NOTE ON NON-QUATERNION NUMBER SYSTEMS*

BY

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SCHEFFERS† has divided all number systems into the quaternion and non-quaternion systems and has shown that the n fundamental units of a non-quaternion system may be so chosen that the multiplication table takes a particularly simple form, which is in turn characteristic of the non-quaternion systems. In this paper I shall show that the choice of the units may be so regulated that the multiplication table becomes still simpler.

SCHEFFER'S form, which we shall call the *regular form*, has the following characteristic properties:

1°. The units are divided into two essentially different classes, the e's and the η's, with the notation:

$$e_1, e_2, \cdots, e_r; \quad \eta_1, \eta_2, \cdots, \eta_s \qquad (r+s=n).$$

2°. The sum of the η's is the modulus (or idemfactor).

3°. As to the products $\eta_\lambda \eta_\mu$, we have the formulas:

$$\eta_\lambda^2 = \eta_\lambda, \quad \eta_\lambda \eta_\mu = 0 \qquad (\lambda, \mu = 1, \cdots, s; \lambda \neq \mu).$$

4°. For each e_k there is one and only one η_λ such that

$$\eta_\lambda e_k = e_k;$$

and for each e_k there is one and only one η_μ such that

$$e_k \eta_\mu = e_k.$$

The particular units η_λ, η_μ thus related to e_k may or may not be distinct. The unit e_k is said to have the *character* (λ, μ).

Every other product of an η with e_k is zero.

5°. The product of two e's contains only e's and, moreover, only e's of lower index than either of the factors—that is, with certain constants γ_{ijs},

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$$\begin{cases} e_i e_j = \sum_{s=1}^{i-1} \gamma_{ijs} e_s & (i \leqq j), \\ e_i e_j = \sum_{s=1}^{j-1} \gamma_{ijs} e_s & (i > j). \end{cases}$$

We note as of especial importance the particular case:

$$e_1 e_i = e_i e_1 = 0 \qquad (i = 1, \cdots, r).$$

6°. (Corollary of 4° and 5°.) If e_i is of character (λ, μ) and e_j of character (μ, ν), the product $e_i e_j$ can contain units of character (λ, ν) only. If e_i is of character (λ, μ) and e_j of character (ρ, σ) where $\rho \neq \mu$, the product $e_i e_j$ is zero.

We shall for the present consider a regular system from which the η's have been deleted, that is, the part consisting of the e's and their products with each other only—this is called a *degenerate system.** A degenerate system is said to be regular if it is in a form to satisfy 5°.

The sequel depends essentially upon the following linear ordering of the products† of the multiplication table.

(*a*) The products are divided into groups and the groups themselves ordered, as follows:

The first group contains all products $e_1 e_j$ and $e_j e_1$ $(j = 1, 2, \cdots, r)$:
The second group contains all products $e_2 e_j$ and $e_j e_2$ $(j = 2, 3, \cdots, r)$:
The k-th group contains all products $e_k e_j$ and $e_j e_k$ $(j = k, k+1, \cdots, r)$:
The $(r-1)$-th group contains all products $e_{r-1} e_j$ and $e_j e_{r-1}$ $(j = r-1, r)$:
The r-th group contains $e_r e_r$.‡

(*b*) Within a group, say the i-th, the products are ordered thus:

$$e_i e_i, \; e_i e_{i+1}, \; e_{i+1} e_i, \; e_i e_{i+2}, \; e_{i+2} e_i, \; \cdots, \; e_i e_r, \; e_r e_i.$$

In any system § there are certain products, in number m $(0 \leqq m < r)$, each of which is linearly independent of the products preceding it in order. These products, which are of fundamental importance to us, will be called *independent products;* they will be denoted in order of precedence by

$$\pi_1, \; \pi_2, \; \pi_3, \; \cdots, \; \pi_m.$$

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* SCHEFFERS'S "ausgeartetes System," loc. cit., p. 308.

† "Product" will henceforth be used to denote the product of two e's unless otherwise stated.

‡ If the multiplication table be written in rectangular array, the first group consists of the first line and column, the second group of the part of the second line and column not contained in the first, etc.

§ "System" will mean "degenerate system" while such systems alone are being considered.

To repeat: π_1 is the first product different from zero; π_2 is the first product which cannot be expressed in terms of π_1; π_3 is the first product which cannot be expressed in terms of π_1 and π_2; and so on.

The set of "independent products" is a particular set of linearly independent quadratic products of the system, of which every set then contains m elements. Of course, under a transformation of the system the number m is invariant but the set of "independent products" need not be.

A system whose multiplication table contains m independent products will be called an *m-product* system.

Theorem I. *Any m-product degenerate non-quaternion number system can be so transformed that* (1) *the transformed system will be regular;* (2) *the m independent products of the transformed system will be the first m units of that system;* (3) *the order of equivalence of independent products and units will be:*

$$\pi_1 = e_1, \ \pi_2 = e_2, \ \cdots, \ \pi_m = e_m .$$

The form indicated by this theorem will be called the *normal form.*

Proof. Assuming that any $(m-1)$-product system can be reduced to the normal form, we shall prove that then any m-product system can be reduced to that form.

Let any regular m-product system be transformed by taking the independent products as the first m units; that is, let

$$\bar{e}_1 = \pi_1, \ \bar{e}_2 = \pi_2, \ \cdots, \ \bar{e}_m = \pi_m,$$

the choice of the remaining new units,

$$\bar{e}_{m+1}, \ \bar{e}_{m+r}, \ \cdots, \ \bar{e}_r,$$

being subject to the usual restrictions in the transformation of a number system, and also to the restriction* that only units of like character shall be united in a transformed unit. We proceed to prove that *the transformed system is regular.*

In the first place, in the transformed system the first m units, and these only, will enter the products. For, the transformed system is an m-product system; hence its products must contain at least m units; again the transformed units are linear combinations of the original units and consequently any product $\bar{e}_i \bar{e}_j$ can be expressed in terms of $\pi_1, \pi_2, \cdots, \pi_m$, that is, in terms of $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_m$. Hence, regularity is preserved in so far as products $\bar{e}_i \bar{e}_j$ $(i > m, j > m)$ are concerned.

It remains to consider the products $\bar{e}_i \bar{e}_j$, where in each product one index is less than or equal to m, that is, the products

$$\pi_i \bar{e}_j, \ \bar{e}_j \pi_i \qquad (i = 1, \cdots, m; j = 1, \cdots, r; i \leqq j)$$

* For the units $\bar{e}_{m+1}, \cdots, \bar{e}_r$ one may indeed choose certain $r - m$ of the original units.

We consider one such pair of products $\pi_i \bar{e}_j$, $\bar{e}_j \pi_i$. In the original system, we have

$$\pi_i = e_s e_t = a_1 e_1 + a_2 e_2 + \cdots + a_{u-1} e_{u-1} \qquad (u \leqq s,\ u \leqq t).$$

Any product in the original system, one of whose factors consists of units of index lower than u precedes the independent product

$$\pi_i = e_s e_t \qquad (u \leqq s,\ u \leqq t),$$

and so, by the definition of independent products, is expressible linearly in terms of $\pi_1, \pi_2, \cdots, \pi_{i-1}$. Hence either product of $\bar{e}_i = \pi_i$ with $\bar{e}_j$ $(j \geqq i)$ can contain only $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{i-1}$. Hence we see that the transformed system is regular.

We now have an m-product regular system in which the units entering products are m in number and are the first m units of the system.* The unit $\bar{e}_1$ being deleted, a regular system remains in which there are $m - 1$ units entering products, and $m - 1$ independent products. In accordance with our assumption the deleted system may be reduced to the normal form; let the reduction be made and denote the new units by e_2, $e_3 \cdots e_r$, the new independent products by π'_1, $\pi'_2, \cdots, \pi'_{m-1}$. Since the deleted system is normal,

$$\pi'_1 = e_2,\ \pi'_2 = e_3,\ \cdots,\ \pi'_{m-1} = e_m.$$

On the restoration of $\bar{e}_1$, which will now be denoted by e_1, we have the original m-product system which we had before deletion but in a new form; it will evidently still be regular.

Since the products $e_1 e_i$, $e_i e_1$ all vanish, if $e_j e_k = e_i$ was an independent product in the deleted system,

$$e_j e_k = e_i + a_i e_1$$

will be an independent product now. Then $m - 1$ of the independent products will be

$$e_2 + a_2 e_1,\ e_3 + a_3 e_1,\ \cdots,\ e_m + a_m e_1.$$

Before searching for the remaining independent product let us make the transformation:

$$\bar{e}_2 = e_2 + a_2 e_1,\ \bar{e}_3 = e_3 + a_3 e_1,\ \cdots,\ \bar{e}_m = e_m + a_m e_1,$$

the remaining units being unchanged. The regularity has not been disturbed and the independent products remain unchanged.

Denoting the transformed units by e_2, e_3, $\cdots$, e_r, we have as $m - 1$ of the independent products the units e_2, e_3, $\cdots$, e_m. The only other unit which enters any product is e_1; hence, the first product which contains e_1 is the remaining independent product; denote this product $e_g e_h$ by π_t.

* We have by this first transformation made a twofold simplification of the regular form of the system.

If e_i is the independent product immediately preceding π_t,

$$e_g e_h = \pi_t = a_1 e_1 + a_2 e_2 + \cdots + a_i e_i \qquad (a_1 \neq 0\,;\ i < g,\ i < h)\,.$$

Now we make the transformation:

$$\bar{e}_1 = e_2\,,\ \bar{e}_2 = e_3\,,\ \cdots,\ \bar{e}_{i-1} = e_i\,,\ \bar{e}_i = \pi_t\,,\ \bar{e}_{i+1} = e_{i+1}\,,\ \cdots,\ \bar{e}_r = e_r\,,$$

and prove that the transformed system will be regular and that its independent products will be

$$\pi_1 = \bar{e}_1\,,\ \pi_2 = \bar{e}_2\,,\ \cdots,\ \pi_{m-1} = \bar{e}_{m-1}\,,\ \pi_m = \bar{e}_m\,,$$

that is, that the transformed system is in the normal form.

The transformation in effect omits the unit e_1 and leaving the remaining units in their original order introduces π_t as the new unit $\bar{e}_i$. The products of the new system are the products of the old (apart from the vanishing e_1-products) and the products involving $\pi_t = \bar{e}_i$. Of these all not involving $\bar{e}_i$ obviously have regular expressions, since in the original system the first product involving e_1 was $e_g e_h = \pi_t = \bar{e}_i$ $(g > i\,,\ h > i)$, and so all products $\bar{e}_k \bar{e}_l$ involving e_1 have $k > i\,,\ l > i$ and may be expressed regularly, the e_1 being replaced by $\bar{e}_i,\ \bar{e}_1,\ \cdots,\ \bar{e}_{i-1}$. And it is easy to see also that the products involving $\bar{e}_i$ are expressible regularly.

Further, in the new system $\bar{e}_1,\ \cdots,\ \bar{e}_{i-1}$ arise as before as sequential independent products from the products preceding the product $e_g e_h$, the products involving $\bar{e}_i$ as a factor obviously being expressible in terms of the products involving $\bar{e}_1,\ \cdots,\ \bar{e}_{i-1}$. Then, as the next independent product,

$$\pi_t = e_g e_h = \bar{e}_i = \bar{e}_g \bar{e}_h \qquad (i < g,\ i < h)$$

enters, and by the intervention of the $\bar{e}_i$, with $\bar{e}_1,\ \cdots,\ \bar{e}_{i-1}$ to replace the original e_1, the $\bar{e}_{i+1},\ \cdots,\ \bar{e}_m$ enter as before as sequential independent products. Thus after the transformation the independent products in order are $\bar{e}_1,\ \cdots,\ \bar{e}_m$.

Thus the general step of the induction proof is established, and it is evident that a zero-product system is normal; hence we have proved that *every system can be brought into the normal form.*

THEOREM II. *The product of two independent products is not an independent product.*

Let e_i and e_j $(i < j)$ be independent products in a normal system and let $e_p e_q = e_j$.

Then

$$e_i e_j = (e_i e_p) e_q = (a_1 e_1 + a_2 e_2 + \cdots + a_{i-1} e_{i-1}) e_q\,.$$

Hence $e_i e_j$ is equal to a sum of products which precede it, and cannot be an independent product.

We have thus far considered the degenerate system, and have defined the normal form for that alone. We call the complete non-quaternion system normal if the system is regular and the degenerate system normal. It is necessary to show that the transformations employed have not destroyed regularity with respect to the products of the e's with the η's in order to extend our results to the complete system.

The regularity of the complete system could be destroyed by transformations of the degenerate system which maintain its own regularity in one way only; that is by uniting in a transformed unit, two or more units of different characters.

In the first transformation of the degenerate system it was explicitly stated that the units united in a transformed unit should all be of the same character.

The other transformations which united two or more units in a transformed unit were made by means of equations of the form:

$$\bar{e}_i = e_j e_k,$$

and all such transformations are permissible, for one sees by 6° that the units contained in a product $e_i e_j$ must all be of the same character. Hence we have:

THEOREM III. *Any non-quaternion system may be reduced to the normal form.*

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