NOTE ON THE FUNCTIONS OF THE FORM

$f(x) \equiv \phi(x) + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$

WHICH IN A GIVEN INTERVAL DIFFER THE LEAST POSSIBLE FROM ZERO*

BY

H. F. BLICHFELDT

TSCHEBYCHEFF has considered the following problem: $f(x)$ is a given function of x and of n arbitrary constants $a_1, a_2, \cdots, a_n$, and is, together with its partial derivatives with respect to $x, a_1, a_2, \cdots, a_n$, continuous in the interval $a \leqq x \leqq b$; to determine the constants $a_1, a_2, \cdots, a_n$ so that the greatest value of $[f(x)]^2$, in the same interval, shall differ as little as possible from zero. †

His solution is as follows:

For any given set of values of the constants $a_1, a_2, \cdots, a_n$, let $x_1, x_2, \cdots, x_\mu$ be all the different values of x for which $[f(x)]^2$, in the interval $a \leqq x \leqq b$, reaches its greatest value L^2. Then must $x_1, x_2, \cdots, x_\mu$ evidently satisfy the two equations

$$\text{(I)} \qquad [f(x)]^2 - L^2 = 0, \quad (x-a)(x-b)f'(x) = 0.$$

If it is now possible to satisfy the μ equations

$$\text{(II)} \qquad \sum_{k=1}^{n} \frac{\partial f(x_i)}{\partial a_k} N_k = f(x_i) = \pm L \qquad (i=1, 2, \cdots, \mu),$$

by finite, determinate values of the n quantities $N_1, \cdots, N_n$, then it will be possible to give to the constants $a_1, \cdots, a_n$ such small increments, proportional to $N_1, \cdots, N_n$, that the greatest absolute value of $f(x)$, for $a \leqq x \leqq b$, becomes less than L, which is taken positive.

---

* Presented to the Society (Chicago) April 14, 1900, under a slightly different title. Received for publication November 21, 1900.

† *Sur les questions de minima qui se rattachent à la représentation approximative des fonctions*, Pétersbourg Mémoires, series 6, vol. 7. The above statement of TSCHEBYCHEFF'S results is taken from J. BERTRAND, *Calcul Différentiel*, p. 512.

Consequently, the set of constants $a_1, \cdots, a_n$, for which the greatest absolute value of $f(x)$ in the given interval is the least possible, must be such that the μ equations (II) are inconsistent for finite values of the constants $N_1, \cdots, N_n$.

This requires, in many cases, that $\mu > n$, only. In particular, if

$$f(x) \equiv x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n,$$

and the interval is $-h \leqq x \leqq +h$, the necessary and sufficient condition imposed upon $x_1, \cdots, x_n$ is that the equations (I),

$$[f(x)]^2 - L^2 = 0, \quad (x^2 - h^2) f'(x) = 0,$$

possess $n+1$ different, common solutions. These solutions are therefore $-h$, $+h$, and all the $n-1$ roots of $f'(x) = 0$, which last are obviously double roots of $[f(x)]^2 - L^2 = 0$. Hence, as $f'(x) = nx^{n-1} +$ etc., we must have identically

$$[f(x)]^2 - L^2 = x^{2n} + \text{etc.} = (x^2 - h^2)\left(\frac{f'(x)}{n}\right)^2,$$

and therefore

$$\frac{f'(x)}{\sqrt{L^2 - [f(x)]^2}} = \frac{n}{\sqrt{h^2 - x^2}},$$

from which we get by integration,

$$f(x) = \frac{h^n}{2^{n-1}} \cos n \cos^{-1}\left(\frac{x}{h}\right) = \frac{1}{2^n}\{[x + \sqrt{x^2 - h^2}]^n + [x - \sqrt{x^2 - h^2}]^n\},$$

the values of L and the constant of integration being readily determined. The result just given is in accord with that of BERTRAND obtained (loc. cit., pp. 514–518) by lengthy considerations involving the theory of continued fractions. *

In general, several sets of constants $a_1, \cdots, a_n$ can be found to satisfy TSCHEBYCHEFF'S conditions. If the function $f(x)$ involves the constants in the following manner:

$$f(x) \equiv \phi(x) + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n,$$

the labor of selecting the required set is much simplified by the following considerations.

Let us by the "maxima" of $[f(x)]^2$ in the interval $a \leqq x \leqq b$ understand those only which are equal to L^2, the greatest value of $[f(x)]^2$ in the given interval. If we classify these maxima as *positive* or *negative* according as the corresponding value of $f(x)$ is $+L$ or $-L$, and plot the curve $y = f(x)$, *we*

* In Liouville's Journal, ser. 2, vol. 19 (1874), pp. 319-347, TSCHEBYCHEFF has solved this problem with the additional condition imposed upon $f(x)$, that it either never decreases or never increases in the given interval.

*shall find at least n alternations of the two kinds of maxima in the given interval.**

The curve $y' = \beta_1 x^{n-1} + \beta_2 x^{n-2} + \cdots + \beta_n$ is continuous and can be made to cross the axis of X at $n-1$ given points. If the curve $y = f(x) \equiv \phi(x) + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$, whose maxima are assumed to have their least possible absolute value, had less than n alternations of the two kinds of maxima, we could construct a curve $y' = \beta_1 x^{n-1} + \beta_2 x^{n-2} + \cdots + \beta_n$ which would have positive ordinates whenever $y = f(x)$ possessed negative maxima, and vice versa, at the same time making the largest of its ordinates in the given interval as small as we please. The maxima of the curve

$$y = f(x) + y' \equiv \phi(x) + (a_1 + \beta_1)x^{n-1} + (a_2 + \beta_2)x^{n-2} + \cdots + (a_n + \beta_n)$$

would thus be less in absolute value than those of $y = f(x)$, contrary to hypothesis.

For instance, the nearest approximation to $\sin x$ of the form $a_1 x + a_2$ in the interval $0 \leqq x \leqq h \leqq \pi/2$ must be such that the curve $y = \sin x - a_1 x - a_2$ shall have two positive maxima including one negative, or vice versa; there being just three maxima in the given interval, namely, for $x = 0$, $\cos^{-1} a_1$, h. These maxima, being given by

$$-a_2, \quad +\sqrt{1 - a_1^2} - a_1 \cos^{-1} a_1 - a_2, \quad \sin h - a_1 h - a_2,$$

must therefore satisfy the relations:

$$(-a_2) + (+\sqrt{1 - a_1^2} - a_1 \cos^{-1} a_1 - a_2) = 0,$$
$$-(-a_2) + (\sin h - a_1 h - a_2) = 0,$$

so that

$$a_1 = \frac{1}{h} \sin h, \quad a_2 = +\tfrac{1}{2}\sqrt{1 - \left(\frac{\sin h}{h}\right)^2} - \frac{\sin h}{2h} \cos^{-1}\left(\frac{\sin h}{h}\right).$$

The approximation is, therefore,

$$\frac{\sin h}{h} x + \tfrac{1}{2}\sqrt{1 - \left(\frac{\sin h}{h}\right)^2} - \frac{\sin h}{2h} \cos^{-1}\left(\frac{\sin h}{h}\right).$$

STANFORD UNIVERSITY, CAL.

* The writer has not access to the original memoirs of TSCHEBYCHEFF, in which this property may have been indicated.