \cdot 126),$$

where the $2 \cdot 126$ substitutions S_i replace ξ_6 by the $2 \cdot 126$ distinct linear functions (f). In the quotient group $O_{6,3}$, S_i and $CS_i \equiv S_iC$ become identical. Denote by S_i' $(i = 1, \cdots, 126)$ the corresponding distinct substitutions of $O_{6,3}$.

* American Journal of Mathematics, l. c., foot of p. 195.

A rectangular array of the substitutions of $O_{6,3}$ is therefore given by the formula:

$$S'_i O_{5,3} \qquad (i = 1, 2, \cdots, 126).$$

To determine the S'_i, we note that the linear functions (f) are of the forms:

$$\pm \xi_i, \quad \pm \xi_i \pm \xi_j \pm \xi_k \pm \xi_l .$$

Since $-f$ is derived from f by the substitution C, we take only one of each pair $\pm f$ in determining the S'_i. For six of the S'_i we may take

$$(a) \qquad I,\ F',\ E'_2F',\ E'^2_1F',\ E'_2E'^2_1F',\ E'_3E'_2E'^2_1F'$$

which replace ξ_6 by ξ_6, ξ_1, ξ_2, ξ_3, ξ_4, ξ_5 respectively.

The substitution $W'F'$ replaces ξ_6 by $w \equiv \xi_1 - \xi_2 - \xi_3 - \xi_4$. Consider the 2^5 products K' of an even number of the C_i $(i = 1, \cdots, 6)$. If a particular K' replace w by ω, the four substitutions K', CK', C_5C_6K', $C_1C_2C_3C_4K'$ replace w by either ω or $-\omega$. Hence, of the 2^5 substitutions K', we need only consider 2^3 representatives, as

$$(k) \qquad \begin{gathered} K'_1 = I, \quad K'_i = C_1C_i\,(i = 2, \cdots, 5), \quad K'_6 = C_1C_2C_3C_5, \\ K'_7 = C_1C_2C_4C_5, \quad K'_8 = C_1C_3C_4C_5 . \end{gathered}$$

The substitutions $K'_jW'F'$ may therefore be taken as eight of the S'_i, distinct from the above six.

Using for the moment the notation 1234 for $\xi_1 + \xi_2 + \xi_3 + \xi_4$, we find that the substitutions

$$(b) \qquad \begin{gathered} I,\ E'_3,\ E'_2E'_3,\ E'_1E'_2E'_3,\ E'^2_1E'_2E'_3,\ F',\ E'_2F',\ E'^2_1F',\ E'_2E'^2_1F', \\ E'_3F',\ E'_2E'_3F',\ E'_1E'_3F',\ E'^2_1E'_3F',\ E'_1E'_2E'_3F',\ E'_2E'^2_1E'_3F', \end{gathered}$$

respectively replace 1234 by 1234, 1235, 1245, 2345, 1345, 2346, 1346, 1246, 1236, 1356, 2456, 1256, 2356, 3456, 1456, giving each of the 15 combinations of 1, 2, 3, 4, 5, 6 four at a time.

It follows that we may take as our 126 substitutions S'_i the six substitutions (a) and the 120 obtained by multiplying the 16 substitutions (b) on the right hand by $K'_jW'F'$ $(j = 1, 2, \cdots, 8)$. We have therefore explained the origin of the table given in § 7. Indeed, from that table we obtain a rectangular array for $O_{6,3}$ by replacing the group G by $O_{5,3}$ and accenting all the letters.

§§ 4–14. *Determination of the abstract group* Γ.

4. Consider the abstract group Γ obtained by the extension of G by an operator F subject to the relations:

$$(12) \qquad F^2 = I, \quad (E_1F)^3 = I,$$

$$(13) \qquad B_1F = FB_1B_4,$$

(14) $$E_2E_1F = FE_1^2E_2,$$

(15) $$E_3E_1F = FE_3E_1,$$

(16) $$WFWF = FWFW,$$

(17) $$WV = VW, \quad V \equiv FE_3E_1^2E_2E_1E_3F \equiv V^{-1},$$

(18) $$FW^2B_1B_4FE_3WE_3FWF = B_1E_3WE_3.$$

We readily verify that the generators E_1', E_2', E_3', B_1', W', F' of $O_{6,3}$ satisfy the relations (12) $\cdots$ (18).* Note that $V \sim V' \equiv (\xi_2\xi_3)(\xi_5\xi_6)$. The order of Γ is therefore at least as great as the order of $O_{6,3}$. To complete the proof of their simple isomorphism it remains only to prove that the order of Γ is at most as great as the order of $O_{6,3}$.

5. From the relations (1), $\cdots$, (8), (12), $\cdots$, (18), we proceed to derive a number of relations needed below. From (14) and (15) we find

(19) $$E_3E_1E_2E_1F = FE_3E_1 \cdot E_1^2E_2 = FE_3E_2.$$

Taking the reciprocal of (19) and multiplying on the right and left by F, we get

(20) $$FE_2E_3 = E_1^2E_2E_1^2E_3F = E_2E_1E_3E_2F.$$

From (20) we find

(21) $$E_1 = E_2FE_2E_3FE_3E_2.$$

From (12) and (14) we get

(22) $$FE_1FE_2F = FE_1FE_1 \cdot E_1^2E_2F = FE_1FE_1FE_2E_1 = E_1^2E_2E_1.$$

From (12) and (7) we find

$$E_3E_2E_1^2FE_1E_2E_3 = E_3E_2E_1 \cdot FE_1^2F \cdot E_2E_3 = E_3FE_1^2E_2E_1^2 \cdot E_2E_1E_2E_3F$$
$$= E_1FE_3E_1 \cdot E_2E_1^2E_2E_3F = E_1FE_3E_1^2E_2E_1E_3F = E_1V:$$

(23) $$E_1V = E_3E_2E_1^2FE_1E_2E_3.$$

(24) $$E_1VE_1V \equiv (E_3E_2E_1^2FE_1E_2E_3)^2 = I.$$

Taking the reciprocal of (13) and applying $B_4^2 = \mathrm{I}$, $B_1B_4 = B_4B_1$, we find

$$FB_1 = B_4B_1F = B_4 \cdot FB_1B_4 = B_4FB_4B_1:$$

(25) $$B_4F = FB_4.$$

* In regard to the relation corresponding to (18) it should be remarked that for orthogonal substitutions there occurs an additional factor C in one member.

Transforming (13) by E_3E_1, which by (15) transforms F into itself, we get

(26) $$B_1B_2F = FB_1B_2B_4.$$

By (13), $B_1B_2F = B_2FB_1B_4$. Hence from (26) we find

(27) $$B_2F = FB_2.$$

Applying (1), (2), (6), (8) and (14), we find *

$$E_2E_1B_1F = B_1B_2B_3E_2E_1F = B_1B_2B_3FE_1^2E_2,$$

$$E_2E_1FB_1B_4 = FE_1^2E_2B_1B_4 = FB_1B_3B_4E_1^2E_2.$$

Equating these products by virtue of (13), we find

(28) $$B_1B_2B_3F = FB_1B_3B_4.$$

Combining (28) with (26) and (28) with (25), we find respectively

(29) $$B_3F = FB_2B_3,$$

(30) $$B_1B_2B_3B_4F = FB_1B_3.$$

Hence F transforms any product formed from B_1, B_2, B_3, B_4 into another such product.

6. Corresponding to the orthogonal substitutions K_i' defined by formulæ (k) of § 3, we have the following operators of Γ:

$$K_1 = I, \qquad K_2 = B_1, \qquad K_3 = B_1B_2, \quad K_4 = B_1B_2B_3,$$

$$K_5 = B_1B_2B_3B_4, \quad K_6 = B_1B_3B_4, \quad K_7 = B_1B_4, \quad K_8 = B_1B_2B_4.$$

Any product derived from E_1, E_2, E_3, B_1 transforms any product derived from B_1, B_2, B_3, B_4 into a product of the B_i, a statement made evident by considering the corresponding substitutions of the isomorphic group $O_{5,3}$. In virtue of the theorem at the end of § 5, a like result holds when the transformer is any operator (for example, V) derived from F, E_1, E_2, E_3, B_1.

Since $B_1B_3W = WB_1B_3$, we have on applying (30),

$$B_1B_3WFG = WFB_1B_2B_3B_4G = WFG.$$

Hence the products $BWFG$, where B runs through the 16 distinct products of the B_1, B_2, B_3, B_4, reduce to the eight distinct products K_jWFG $(j = 1, \cdots, 8)$. For example, $B_2WFG = K_4WFG$.

* In establishing identities between operators of G it is frequently simpler to work with the corresponding substitutions of the simply isomorphic group $O_{5,3}$.

7. Consider the following set of operators belonging to Γ:

$$
\begin{array}{lll}
R_0 \equiv G & R_1 \equiv FG & R_2 \equiv E_2FG \\
R_3 \equiv E_1^2FG & R_4 \equiv E_2E_1^2FG & R_5 \equiv E_3E_2E_1^2FG \\
R_{1j} \equiv K_jWFG & R_{2j} \equiv E_3K_jWFG & R_{3j} \equiv E_2E_3K_jWFG \\
R_{4j} \equiv E_1E_2E_3K_jWFG & R_{5j} \equiv E_1^2E_2E_3K_jWFG & R_{6j} \equiv FK_jWFG \\
R_{7j} \equiv E_2Fk_jWFG & R_{8j} \equiv E_1^2FK_jWFG & R_{9j} \equiv E_2E_1^2FK_jWFG \\
R_{10j} \equiv E_3FK_jWFG & R_{11j} \equiv E_2E_3FK_jWFG & R_{12j} \equiv E_1E_3FK_jWFG \\
R_{13j} \equiv E_1^2E_3FK_jWFG & R_{14j} \equiv E_1E_2E_3FK_jWFG & R_{15j} \equiv E_2E_1^2E_3FK_jWFG
\end{array}
$$

where $j = 1, 2, \cdots, 8$.

From the developments given below it will follow that this table is a rectangular array of the operators of the abstract group Γ with G as first row and therefore that Γ *is holoedrically isomorphic with* $O_{6,3}$. Indeed, in §§ 10–14, we prove that the 126 rows of our table are merely permuted amongst themselves on applying as a left hand multiplier any of the generators E_1, E_2, E_3, B_1, W, F and therefore for an arbitrary operator of Γ. Since the row R_0 contains the identity, it will follow that an arbitrary operator of Γ belongs to the above table. The order of Γ is thus not greater than that of $O_{6,3}$.

8. Lemma I.—*The rows* R_{1j} $(j = 1, \cdots, 8)$ *are merely permuted upon applying as a left hand multiplier either* E_1 *or* E_2.

$$
\begin{aligned}
E_2R_{1j} \equiv E_2K_jWFG = K_iE_2WFG = K_iB_3WB_1FG \\
= K_tWFB_1B_4G = K_tWFG \equiv R_{1t}.
\end{aligned}
$$

$$E_2E_1K_jWFG = K_rE_2E_1WFG = K_rWFE_1^2E_2G = K_rWFG \equiv R_{1r}.$$

$$E_1^2K_sWFG = E_2E_1 \cdot E_2E_1 \cdot E_2K_sWFG = K_jWFG,$$

by the preceding results. Hence

$$E_1R_{1j} \equiv E_1K_jWFG = K_sWFG \equiv R_{1s}.$$

9. Lemma II.—*The rows* R_{1j} *are permuted upon applying as a left hand multiplier the operator* V *defined by* (17).

$$
\begin{aligned}
VR_{1j} &\equiv VK_jWFG = K_iVWFG & \text{[by § 6]} \\
&= K_iWVFG = K_iWFE_3E_1^2E_2E_1E_3 & \text{[by (17)]} \\
&= K_iWFG \equiv R_{1i}.
\end{aligned}
$$

10. Theorem.—*The application of* E_2 *as a left hand multiplier permutes the* 126 *rows.*

By inspection we see that E_2 interchanges R_1 with R_2, R_3 with R_4, R_{2j} with R_{3j}, R_{6j} with R_{7j}, R_{8j} with R_{9j}, R_{10j} with R_{11j}, R_{13j} with R_{15j}. Furthermore,

$$E_2R_5 \equiv E_2E_3E_2E_1^2FG = E_3E_2E_3E_1^2FG = E_3E_2E_1^2E_3E_1FG$$
$$= E_3E_2E_1^2FE_3E_1G = E_3E_2E_1^2FG \equiv R_5 \qquad \text{[by use of (15)]}.$$
$$E_2R_{4j} \equiv E_2E_1E_2E_3K_jWFG = E_1^2E_2E_3E_1K_jWFG = R_{5s},$$

upon applying lemma I to replace E_1R_{1j} by R_{1s}.

The condition for the identity $E_2R_{12j} = R_{12t}$ is

$$E_2E_1E_3FK_jWFG = E_1E_3FK_tWFG,$$

or

$$FE_3E_1^2 \cdot E_2E_1E_3FK_jWFG = K_tWFG,$$

which is satisfied in virtue of lemma II.

Likewise, the condition for the identity $E_2R_{14j} = R_{14t}$ is

$$(E_1E_2E_3F)^{-1}E_2(E_1E_2E_3F)K_jWFG = K_tWFG,$$

which is satisfied by lemma II since we have

$$FE_3E_2E_1^2E_2E_1E_2E_3F = FE_3E_1^2E_2E_1E_3F \equiv V.$$

11. Theorem.—*The application of* E_3 *as a left hand multiplier permutes the* 126 *rows.*

By inspection E_3 interchanges R_4 with R_5, R_{1j} with R_{2j}, R_{6j} with R_{10j}, R_{8j} with R_{12j}.

$$E_3R_1 \equiv E_3FG = E_2FE_2E_1E_2E_3G = E_2FG \equiv R_2 \qquad \text{[by (21)]}.$$
$$E_3R_3 \equiv E_3E_1^2FG = E_1^2E_3E_1FG = E_1^2FE_3E_1G = E_1^2FG \equiv R_3.$$
$$E_3R_{3j} \equiv E_2E_3E_2K_jWFG = E_2E_3K_tWFG \equiv R_{3t} \qquad \text{[by lemma I]}.$$
$$E_3R_{4j} \equiv E_3E_1E_2E_3K_jWFG = E_1^2E_2E_3E_2K_jWFG$$
$$= E_1^2E_2E_3K_tWFG \equiv R_{5t} \qquad \text{[by lemma I]}.$$
$$E_3R_{7j} \equiv E_3E_2FK_jWFG = E_1^2E_3E_1^2E_2FK_jWFG$$
$$= E_1^2E_3FE_2E_1K_jWFG = E_1^2E_3FK_sWFG \equiv R_{13s},$$

upon applying (14) and lemma I.

The condition for the identity $E_3R_{9j} = R_{9t}$ is

$$(E_2E_1^2F)^{-1}E_3(E_2E_1^2F)K_jWFG = K_tWFG,$$

and it is satisfied in virtue of lemma II since

$$FE_1E_2E_3E_2E_1^2F = FE_1E_3E_2E_3E_1^2F = FE_3E_1^2E_2E_1E_3F \equiv V.$$

$$E_3R_{11j} \equiv E_3E_2E_3FK_jWFG = E_2E_3E_2FK_jWFG$$
$$= E_2E_3E_2 \cdot E_2E_1FK_iWFG \quad \text{[by lemma I and (14)]}$$
$$= E_2E_1^2E_3FK_iWFG = R_{15i}.$$

The condition for the identity $E_3R_{14j} = R_{14t}$ is

$$(E_1E_2E_3F)^{-1}E_3(E_1E_2E_3F)K_jWFG = K_tWFG,$$

and it is satisfied by lemmas I and II since we have

$$FE_3E_2E_1^2E_3E_1E_2E_3F = FE_3E_2E_1E_3E_2E_3F$$
$$= FE_3E_2E_1E_2E_3E_2F = FE_3E_1^2E_2E_1^2E_3E_2F$$
$$= FE_3E_1^2E_2E_1E_3 \cdot E_1^2E_2F = FE_3E_1^2E_2E_1E_3FE_2E_1 = VE_2E_1.$$

12. Theorem.—*The application of F as a left hand multiplier permutes the 126 rows.*

$$FR_0 = R_1.$$
$$FR_{1j} = R_{6j}.$$
$$FR_2 \equiv FE_2FG = E_1^2FE_1^2E_2E_1G = E_1^2FG \equiv R_3 \quad \text{[by (22)]}.$$
$$FR_4 \equiv FE_2E_1^2FG = E_1^2E_2FE_1FG \quad \text{[by (14)]}$$
$$= E_1^2E_2 \cdot E_1^2FE_1^2G = E_2E_1E_2FE_1^2G \quad \text{[by (12)]}$$
$$= E_2E_1^2E_1^2E_2FE_1^2G = E_2E_1^2FE_2G = R_4 \quad \text{[by (14)]}.$$
$$FR_5 \equiv FE_3E_2E_1^2FG = E_3E_2FE_2 \cdot FG \quad \text{[by (21)]}$$
$$= E_3E_2 \cdot FR_2 = E_3E_2R_3 \equiv E_3E_2E_1^2FG \equiv R_5.$$
$$FR_{2j} \equiv FE_3K_jWFG = E_3E_1FE_1^2K_jWFG \quad \text{[by (21)]}$$
$$= E_1^2E_3FK_tWFG \equiv R_{13t} \quad \text{[by lemma I]}.$$

The condition $FR_{3j} = R_{14t}$ or

$$(E_1E_2E_3F)^{-1}F(E_2E_3)K_jWFG = K_tWFG$$

is satisfied by lemma II, since we have, by (20),

$$FE_3E_2E_1^2FE_2E_3 = FE_3E_2E_1^2 \cdot E_1^2E_2E_1^2E_3F = FE_3E_1^2E_2E_1E_3F = V.$$

$$FR_{5j} \equiv FE_1^2E_2E_3K_jWFG = E_2E_3FK_tWFG \equiv R_{11t},$$

since by (14) and (15) we have

$$FE_1^2E_2E_3 = E_2E_1FE_3 = E_2E_1 \cdot E_3E_1FE_1^2 = E_2E_3FE_1^2 .$$

The condition for the identity $FR_{4j} = R_{4t}$ is satisfied by lemmas I and II:

$$(E_1E_2E_3)^{-1}FE_1E_2E_3K_jWFG \equiv E_1VK_jWFG = K_tWFG \quad \text{[by (23)]}.$$

$$FR_{7j} \equiv FE_2FK_jWFG = E_1^2FE_1^2E_2E_1K_jWFG \quad \text{[by (22)]}$$

$$= E_1^2FK_tWFG \equiv R_{8t} \quad \text{[by lemma I]}.$$

$$FR_{9j} \equiv FE_2E_1^2FK_jWFG = R_{9t} \equiv E_2E_1^2FK_tWFG ,$$

since

$$FE_2E_1^2F = E_1^2E_2FE_1F = E_1^2E_2 \cdot E_1^2FE_1^2$$

$$= E_2E_1^2E_1^2E_2FE_1^2 = E_2E_1^2FE_2E_1 \cdot E_1^2 .$$

$$FR_{10j} \equiv FE_3FK_jWFG = R_{12t} \equiv E_1E_3FK_tWFG ,$$

since

$$FE_3F = FE_3E_1 \cdot E_1^2F = E_3E_1 \cdot FE_1^2F = E_3E_1^2FE_1 = E_1E_3FE_1 .$$

The condition for the identity $FR_{15j} = R_{15k}$, viz.,

$$(E_2E_1^2E_3F)^{-1}F(E_2E_1^2E_3F)K_jWFG = K_kWFG ,$$

is seen to be satisfied in virtue of lemmas I and II as follows:

$$FE_3E_1E_2FE_2E_1^2E_3F = E_3E_1FE_2FE_2E_1^2E_3F = E_3 \cdot FE_1^2E_2E_1 \cdot E_2E_1^2E_3F$$

$$= E_1E_3E_1 \cdot FE_1E_2E_1E_3F \quad \text{[by (22)]}$$

$$= E_1FE_3E_1^2E_2E_1E_3F = E_1V .$$

13. By (21), E_1 is expressed as a product of the E_2, E_3, F. Hence by §§ 10–12, E_1 permutes the 126 rows when applied as a left hand multiplier.

From the following relations given under formulæ (8), (2), (13) above,

$$B_1E_1 = E_1B_1B_2 , \quad B_1E_2 = E_2B_1 , \quad B_1E_3 = E_3B_1 , \quad B_1F = B_1B_4 ,$$

and from the remarks in § 6 concerning the $K_j\,(j = 1, \cdots, 8)$, it follows that B_1 applied as a left hand multiplier permutes the 126 rows. Then by (6) a like result holds for B_2, B_3 and B_4.

14. Theorem.—*The application of W as a left hand multiplier permutes the* 126 *rows.*

$$WR_0 = R_0 .$$

$$WR_1 \equiv WFG \equiv R_{11} .$$

$$WR_2 \equiv WE_2FG = B_3WFG = B_1WB_1B_3FG = K_2WFG = R_{12} .$$

$$WR_3 \equiv WE_1^2FG = E_1^2 WB_1FG = E_1^2 WFG = K_3 WFG \equiv R_{13}$$
[by lemma I].
$$WR_4 = WE_2E_1^2FG = E_2E_1 WE_1FG = E_2E_1B_3E_2E_1 WFG$$
$$= B_2E_2E_1 \cdot E_2E_1 WFG = B_2 WE_1^2E_2FG = K_4 WFG \equiv R_{14}.$$

The condition for the identity $WR_5 = R_5$ is that $(E_3E_2E_1^2F) WE_3E_2E_1^2F$ shall belong to G. We shall verify that it equals $E_1E_2E_3E_2 WE_2E_3E_2E_1^2$. The condition for this equality may be written

$$(E_1E_2E_3E_2)^{-1}FE_1E_2E_3 WE_3E_2E_1^2E(E_2E_3E_2E_1^2)^{-1} = W.$$

By (23), the left member is equal to

$$E_2E_1 VWE_1 VE_2 = E_2E_1 WVE_1 VE_2 \qquad \text{[by (17)]}$$
$$= E_2E_1 WE_1^2E_2 = W \qquad \text{[by (24) and (7)]}.$$

Each row R_{ij} is of the form $AK_j WFG$, where A denotes a product built from E_1, E_2, E_3, F. But E_1, E_2, E_3 each transform K_j into some K_i, a statement made evident by considering the isomorphic orthogonal substitutions. Furthermore, by § 6, F transforms K_j into some K_i. Hence each row R_{ij} may be given the form $K_tA\, WFG$.

In virtue of the following relations between orthogonal substitutions,

$$W'C_1C_2 = C_3C_4E_2' W', \qquad W'C_1C_3 = C_2C_4E_1'^2E_2'E_1' W',$$
$$W'C_1C_4 = C_2C_3E_1'E_2'E_1'^2 W',$$
$$W'C_1C_5 = C_1C_5 W'^2, \qquad W'C_1C_2C_3C_5 = C_1C_5E'E_2'E_1'^2 W'^2,$$
$$W'C_1C_2C_4C_5 = C_1C_5E_1'^2E_2'E_1' W'^2, \qquad W'C_1C_3C_4C_5 = C_1C_5E_2' W'^2,$$

we have in the isomorphic group G the general relation

$$WK_t = K_sA' W^{\pm 1}$$

where A' is derived from E_1 and E_2. Hence

$$WR_{ij} = WK_tA\, WFG = K_sA' W^{\pm 1}A\, WFG.$$

But, by §§ 10 and 13, K_s or A' when applied as left hand multiplier permutes the 126 rows. Hence it remains only to prove that W (and hence also W^{-1}) permutes the rows $R_{i1} \equiv A\, WFG$ when applied as a left hand multiplier.

$$WR_{11} \equiv W^2FG = B_1B_2B_3B_4 WB_1B_2B_3B_4FG = B_1B_2B_3B_4 WFG$$
$$= K_5 WFG = R_{15}.$$

$$WR_{61} \equiv WFWFG = FWFWG = FWFG = R_{61} \qquad \text{[by (16)]}.$$

The condition for the identity $WR_{21} = R_{33}$ is that the operator

$$(E_2E_3K_3WF)^{-1}WE_3WF \equiv F(W^2K_3E_3E_2WE_3W)F$$

shall belong to G. To it corresponds in the orthogonal group a substitution which corresponds to

$$S \equiv B_2B_3E_1E_2E_3E_2W^2E_3E_2E_1^2.$$

We proceed to prove that in the abstract group Γ:

$$FSF = W^2K_3E_3E_2WE_3W \equiv T_1.$$

Now

$$\begin{aligned}
FSF &= B_3FE_1E_2E_3 \cdot E_2W^2E_3E_2E_1^2F \\
&= B_3E_1E_2E_3E_1V \cdot E_2W^2E_3E_2E_1^2F && \text{[by (23)]} \\
&= B_3E_1E_2E_3VW^2E_1^2E_2 \cdot E_3E_2E_1^2F && \text{[by (24) and (7)]} \\
&= B_3E_1E_2E_3W^2E_3E_2E_1^2FE_1E_2E_3 \cdot E_2E_3E_2E_1^2F && \text{[by (17) and (23)]} \\
&= B_3E_1E_2E_3W^2E_3E_2E_1^2FE_3E_1F \\
&= B_3E_1E_2E_3W^2E_3E_2E_1E_3 \equiv T_2 && \text{[by (15)]}.
\end{aligned}$$

It remains to prove that the products denoted by T_1 and T_2 are equal. Each belongs to the group G; they will therefore be identical if the corresponding orthogonal substitutions are identical. But to both T_1 and T_2 there corresponds the same orthogonal substitution, viz.,

$$\xi_1' = \xi_1, \quad \xi_2' = -\xi_2 + \xi_3 - \xi_4 + \xi_5, \quad \xi_3' = \xi_2 - \xi_3 - \xi_4 + \xi_5,$$
$$\xi_4' = -\xi_2 - \xi_3 - \xi_4 - \xi_5, \quad \xi_5' = -\xi_2 - \xi_3 + \xi_4 + \xi_5.$$

The condition for the identity $WR_{10\,1} = R_{10\,7}$ is that the product

$$(E_3FB_1B_4WF)^{-1}WE_3FWF$$

shall belong to G. It is satisfied in virtue of relation (18).

In view of the following relations derived from (3) and (8),

$$WE_2 = B_3W, \quad WE_1 = B_3E_2E_1W, \quad WE_1^2 = E_1^2B_3E_2W,$$

$WR_{3\,1}$, $WR_{4\,1}$ and $WR_{5\,1}$ are each of the form

$$DWE_3WFG \equiv DWR_{2\,1} = DR_{3\,3},$$

where D is derived from B_3, E_1, E_2. Also $WR_{7\,1}$, $WR_{8\,1}$, $WR_{9\,1}$ are each of the form

$$DWFWFG \equiv DWR_{6\,1} = DR_{6\,1}.$$

Finally, the products WR_{i1} $(i = 11, 12, 13, 14, 15)$ are each of the form

$$DWE_3FWFG \equiv DWR_{10\,1} = DR_{10\,7}.$$

By the results of §§ 10 and 13, each of the products $DR_{3\,3}$, $DR_{6\,1}$, $DR_{10\,7}$ equals some row R_i or R_{ij}.

§§ 15–17. *Isomorphism and correspondence of generators between the orthogonal and hyperabelian groups.*

15. The simple group $H_{4,3}$ of order 25920, which is derived from the decomposition of the Abelian group of modulus 3 on four indices, is simply isomorphic with the simple subgroup $O_{5,3}$ of the quinary orthogonal group of modulus 3.* We proceed to determine the operators of the former group which correspond to the generators E_1', E_2', E_3', C_1C_2, W' of the latter. We first determine the operators of $O_{5,3}$ which correspond to the generators † $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$, $\bar{B}_4$, $\bar{B}_5$, $\bar{B}$ of $H_{4,3}$ given on pages 65 and 67 of volume 31 of the Proceedings of the London Mathematical Society. Of the two possible forms for $\bar{B}$, we choose that one given by $\gamma_1 \equiv 1 \pmod 3$, viz.,

$$\bar{B} = \pm \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{vmatrix}.$$

By the general correspondence set up in §5 of the paper cited in the foot-note, we find that $\bar{B}$ corresponds to the substitution

$$(31)\qquad \begin{array}{r|ccccc} & \xi_1 & Y_{13} & Y_{14} & Y_{23} & Y_{24} \\ \hline \xi_1' = & 1 & 0 & 0 & 0 & 0 \\ Y_{13}' = & 0 & 1 & 0 & 1 & 0 \\ Y_{14}' = & 0 & 2 & 1 & 2 & 1 \\ Y_{23}' = & 0 & 0 & 0 & 1 & 0 \\ Y_{24}' = & 0 & 0 & 0 & 2 & 1 \end{array}$$

leaving invariant modulo 3 the function

$$\phi \equiv \xi_1^2 + Y_{13}Y_{24} - Y_{14}Y_{23}.$$

* Transactions of the American Mathematical Society, vol. 1, p. 95.

† The substitutions $\bar{B}_i$ enter §15 alone and are to be distinguished from the earlier B_i.

We introduce the new indices

$$(32)\qquad \begin{aligned} \xi_2 &\equiv -Y_{13}-Y_{24}, & \xi_3 &\equiv Y_{13}+Y_{14}+Y_{23}-Y_{24},\\ \xi_4 &\equiv -Y_{13}+Y_{14}+Y_{23}+Y_{24}, & \xi_5 &\equiv -Y_{14}+Y_{23}. \end{aligned}$$

Then

$$\phi \equiv \xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2+\xi_5^2 \pmod 3.$$

Solving modulo 3 the relations (32), we find

$$\begin{aligned} Y_{13} &\equiv \xi_2+\xi_3-\xi_4, & Y_{14} &\equiv \xi_3+\xi_4+\xi_5,\\ Y_{24} &\equiv \xi_2-\xi_3+\xi_4, & Y_{23} &\equiv \xi_3+\xi_4-\xi_5. \end{aligned}$$

Expressing in terms of the new indices ξ_i the substitution (31) to which $\bar{B}$ corresponds, we obtain the result:

$$(33)\qquad \bar{B}\sim B' \equiv C_4C_5(\xi_3\xi_5\xi_4).$$

Proceeding similarly with the substitutions $\bar{B}_1, \cdots, \bar{B}_5$, we find that

$$(34)\qquad \bar{B}_1\sim B_1' \equiv C_3C_4E_3'E_1'E_2'E_3'W'E_3'E_2'E_1'^2E_3'C_2C_5,$$

$$(35)\qquad \bar{B}_2\sim B_2' \equiv C_2C_3C_4C_5,$$

$$(36)\qquad \bar{B}_3\sim B_3' \equiv C_3C_4B_1'C_3C_4,$$

$$(37)\qquad \bar{B}_4\sim B_4' \equiv C_4C_5E_3'B_1'E_3'C_4C_5,$$

$$(38)\qquad \bar{B}_5\sim B_5' \equiv C_1C_3C_4C_5,$$

where the substitutions B_1', B_4' are in matricular notation

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -1\\ 0 & -1 & 0 & 0 & 0\\ 1 & 0 & 1 & -1 & 1\\ 1 & 0 & -1 & 1 & 1\\ -1 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} -1 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & 1 & -1\\ 0 & 1 & 1 & -1 & 1\\ 0 & 1 & -1 & 1 & 1\\ 0 & -1 & 1 & 1 & 1 \end{bmatrix}.$$

We proceed next to express the generators E_1', E_2', E_3', C_1C_2, W' of the orthogonal group $O_{5,3}$ in terms of B', B_1', $\cdots$, B_5', which correspond to the generators of $H_{4,3}$.

$$W' = B_2'B_3'B_4'B_3',\quad E_3' = B'B_5'B'W'B_5'W'B',\quad E_2' = C_1C_5W'B_5'W',$$

$$C_1C_5 = B'W'B_5'W'E_3',\quad E_1' = C_3C_5B_1'B'^2B_3'B'E_3'C_1C_2C_1C_5,$$

$$C_1C_5 = B_2'B_5',\quad C_3C_5 = E_3'C_3C_4E_3',\quad C_3C_4 = B_5'C_1C_5.$$

The corresponding Abelian substitutions have corresponding relations. The substitutions $\bar{W}$, $\bar{W}^2 \equiv \bar{W}^{-1}$; $\bar{E}_3$, $\bar{E}_2$, $\bar{E}_1$; $\bar{C}_1\bar{C}_5$, $\bar{C}_3\bar{C}_4$, $\bar{C}_3\bar{C}_5$, $\bar{C}_1\bar{C}_2$, $\bar{C}_2\bar{C}_3$ are found to be the following:

$$\begin{bmatrix} 0 & -1 & - & \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix};$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

These calculations have been checked in several ways.

16. THEOREM.—*If the hyperabelian group* $HA_{4,3^2}$ *be isomorphic with the orthogonal group* $O_{6,3}$ *in such a manner that the correspondences of* §15 *hold between the operators of their respective subgroups* $H_{4,3}$ *and* $O_{5,3}$, *then the substitution of the group* $HA_{4,3^2}$,

$$(39) \qquad \bar{I} \equiv \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I^{-3} & 0 & 0 \\ 0 & 0 & I^{-1} & 0 \\ 0 & 0 & 0 & I^3 \end{bmatrix},$$

in which the mark I is a suitably chosen root of the congruence

$$x^2 \equiv x + 1 \quad (mod\ 3),$$

must correspond to the substitution $C_1 C_2(\xi_1\xi_6)(\xi_3\xi_4)$ which extends $O_{5,3}$ to $O_{6,3}$.

In the quotient group $HA_{4,3^2}$, every hyperabelian substitution multiplying all four indices by the same factor corresponds to the identity, viz., the powers of

$$(40) \qquad \begin{vmatrix} I^2 & 0 & 0 & 0 \\ 0 & I^2 & 0 & 0 \\ 0 & 0 & I^2 & 0 \\ 0 & 0 & 0 & I^2 \end{vmatrix} \equiv \begin{vmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{vmatrix} = 1,$$

where we have set

$$(41) \qquad I^2 \equiv i, \quad I \equiv i - 1, \quad i^2 \equiv -1 \quad (\text{mod } 3).$$

Then since $I^8 = 1$, one has easily

$$(42) \qquad \bar{I}^2 = \bar{C}_2\bar{C}_3\bar{C}_4\bar{C}_5,$$

so that $\bar{I}$ is of period four. We readily verify the following relations:

$$(43) \qquad \begin{gathered} \bar{I}\bar{C}_3\bar{C}_4 = \bar{C}_3\bar{C}_4\bar{I}, \quad \bar{I}\bar{C}_1\bar{C}_2 = \bar{C}_1\bar{C}_2\bar{I}^{-1}, \quad \bar{I}\bar{C}_1\bar{C}_5 = \bar{C}_1\bar{C}_5\bar{I}^{-1}, \\ \bar{I}\bar{C}_2\bar{C}_3 = \bar{C}_3\bar{C}_5\bar{I}^{-1}, \quad \bar{E}_2\bar{E}_3\bar{I}\bar{E}_2\bar{E}_3 = \bar{I}, \quad \bar{E}_2\bar{E}_1\bar{I}\bar{E}_2\bar{E}_1 = \bar{C}_3\bar{C}_5\bar{I}^{-1}. \end{gathered}$$

Suppose that $O_{6,3}$ contains a substitution I' which combines with the generators E_1', E_2', E_3', C_1C_2, W' of $O_{5,3}$ according to the same laws by which $\bar{I}$ combines with $\bar{E}_1$, $\bar{E}_2$, $\bar{E}_3$, $\bar{C}_1\bar{C}_2$, $\bar{W}$ of $H_{4,3}$. Assume for I' the most general form possible, viz.,

$$I' \equiv \pm \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{16} \\ a_{21} & a_{22} & \cdots & a_{26} \\ \cdot & \cdot & \cdot & \cdot \\ a_{61} & a_{62} & \cdots & a_{66} \end{vmatrix},$$

the simultaneous change of sign of every coefficient leaving I' unchanged.

In virtue of the relation corresponding to (42), viz.,

$$I'^{-1} = I' C_2C_3C_4C_5,$$

we find that

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{61} \\ a_{12} & a_{22} & \cdots & a_{62} \\ \cdot & \cdot & \cdot & \cdot \\ a_{15} & a_{25} & \cdots & a_{65} \\ a_{16} & a_{26} & \cdots & a_{66} \end{bmatrix} = \pm \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{16} \\ -a_{21} & -a_{22} & \cdots & -a_{26} \\ \cdot & \cdot & \cdot & \cdot \\ -a_{51} & -a_{52} & \cdots & -a_{56} \\ a_{61} & a_{62} & \cdots & a_{66} \end{bmatrix}.$$

According to the sign $\pm$, we find for I'_+, I'_- the respective values:

$$\pm \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & a_{16} \\ 0 & 0 & a_{23} & a_{24} & a_{25} & 0 \\ 0 & -a_{23} & 0 & a_{34} & a_{35} & 0 \\ 0 & -a_{24} & -a_{34} & 0 & a_{45} & 0 \\ 0 & -a_{25} & -a_{35} & -a_{45} & 0 & 0 \\ a_{16} & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix}, \quad \pm \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a_{16} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & 0 \\ 0 & a_{23} & a_{33} & a_{34} & a_{35} & 0 \\ 0 & a_{24} & a_{34} & a_{44} & a_{45} & 0 \\ 0 & a_{25} & a_{35} & a_{45} & a_{55} & 0 \\ -a_{16} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By the relation corresponding to the first relation (43), I' must be commutative with C_3C_4. We find in consequence for I'_+, I'_- the values:

$$\pm \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & a_{16} \\ 0 & 0 & 0 & 0 & a_{25} & 0 \\ 0 & 0 & 0 & a_{34} & 0 & 0 \\ 0 & 0 & -a_{34} & 0 & 0 & 0 \\ 0 & -a_{25} & 0 & 0 & 0 & 0 \\ a_{16} & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix}, \quad \pm \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a_{16} \\ 0 & a_{22} & 0 & 0 & a_{25} & 0 \\ 0 & 0 & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{34} & a_{44} & 0 & 0 \\ 0 & a_{25} & 0 & 0 & a_{55} & 0 \\ -a_{16} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

The resulting substitution I'_+ does not satisfy the relation corresponding to (43),

$$I'C_1C_2 = C_1C_2I'^{-1},$$

and is therefore excluded. And I'_- satisfies it if and only if $a_{25} = 0$.

The relation $I'C_1C_5 = C_1C_5I'^{-1}$ is then satisfied by I'_-. But the relation:

$$I'C_2C_3 = C_3C_5I'^{-1},$$

requires that $a_{33} = a_{44} = 0$ in I'_-. The relations:

$$E'_2E'_3I'_-E'_2E'_3 = I'_-, \quad E'_2E'_1I'_-E'_2E'_1 = C_3C_5I'^{-1}_-,$$

require in succession that $a_{34} = a_{55}$, $a_{34} = -a_{22}$. Evidently $a_{16} \not\equiv 0$, so that $a_{16} \equiv \pm 1 \pmod 3$. By suitable choice of the sign $\pm$ in front of the matrix for I'_-, we may take $a_{16} = +1$. Setting, for brevity, $a_{34} = \gamma$, we have

$$I'_- \equiv \pm \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, according as $\gamma = \pm 1$, we have

$$I'^{\pm 1}_- = C_1 C_2 (\xi_1 \xi_6)(\xi_3 \xi_4).$$

But, by (39),

$$\bar{I}^{-1} \equiv \bar{I}^3 = \begin{bmatrix} I^3 & 0 & 0 & 0 \\ 0 & I^{-9} & 0 & 0 \\ 0 & 0 & I^{-3} & 0 \\ 0 & 0 & 0 & I^9 \end{bmatrix}$$

is derived from $\bar{I}$ by replacing I by I^3. But I is defined as one root of the irreducible congruence:

$$I^2 \equiv I + 1 \pmod 3,$$

whose second root is I^3. Hence, by a proper choice of notation for the root I, we may set

$$I'_- = C_1 C_2 (\xi_1 \xi_6)(\xi_3 \xi_4) = C_1 C_2 F' E_1'^2 E_2'.$$

Hence

$$F' = C_1 C_2 I'_- E_2' E_1'.$$

It follows that to F' must correspond the hyperabelian substitution

$$(44) \qquad \bar{F} \equiv \bar{C}_1 \bar{C}_2 \bar{I} \bar{E}_2 \bar{E}_1 \equiv \begin{bmatrix} 0 & 0 & -I & 0 \\ 0 & 0 & -I & I \\ I^3 & 0 & 0 & 0 \\ I^3 & -I^3 & 0 & 0 \end{bmatrix}.$$

17. THEOREM.—*The hyperabelian group* $HA_{4,\,3^2}$ *is holoedrically isomorphic* with the abstract group* Γ.

From the simple isomorphism of $H_{4,\,3}$ with G, it follows that the operators $\bar{E}_1, \bar{E}_2, \bar{E}_3, \bar{B}_1 \equiv \bar{C}_1\bar{C}_2, \bar{W}$ satisfy the generational relations (1), $\cdots$, (6) of G, a result capable of verification by direct calculation. It therefore remains only to prove that the operators $\bar{F}, \bar{E}_1, \bar{E}_2, \bar{B}_1 \equiv \bar{C}_1\bar{C}_2, \bar{W}$ satisfy the generational relations (12), $\cdots$, (18). This result may be verified by simple calculations. We note the auxiliary formulæ:

$$C_4C_5 = \begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad E_3WE_3 = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix},$$

$$W^2C_1C_2C_4C_5 = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & -i & 0 \\ 1 & -1 & -i & i \\ i & 0 & -1 & 0 \\ i & -i & -1 & 1 \end{pmatrix}.$$

We have therefore proved that the simple groups $HA_{4,\,3^2}$ and $O_{6,\,3}$ are holoedrically isomorphic.

UNIVERSITY OF TEXAS,
January 12, 1900.

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*Addition: May 5, 1900. In the May number of the Bulletin of the Society the writer establishes the holoedric isomorphism of $O_{6,\,p^n}$ and $HA_{4,\,p^{2n}}$ for any p^n of the form $4l-1$. As the method there used consists in the transformation of the defining invariant of the former group into that of the second compound of the latter group, it gives no direct knowledge of the correspondences of the generators of the isomorphic groups. For $p^n = 3$, §§ 15–16 of the present paper enable us to pass readily from an arbitrary substitution of either group to the corresponding substitution of the other.