ON THE DETERMINATION OF SURFACES CAPABLE OF CONFORMAL REPRESENTATION UPON THE PLANE IN SUCH A MANNER THAT GEODETIC LINES ARE REPRESENTED BY ALGEBRAIC CURVES*

BY

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Introduction.

BELTRAMI has shown † that surfaces of constant curvature can be built upon the plane in such a manner that the geodetic lines shall go over into straight lines, and that this result is true for no other surface. He considered this as the simplest case of building one surface upon another in such a manner that the geodetic lines of one surface shall go over into the geodetic lines of the other. The general question was later solved by DINI. ‡ It is an immediate consequence of BELTRAMI'S memoirs that surfaces of constant curvature are the only surfaces that can be built conformally upon the plane in such a manner that the geodetic lines shall go over into straight lines or arcs of circles. § This latter fact suggests a generalization of BELTRAMI'S problem different from the one which he had in mind, i. e., to so build a surface conformally upon the plane in such a manner that the geodetic lines shall go over into algebraic curves. It is proposed to consider that question.

§ 1.

We consider a doubly infinite system of algebraic curves in the plane:

$$(1) \qquad F_3(x, y) + AF_2(x, y) + BF_1(x, y) = 0,$$

where A and B are the parameters of the system.

We think of the surface as given by

$$x_i = \phi_i(\mu, \nu) \qquad (i=1, 2, 3),$$

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† Annali di Matematica, vol. 7, 1866.

‡ Annali di Matematica, ser. 2, vol. 3, 1870.

§ For an independent proof of this proposition see F. BUSSE, *Inaugural Dissertation*, Berlin, 1896.

where μ and ν are rectangular surface coördinates, i. e., such that the systems of curves $\mu=$ const., $\nu=$ const., intersect orthogonally.

Let the relation between the surface and the plane be given by

$$x=\psi_1(\mu,\nu), \quad y=\psi_2(\mu,\nu).$$

Then we write $f_i(\mu,\nu)$ for $F_i(\psi_1,\psi_2)$ and equation (1) becomes

$$f_3(\mu,\nu)+Af_2(\mu,\nu)+Bf_1(\mu,\nu)=0, \tag{2}$$

and from this we find:

$$f_3'(\mu,\nu)+Af_2'(\mu,\nu)+Bf_1'(\mu,\nu)=0, \tag{3}$$

$$f_3''(\mu,\nu)+Af_2''(\mu,\nu)+Bf_1''(\mu,\nu)=0, \tag{4}$$

where

$$f_i'(\mu,\nu)=\frac{\partial f_i(\mu,\nu)}{\partial\mu}d\mu+\frac{\partial f_i(\mu,\nu)}{\partial\nu}d\nu,$$

$$f_i''(\mu,\nu)=\frac{\partial^2 f_i(\mu,\nu)}{\partial\mu^2}d\mu^2+2\frac{\partial^2 f_i(\mu,\nu)}{\partial\mu\partial\nu}d\mu d\nu+\frac{\partial^2 f_i(\mu,\nu)}{\partial\nu^2}d\nu^2$$
$$+\frac{\partial f_i(\mu,\nu)}{\partial\mu}d^2\mu+\frac{\partial f_i(\mu,\nu)}{\partial\nu}d^2\nu.$$

From (2), (3) and (4) we obtain

$$\begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}=0. \tag{5}$$

We write

$$n_{i1}=\frac{\partial f_i}{\partial\mu}, \quad n_{i2}=\frac{\partial f_i}{\partial\nu}; \quad m_{i1}=\frac{\partial^2 f_i}{\partial\mu^2}, \quad m_{i2}=\frac{\partial^2 f_i}{\partial\nu^2}, \quad m_{i3}=\frac{\partial^2 f_i}{\partial\mu\partial\nu}.$$

Making use of these abbreviations, substituting for f_i' and f_i'' their values, and expanding in terms of the first row, we have from equation (5):

$$f_1\begin{vmatrix} n_{21}d\mu+n_{22}d\nu & m_{21}d\mu^2+m_{22}d\nu^2+2m_{23}d\mu d\nu+n_{21}d^2\mu+n_{22}d^2\nu \\ n_{31}d\mu+n_{32}d\nu & m_{31}d\mu^2+m_{32}d\nu^2+2m_{33}d\mu d\nu+n_{31}d^2\mu+n_{32}d^2\nu \end{vmatrix}$$
$$-f_2\begin{vmatrix} n_{11}d\mu+n_{12}d\nu & m_{11}d\mu^2+m_{12}d\nu^2+2m_{13}d\mu d\nu+n_{11}d^2\mu+n_{12}d^2\nu \\ n_{31}d\mu+n_{32}d\nu & m_{31}d\mu^2+m_{32}d\nu^2+2m_{33}d\mu d\nu+n_{31}d^2\mu+n_{32}d^2\nu \end{vmatrix}$$
$$+f_3\begin{vmatrix} n_{11}d\mu+n_{12}d\nu & m_{11}d\mu^2+m_{12}d\nu^2+2m_{13}d\mu d\nu+n_{11}d^2\mu+n_{12}d^2\nu \\ n_{21}d\mu+n_{22}d\nu & m_{21}d\mu^2+m_{22}d\nu^2+2m_{23}d\mu d\nu+n_{21}d^2\mu+n_{22}d^2\nu \end{vmatrix}=0,$$

or

$$f_1\theta(23)-f_2\theta(13)+f_3\theta(12)=0,$$

where

$$\theta(23) = \begin{vmatrix} n_{21}m_{21} \\ n_{31}m_{31} \end{vmatrix} d\mu^3 + \begin{vmatrix} n_{21}m_{22} \\ n_{31}m_{32} \end{vmatrix} d\mu d\nu^2 + 2\begin{vmatrix} n_{21}m_{23} \\ n_{31}m_{33} \end{vmatrix} d\mu^2 d\nu + \begin{vmatrix} n_{21}n_{22} \\ n_{31}n_{32} \end{vmatrix} d\mu d^2\nu$$

$$+ \begin{vmatrix} n_{22}m_{21} \\ n_{32}m_{31} \end{vmatrix} d\mu^2 d\nu + \begin{vmatrix} n_{22}m_{22} \\ n_{32}m_{32} \end{vmatrix} d\nu^3 + 2\begin{vmatrix} n_{22}m_{23} \\ n_{32}m_{33} \end{vmatrix} d\mu d\nu^2 + \begin{vmatrix} n_{22}n_{21} \\ n_{32}n_{31} \end{vmatrix} d\nu d^2\mu ,$$

with corresponding expressions for $\theta(13)$, $\theta(12)$. Adding like terms, we have for terms in $d\mu^3$:

$$\left\{ f_1(\mu, \nu) \begin{vmatrix} n_{21}m_{21} \\ n_{31}m_{31} \end{vmatrix} - f_2(\mu, \nu) \begin{vmatrix} n_{11}m_{11} \\ n_{31}m_{31} \end{vmatrix} + f_2(\mu, \nu) \begin{vmatrix} n_{11}m_{11} \\ n_{21}m_{21} \end{vmatrix} \right\} d\mu^3 ,$$

or

$$(6) \qquad \begin{vmatrix} f_1(\mu, \nu) & n_{11} & m_{11} \\ f_2(\mu, \nu) & n_{21} & m_{21} \\ f_3(\mu, \nu) & n_{31} & m_{31} \end{vmatrix} d\mu^3 .$$

And we easily see that we have like determinants for the coefficients of the other terms. Indeed if we write [11] for the coefficient of $d\mu^3$, our differential equation is:

$$(7) \qquad [11] d\mu^3 + [22] d\nu^3 + \{2[13] + [21]\} d\mu^2 d\nu + \{2[23] + [12]\} d\mu d\nu^2 + \Delta(d\mu d^2\nu - d\nu d^2\mu) = 0 ,$$

where Δ is the determinant $|f_1 n_{21} n_{32}|$.

For shortness we write equation (7) in the form:

$$(8) \qquad a_1 d\mu^3 + a_4 d\nu^3 + a_2 d\mu^2 d\nu + a_3 d\mu d\nu^2 + a_5(d\mu d^2\nu - d\nu d^2\mu) = 0 .$$

§ 2.

The geodetic lines of our surface are given by the differential equation:

$$(9) \qquad \tfrac{1}{2} G \frac{\partial G}{\partial \mu} d\nu^3 - \tfrac{1}{2} E \frac{\partial E}{\partial \nu} d\mu^3 + \left[E \frac{\partial G}{\partial \mu} - \tfrac{1}{2} G \frac{\partial E}{\partial \mu} \right] d\mu^2 d\nu + \left[G \frac{\partial E}{\partial \nu} - \tfrac{1}{2} E \frac{\partial G}{\partial \nu} \right] d\mu d\nu^2 + EG(d\mu d^2\nu - d\nu d^2\mu) = 0 ;$$

we are to compare this with equation (8). This comparison gives the following system of partial differential equations:

$$(10) \qquad \begin{gathered} E\frac{\partial G}{\partial \mu} - \tfrac{1}{2} G \frac{\partial E}{\partial \mu} = \lambda a_2, \quad G\frac{\partial E}{\partial \nu} - \tfrac{1}{2} E \frac{\partial G}{\partial \nu} = -\lambda a_3, \\ \tfrac{1}{2} G \frac{\partial G}{\partial \mu} = \lambda a_4, \quad \tfrac{1}{2} E \frac{\partial E}{\partial \nu} = -\lambda a_1, \quad EG = \lambda a_5 . \end{gathered}$$

From this system we eliminate λ by means of the last equation and obtain:

$$(11)\qquad \begin{gathered} E\frac{\partial G}{\partial \mu} - \tfrac{1}{2} G \frac{\partial E}{\partial \mu} = EG\frac{a_2}{a_5}, \quad G\frac{\partial E}{\partial \nu} - \tfrac{1}{2} E\frac{\partial G}{\partial \nu} = -EG\frac{a_3}{a_5}, \\ \tfrac{1}{2} G\frac{\partial G}{\partial \mu} = EG\frac{a_4}{a_5}, \quad \tfrac{1}{2} E\frac{\partial E}{\partial \nu} = -EG\frac{a_1}{a_5}. \end{gathered}$$

From these we have:

$$(12)\qquad \begin{aligned} &1. && \frac{\partial \log G}{\partial \mu} - \frac{1}{2}\frac{\partial \log E}{\partial \mu} = \frac{a_2}{a_5}, \\ &2. && \frac{\partial \log E}{\partial \nu} - \frac{1}{2}\frac{\partial \log G}{\partial \nu} = -\frac{a_3}{a_5}, \\ &3. && \frac{\partial \log E}{\partial \nu} = -2\frac{a_1}{a_5}\frac{G}{E}, \\ &4. && \frac{\partial \log G}{\partial \mu} = 2\frac{a_4}{a_5}\frac{E}{G}. \end{aligned}$$

Integrating 1 and 2 we have:

$$(13)\qquad \begin{aligned} &1. && \log G - \tfrac{1}{2}\log E = \log F_1 + \log V_1, \\ &2. && \log E - \tfrac{1}{2}\log G = -\log F_2 + \log U_1, \end{aligned}$$

where $F_1 = e^{\int \frac{a_2}{a_5} d\mu}$ and $F_2 = e^{\int \frac{a_3}{a_5} d\mu}$; U_1 is a function of μ only and V_1 of ν only. Then (13) gives

$$(14)\qquad \begin{aligned} &1. && \frac{G}{E^{\frac{1}{2}}} = F_1 V_1, \\ &2. && \frac{E}{G^{\frac{1}{2}}} = F_2^{-1} U_1. \end{aligned}$$

Squaring each of these equations and multiplying it by the other we find

$$(15)\qquad \begin{aligned} &1. && E = U_1^{\frac{4}{3}} V_1^{\frac{2}{3}} F_1^{\frac{2}{3}} F_2^{-\frac{4}{3}}, \\ &2. && G = U_1^{\frac{2}{3}} V_1^{\frac{4}{3}} F_1^{\frac{4}{3}} F_2^{-\frac{2}{3}}. \end{aligned}$$

From these last two equations we have the following values:

$$(16)\qquad \begin{aligned} &1. && \frac{E}{G} = U_1^{\frac{2}{3}} V_1^{-\frac{2}{3}} F_1^{-\frac{2}{3}} F^{-} \\ &2. && \frac{\partial \log G}{\partial \mu} = \frac{2}{3}\frac{1}{U_1}\frac{dU_1}{d\mu} + \frac{4}{3}\frac{a_2}{a_5} - \frac{2}{3}\frac{\partial \log F_2}{\partial \mu}, \\ &3. && \frac{\partial \log E}{\partial \nu} = \frac{2}{3}\frac{1}{V_1}\frac{dV_1}{\partial \nu} - \frac{4}{3}\frac{a_3}{a_3} + \frac{2}{3}\frac{\partial \log F_1}{\partial \nu}. \end{aligned}$$

We substitute these values in equations (3) and (4) of the system (12) and obtain:

$$(17)\qquad \frac{dU_1}{d\mu} + \left[2\frac{a_2}{a_5} - \frac{\partial \log F_2}{\partial \mu}\right] U_1 = 3\frac{a_4}{a_5} V_1^{-\frac{2}{3}} F_1^{-\frac{2}{3}} F_2^{-\frac{2}{3}} U_1^{\frac{5}{3}},$$

$$(18)\qquad \frac{dV_1}{d\nu} + \left[-2\cdot\frac{a_3}{a_5} + \frac{\partial \log F_1}{\partial \nu}\right] V_1 = -3\frac{a_1}{a_5} U_1^{-\frac{2}{3}} F_1^{\frac{2}{3}} F_2^{\frac{2}{3}} V_1^{\frac{5}{3}}.$$

The integrals of these are:

$$(19)\qquad \begin{aligned} &1. \qquad \frac{V_1^{\frac{2}{3}}}{U_1^{\frac{2}{3}}} = 2F_1^{\frac{4}{3}} F_2^{-\frac{2}{3}}(F_3 + V_2),\\ &2. \qquad \frac{U_1^{\frac{2}{3}}}{V_1^{\frac{2}{3}}} = 2F_1^{\frac{2}{3}} F_2^{-\frac{4}{3}}(F_4 + U_2), \end{aligned}$$

where

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2} d\mu, \quad F_4 = \int \frac{a_1}{a_5} F_2^2 d\nu$$

and U_2 and V_2 are functions of μ only and of ν only respectively.

These equations are *sufficient to say whether or not a solution exists and to find such solutions if they do exist.* It is proposed to illustrate this by an example (pp. 156–159) before going farther with the general theory.

EXAMPLE. We take $f_3 = f_3(\nu)$, a function of ν only; also

$$f_2(\mu, \nu) = \mu^n \nu^m, \quad f(\mu, \nu) = \mu^n \nu^\beta.$$

Then after some calculation we find the following values:

$$(20)\qquad \begin{aligned} a_1 &= 0,\\ a_2 &= \mu^{2n-2}\nu^{\beta+m-1} n(n+1)(\beta - m) f_3(\nu),\\ a_3 &= \mu^{2n-1}\nu^{\beta+m-2}\left[n(\beta-m)(m+\beta-1)f_3(\nu) - 2n(\beta-m)\nu\frac{df_3(\nu)}{d\nu}\right],\\ a_4 &= \mu^{2n}\nu^{\beta+m-3}(\beta-m)\left[\beta m f_3(\nu) - (\beta+m-1)\nu\frac{df_3(\nu)}{d\nu} + \nu^2\frac{d^2 f_3(\nu)}{d\nu^2}\right],\\ a_5 &= \mu^{2n-1}\nu^{\beta+m-1} n(\beta-m) f_3(\nu). \end{aligned}$$

From these we find:

$$(21)\qquad \begin{aligned} \frac{a_2}{a_5} &= \frac{n+1}{\mu},\\ \frac{a_3}{a_5} &= \frac{m+\beta-1}{\nu} - 2\frac{1}{f_3(\nu)}\frac{df_3(\nu)}{d\nu},\\ \frac{a_1}{a_5} &= 0,\\ \frac{a_4}{a_5} &= \frac{\mu}{n}\left[\frac{\beta m}{\nu^2} - (\beta+m-1)\frac{1}{\nu}\frac{1}{f_3(\nu)}\frac{df_3(\nu)}{d\nu} + \frac{1}{f_3(\nu)}\frac{d^2 f_3(\nu)}{d\nu^2}\right] = \frac{\mu}{n}\theta(\nu). \end{aligned}$$

Then we have:

$$\int \frac{a_2}{a_5}\, d\mu = \int (n+1)\frac{d\mu}{\mu} = \log \mu^{n+1},$$

$$\int \frac{a_3}{a_5}\, d\nu = \int (m+\beta-1)\frac{d\nu}{\nu} - 2\int d\log f_3(\nu) d\nu = \log \frac{\nu^{m+\beta-1}}{f_3^2(\nu)}.$$

Therefore

$$F_1 = \mu^{n+1}, \quad F_2 = \frac{\nu^{m+\beta-1}}{f_3^2(\nu)}, \quad F_4 = 0,$$

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2} d\mu = \int -\theta(\nu)\mu^{-2(n+1)+1} d\mu = \frac{\theta(\nu)}{2n^2}\mu^{-2n}.$$

Introducing these values into (19) we have:

$$(22)\quad \begin{aligned} &1. \quad V_1^{\frac{2}{3}} U_1^{-\frac{2}{3}} = 2\mu^{\frac{4}{3}(n+1)}\nu^{-\frac{2}{3}(m+\beta-1)} f_3^{\frac{4}{3}}(\nu)\left[\frac{\theta(\nu)\mu^{-2n} + 2n^2 V_2}{2n^2}\right]\\ &2. \quad V_1^{-\frac{2}{3}} U_1^{\frac{2}{3}} = 2\mu^{\frac{2}{3}(n+1)}\nu^{-\frac{4}{3}(m+\beta-1)} f_3^{\frac{8}{3}}(\nu)\, U_2. \end{aligned}$$

This requires either

$$(a) \qquad V_2 = K\theta(\nu) \qquad (K = \text{constant}),$$

or

$$(b) \qquad \theta(\nu) = 0.$$

Case (a). If $V_2 = K\theta(\nu)$, then we find:

$$U_1^{\frac{2}{3}} = \lambda\mu^{-\frac{4}{3}(n+1)}[\mu^{-2n} + 2n^2K]^{-1},$$

$$U_1^{\frac{2}{3}} = a\mu^{\frac{2}{3}(n+1)} U_2,$$

which require that

$$U_2 = \frac{\lambda}{a}\mu^{-2(n+1)}[\mu^{-2n} + 2n^2K]^{-1};$$

therefore

$$U_1 = \frac{\lambda^{\frac{3}{2}}\mu^{n-2}}{[2n^2K\mu^{2n} + 1]^{\frac{3}{2}}}.$$

And for V_1 we find:

$$V_1^{\frac{2}{3}} = \lambda'\nu^{-\frac{2}{3}(m+\beta-1)} f_3^{\frac{4}{3}}(\nu)\theta(\nu),$$

$$V_1^{\frac{2}{3}} = a'\nu^{\frac{4}{3}(m+\beta-1)} f_3^{-\frac{8}{3}}(\nu).$$

These require that:

$$\theta(\nu) = \frac{a'}{\lambda'}\frac{\nu^{2(m+\beta-1)}}{f_3^4(\nu)};$$

Taking $a' = \lambda'$, as we may do, and restoring the value of $\theta(\nu)$ we have the following differential equation to determine $f_3(\nu)$:

$$\frac{d^2 f_3(\nu)}{d\nu^2} - (\beta + m - 1)\nu^{-1}\frac{df_3(\nu)}{d\nu} + \beta m \nu^{-2} f_3(\nu) = \nu^{2m+\beta-1} f_3^{-3}(\nu). \tag{23}$$

Before solving this differential equation we consider case (b). If $\theta(\nu) = 0$ we find:

$$U_1^{\frac{2}{3}} = \lambda\mu^{-\frac{4}{3}(n+1)},$$

$$U_1^{\frac{2}{3}} = a\mu^{\frac{2}{3}(n+1)} U_2,$$

giving

$$U_2 = \frac{\lambda}{\mu}\mu^{-2(n+1)};$$

also

$$V_1^{\frac{2}{3}} = \lambda' \nu^{-\frac{2}{3}(m+\beta-1)} f_3^{\frac{4}{3}}(\nu) V_2,$$

$$V_1^{\frac{2}{3}} = a' \nu^{\frac{4}{3}(m+\beta-1)} f_1^{-\frac{8}{3}}(\nu),$$

giving

$$V_2 = \frac{a'}{\lambda'}\frac{\nu^{2(m+\beta-1)}}{f_3^4(\nu)}.$$

And $f_3(\nu)$ is given by

$$\frac{d^2 f_3(\nu)}{d\nu^2} - (\beta + m - 1)\nu^{-1}\frac{df_3(\nu)}{d\nu} + \frac{\beta m}{\nu^2} f_3(\nu) = 0. \tag{24}$$

It remains then to see if $f_3(\nu)$ can be determined algebraically.

Since equation (24) is (23) with its right hand number put equal to zero we can consider them together.

We put

$$\nu = e^{\vartheta}, \quad f_3(\nu) = e^{\frac{(m+\beta)\vartheta}{2}} z, \quad \theta = \frac{dz}{d\vartheta}.$$

After some reduction equations (23) and (24) reduce to the forms:

$$\theta\frac{d\theta}{dz} - \frac{(m-\beta)^2}{4} z = z^{-3}, \tag{25}$$

$$\theta\frac{d\theta}{dz} - \frac{(m-\beta)^2}{4} z = 0. \tag{26}$$

We consider the latter and simpler case first.

It gives:

$$\theta = \tfrac{1}{2}\sqrt{(m-\beta)^2 z^2 + 4c}.$$

Hence

$$\vartheta = \int \frac{2dz}{\sqrt{(m-\beta)^2 z^2 + 4c}}.$$

Therefore

$$\vartheta = \log\left[\frac{2(m-\beta)^2 z + 2(m-\beta)\sqrt{(m-\beta)^2 z^2 + 4c_1}}{2(m-\beta)^2 z - 2(m-\beta)\sqrt{(m-\beta)^2 z^2 + 4c_1}}\right]^{\frac{1}{m-\beta}} + \log c_2.$$

Remembering that $\vartheta = \log \nu$ and that $z = f_3(\nu)/\nu^{\frac{m+\beta}{2}}$, we find after some reduction:

$$f_3^2(\nu) = \frac{4c_1\nu^{m+\beta}}{(m-\beta)^2\{(\nu^{m-\beta}-c_2)^2-1\}};$$

and for the equation of the curve:

$$(m-\beta)^2\{(\nu^{m-\beta}-c_3)^2-1\}(A\mu^n\nu^m + B\mu^n\nu^\beta)^2 - 4c_1\nu^{m+\beta} = 0.$$

Hence

$$V_1 = K_1[\nu^{2(m-\beta)-1} - 2c_2\nu^{m+\beta-1} + (c_2^2-1)\nu^{-1}]^2,$$

$$U_1 = K_2\mu^{-2(n+1)} \qquad (K_1 \text{ and } K_2 \text{ constant});$$

these give:

$$E = p\mu^{-2(n+1)},$$

$$G = q[\nu^{2(m-\beta)-1} - 2c_2\nu^{m-\beta-1} + (c_2^2-1)\nu^{-1}]^2,$$

where p and q are constant.

In regard to the other integral:

$$\vartheta = \int \frac{2zdz}{2cz + \sqrt{(m-\beta)^2z^4-4}},$$

it appears that in order to keep the exponents rational and the coefficients real it is necessary to take $c = 0$.

Then we find finally:

$$f_3^2(\nu) = \frac{4K^2\nu^{2\beta}+\nu^{2m}}{2K(m-\beta)},$$

where K is constant, and the curve is given by the equation:

$$4K^2\nu^{2\beta} + \nu^{2m} = 2K(m-\beta)(A\mu^n\nu^m + B\mu^n\nu^\beta)^2.$$

To revert to the general theory, we may say that if a solution exists, the right hand member of equation 1 of (19) must be of the form

$$\frac{\Theta(\nu)}{\Theta(\mu)},$$

and then

$$V_1^{\frac{1}{2}} = K\Theta(\nu), \quad U_1^{\frac{1}{2}} = K\Theta(\mu).$$

The functions $\psi_1(\mu, \nu)$ and $\psi_2(\mu, \nu)$ are given by the equations:

$$\frac{F_1(\psi_1, \psi_2)}{F_3(\psi_1, \psi_2)} = \frac{f_1(\mu, \nu)}{f_3(\mu, \nu)}, \qquad \frac{F_2(\psi_1, \psi_2)}{F_3(\psi_1, \psi_2)} = \frac{f_2(\mu, \nu)}{f_3(\mu, \nu)}.$$

Since we desire a conformal representation we shall take

$$\psi_1 = \mu, \quad \psi_2 = \nu.$$

But it is plain that by such a choice certain conditions are imposed by equations (19) and we proceed to consider those conditions.

§ 3.

Calculation of V_2 and U_2.

If we represent the right hand members (19) by R and S respectively, then we must have:

$$1. \qquad RS = 1,$$

$$(27) \quad 2. \qquad \frac{\partial^2 \log R}{\partial\mu\partial\nu} = 0,$$

$$3. \qquad \frac{\partial^2 \log S}{\partial\mu\partial\nu} = 0.$$

From the second of these we find:

$$(28) \qquad \tfrac{4}{3}\frac{\partial}{\partial\nu}\frac{a_2}{a_5} - \tfrac{2}{3}\frac{\partial}{\partial\mu}\frac{a_3}{a_5} + \frac{\partial^2 \log(F_3 + V_2)}{\partial\mu\partial\nu} = 0.$$

Considering the last term of this equation we find

$$\frac{\partial^2 \log(F_3 + V_2)}{\partial\mu\partial\nu} = \frac{\partial}{\partial\nu}\frac{\partial F_3/\partial\nu}{F_3 + V_2} = \frac{\partial}{\partial\nu}\left\{-4\frac{a_4}{a_5}F_2^{-2}(F_4 + U_2)\right\}$$

$$= -4F_2^{-2}(F_4 + U_2)\frac{\partial}{\partial\nu}\frac{a_4}{a_5} + 8\frac{a_3}{a_5}\frac{a_4}{a_5}F_2^{-2}(F_4 + U_2) - 4\frac{a_1}{a_5}\frac{a_4}{a_5},$$

since

$$\frac{\partial F_3}{\partial\mu} = -\frac{a_4}{a_5}F_1^{-2} \quad \text{and} \quad 4F_1^2F_2^{-2}(F_3 + V_2)(F_4 + U_2) = 1.$$

Whence from (28) we find:

$$(29) \qquad 2F_2^{-2}(F_4 + U_2) = \frac{\tfrac{2}{3}\frac{\partial}{\partial\mu}\frac{a_3}{a_5} - \tfrac{4}{3}\frac{\partial}{\partial\nu}\frac{a_2}{a_5} + 4\frac{a_1}{a_5}\frac{a_4}{a_5}}{4\frac{a_3}{a_5}\frac{a_4}{a_5} - 2\frac{\partial}{\partial\nu}\frac{a_4}{a_5}};$$

and from 3 of (27) we find in like manner:

$$(30) \qquad 2F_1^2(F_3 + V_2) = \frac{\tfrac{4}{3}\frac{\partial}{\partial\mu}\frac{a_3}{a_5} - \tfrac{2}{3}\frac{\partial}{\partial\nu}\frac{a_2}{a_5} + 4\frac{a_1}{a_5}\frac{a_4}{a_5}}{4\frac{a_1}{a_5}\frac{a_2}{a_5} + 2\frac{\partial}{\partial\mu}\frac{a_1}{a_5}}.$$

If we represent the right hand members of (29) and (30) by Δ_1 and Δ_2 respectively, then equations (19) become:

$$
(31) \qquad \begin{aligned}
\frac{V_1^{\frac{2}{3}}}{U_1^{\frac{2}{3}}} &= F_1^{-\frac{2}{3}} F_2^{-\frac{2}{3}} \Delta_1 , \\[2ex]
\frac{U_1^{\frac{2}{3}}}{V_1^{\frac{2}{3}}} &= F_1^{\frac{2}{3}} F_2^{\frac{2}{3}} \Delta_2 .
\end{aligned}
$$

§ 4.

Partial differential equations satisfied by $f_i(\mu, \nu)$.

As a first condition we have

$$\Delta_1 \Delta_2 = 1 .$$

Differentiating (29) with respect to ν, and using (19) we have

$$2 \frac{a_1}{a_5} F_2^2 = F_2^2 \frac{\partial \Delta_1}{\partial \nu} + 2 F_2^2 \frac{a_3}{a_5} \Delta_1 ,$$

or

$$\frac{\partial \Delta_1}{\partial \nu} + 2 \frac{a_3}{a_5} \Delta_1 = 2 \frac{a_1}{a_5} ;$$

and in like manner from (30) we find

$$\frac{\partial \Delta_2}{\partial \mu} - 2 \frac{a_2}{a_5} \Delta_2 = - 2 \frac{a_4}{a_5} .$$

Hence we have the system:

$$
(32) \qquad \begin{aligned}
&\Delta_1 \Delta_2 = 1 , \\
&\frac{\partial \Delta_1}{\partial \nu} + 2 \frac{a_3}{a_5} \Delta_1 = 2 \frac{a_1}{a_5} , \\
&\frac{\partial \Delta_2}{\partial \mu} - 2 \frac{a_2}{a_5} \Delta_2 = - 2 \frac{a_4}{a_5} .
\end{aligned}
$$

These are the necessary, and, as is easily seen, sufficient conditions to impose upon $f_i(\mu, \nu)$ in order that a solution may exist for the rectangular coördinate system (μ, ν).

§ 5.

We consider next a somewhat general solution of our problem. We have written

$$\int \frac{a_2}{a_5} d\mu = \log F_1 , \quad \int \frac{a_3}{a_5} d\nu = \log F_2 .$$

We consider one case where these integrals actually yield logarithms, i. e., $\partial a_5 / \partial \mu = \lambda a_2$ and $\partial a_5 / \partial \nu = a a_3$. To simplify the expressions for a_i we divide equation

(2) through by $f_2(\mu, \nu)$ and then consider $f_3' = 1$, $f_2' = f_2/f_3$, $f_3' = f_1/f_3$. This amounts to putting $f_3 = 1$ in our formulas. We find:

$$a_1 = \begin{vmatrix} \frac{\partial f_2}{\partial \mu} & \frac{\partial^2 f_2}{\partial \mu^2} \\ \frac{\partial f_3}{\partial \mu} & \frac{\partial^2 f_3}{\partial \mu^2} \end{vmatrix}, \quad a_4 = \begin{vmatrix} \frac{\partial f_2}{\partial \nu} & \frac{\partial^2 f_2}{\partial \nu^2} \\ \frac{\partial f_3}{\partial \nu} & \frac{\partial^2 f_3}{\partial \nu^2} \end{vmatrix}, \quad a_5 = \begin{vmatrix} \frac{\partial f_2}{\partial \mu} & \frac{\partial f_2}{\partial \nu} \\ \frac{\partial f_3}{\partial \mu} & \frac{\partial f_3}{\partial \nu} \end{vmatrix},$$

$$a_2 = 2\begin{vmatrix} \frac{\partial f_2}{\partial \mu} & \frac{\partial^2 f_2}{\partial \mu \partial \nu} \\ \frac{\partial f_3}{\partial \mu} & \frac{\partial^2 f_3}{\partial \mu \partial \nu} \end{vmatrix} + \begin{vmatrix} \frac{\partial f_2}{\partial \nu} & \frac{\partial^2 f_2}{\partial \mu^2} \\ \frac{\partial f_3}{\partial \nu} & \frac{\partial^2 f_3}{\partial \mu^2} \end{vmatrix}, \quad a_3 = 2\begin{vmatrix} \frac{\partial f_2}{\partial \nu} & \frac{\partial^2 f_2}{\partial \mu \partial \nu} \\ \frac{\partial f_3}{\partial \nu} & \frac{\partial^2 f_3}{\partial \mu \partial \nu} \end{vmatrix} + \begin{vmatrix} \frac{\partial f_2}{\partial \mu} & \frac{\partial^2 f_2}{\partial \nu^2} \\ \frac{\partial f_3}{\partial \mu} & \frac{\partial^2 f_3}{\partial \nu^2} \end{vmatrix}.$$

The case to be considered is found by taking $\partial^2 f_2/\partial\mu\partial\nu = 0$ and $\partial^2 f_3/\partial\mu\partial\nu = 0$, that is, $f_2 = \phi_1(\mu) + \phi_2(\nu)$ and $f_3 = \psi_1(\mu) + \psi_2(\nu)$. Then we shall have:

$$a_5 = \phi_1'(\mu)\,\psi_2'(\nu) - \psi_1'(\mu)\,\phi_2'(\nu),$$

$$a_1 = \phi_1'(\mu)\,\psi_1''(\mu) - \psi_1'(\mu)\,\phi_1''(\mu),$$

$$a_4 = \phi_2'(\nu)\,\psi_2''(\nu) - \psi_2'(\nu)\,\phi_2''(\nu),$$

$$a_2 = \phi_2'(\nu)\,\psi_1''(\mu) - \psi_2'(\nu)\,\phi_1''(\mu),$$

$$a_3 = \phi_1'(\mu)\,\psi_2''(\nu) - \psi_1'(\mu)\,\phi_2''(\nu).$$

Hence

(40) $$\int \frac{a_2}{a_5}\,d\mu = -\log\{\phi_1'(\mu)\,\psi_2'(\nu) - \psi_1'(\mu)\,\phi_2'(\nu)\},$$

(41) $$\int \frac{a_3}{a_5}\,d\nu = \log\{\phi_1'(\mu)\,\psi_2'(\nu) - \psi_1'(\mu)\,\phi_2'(\nu)\},$$

Therefore

$$F_1 = 1 \div \begin{vmatrix} \phi_1'(\mu) & \phi_2'(\nu) \\ \psi_1'(\mu) & \psi_2'(\nu) \end{vmatrix} = \frac{1}{a_5}, \qquad F_2 = \begin{vmatrix} \phi_1'(\mu) & \phi_2'(\nu) \\ \psi_1'(\mu) & \psi_2'(\nu) \end{vmatrix} = a_5.$$

Hence

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2}\,d\mu = -a_4 \int a_5\,d\mu = -\begin{vmatrix} \phi_1(\mu) & \phi_2'(\nu) \\ \psi_1(\mu) & \psi_2'(\nu) \end{vmatrix} \cdot \begin{vmatrix} \phi_2'(\nu) & \phi_2''(\nu) \\ \psi_2'(\nu) & \psi_2''(\nu) \end{vmatrix},$$

$$F_4 = \int \frac{a_1}{a_5} F_2^2\,d\nu = a_1 \int a_5\,d\nu = \begin{vmatrix} \phi_1'(\mu) & \phi_2(\nu) \\ \psi_1'(\mu) & \psi_2(\nu) \end{vmatrix} \cdot \begin{vmatrix} \phi_1'(\mu) & \phi_1''(\mu) \\ \psi_1'(\mu) & \psi_1''(\mu) \end{vmatrix}.$$

In order to separate the variables it is necessary that either

$$\text{(a)} \quad \phi_1(\mu) = \psi_1(\mu) \quad \text{or} \quad \text{(b)} \quad \phi_2(\nu) = \psi_2(\nu),$$

and since the equations and expressions are symmetrical in μ and ν it will only be necessary to consider case (a) and afterwards interchange μ and ν in the result.

First of all we have:

$$F_4 = 0\,, \quad F_1 = \frac{1}{\phi'(\mu)} \frac{1}{\psi_2'(\nu) - \phi_2'(\nu)}\,, \quad F_2 = \phi'(\mu)\,\{\psi_2'(\nu) - \phi_2'(\nu)\},$$

$$F_3 = -\,\phi(\mu)\,\{\psi_2'(\nu) - \phi_2'(\nu)\} \begin{vmatrix} \phi_2'(\nu) & \phi_2''(\nu) \\ \psi_2'(\nu) & \psi_2''(\nu) \end{vmatrix}.$$

Then from (19) we have:

$$\frac{\frac{1}{2}\,\phi'^2(\mu)\{\psi_2'(\nu) - \phi_2'(\nu)\}}{\dfrac{V_2}{\{\psi_2'(\nu) - \phi_2'(\nu)\}} - \phi(\mu) \begin{vmatrix} \phi_2'(\nu) & \phi_2''(\nu) \\ \psi_2'(\nu) & \psi_2''(\nu) \end{vmatrix}} = \frac{2\,U_2}{\phi'^2(\mu)\{\psi_2'(\nu) - \phi_2'(\nu)\}}\,. \tag{42}$$

As V_1 must vanish and $U_2 = \phi_1'^4(\mu)/4\phi(\mu)$, we find

$$U_1 = \frac{1}{\sqrt{8}} \frac{\phi'^3(\mu)}{\phi^{\frac{3}{2}}(\mu)}\,.$$

Also

$$\begin{vmatrix} \phi_2''(\nu) & \phi_2'(\nu) \\ \psi_2''(\nu) & \psi_2'(\nu) \end{vmatrix} = \{\psi_2'(\nu) - \phi_2'(\nu)\}^3,$$

or, if we write x for $\psi_2'(\nu)$ and y for $\phi_2'(\nu)$,

$$x\frac{dy}{d\nu} - y\frac{dx}{d\nu} = (x - y)^3.$$

Put $x - y = q$ and our differential equation is:

$$q\frac{dx}{d\nu} - x\frac{dq}{d\nu} = q^3\,,$$

or

$$\frac{d}{d\nu}\frac{x}{q} = q\,.$$

Therefore

$$\frac{x}{q} = \int q\,d\nu = \psi_2(\nu) - \phi_2(\nu) + c\,;$$

hence

$$\frac{\psi_2'(\nu)}{\psi_2'(\nu) - \phi_2'(\nu)} = \psi_2(\nu) - \phi_2(\nu) + c_1\,,$$

and integrating this we have:

$$\psi_2(\nu) = \tfrac{1}{2}\,[\psi_2(\nu) - \phi_2(\nu)]^2 + c_1[\psi_2(\nu) + \phi_2(\nu)] + c_2\,.$$

Remembering that $\phi_2(\nu) = f_1(\nu)/f_3(\nu)$ and $\phi(\mu) = f_1(\mu)/f_3(\mu)$ we have, after some reduction:

$$
\begin{aligned}
(43) \qquad & B^2[K_2 f_1(\nu) + K_3 f_3(\nu)] f_3(\nu) f_3^2(\mu) \\
& = \{(K_1 B + 1) f_3(\nu) f_3(\mu) + (A + B)[f_1(\mu) f_3(\nu) + f_3(\mu) f_1(\nu)]\}^2,
\end{aligned}
$$

for the equation of our curve. Here the f_i of μ and of ν are *any algebraic functions whatever*. As mentioned above we find a solution for case (b) by interchanging μ and ν in (43).

§ 6.

Conditions for the equality of E and G.

The work thus far is for $F = 0$, i. e., μ and ν are rectangular surface coördinates. It remains to consider the case $E = G$.

From our system of partial differential equations (10) we find that a_1 must be equal to a_3 and a_2 to a_4. It remains to consider the system of equations (32) which were the necessary and sufficient conditions to impose on the $f_i(\mu, \nu)$. First we consider the values of F_3 and F_4. We recall that

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2} d\mu, \quad F_4 = \int \frac{a_1}{a_5} F_2^{-2} d\nu,$$

$$\log F_1 = \int \frac{a_2}{a_5} d\nu, \quad \log F_2 = \int \frac{a_3}{a_5} d\nu.$$

Hence also

$$\frac{\partial F_1}{\partial \mu} = \frac{a_2}{a_5} F_1, \quad \frac{\partial F_2}{\partial \nu} = \frac{a_3}{a_5} F_2.$$

We have then

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2} d\mu = \int -\frac{a_2}{a_5} F_1^{-2} d\mu = \int - F_1^{-3} \frac{\partial F_1}{\partial \mu} d\nu = \tfrac{1}{2} F_1^{-2}.$$

Also

$$F_4 = \int \frac{a_3}{a_5} F_2^2 d\nu = \int F_2 \frac{\partial F_2}{\partial \nu} d\nu = \tfrac{1}{2} F_2^2.$$

Hence the right hand members of (29) and (30) become respectively,

$$2F_2^{-2}(\tfrac{1}{2}F_2^2 + U_2) = 1 + 2F_2^{-2} U_2 = \Delta_1 = \frac{\frac{2}{3}\frac{\partial}{\partial \mu}\frac{a_3}{a_5} - \frac{4}{3}\frac{\partial}{\partial \nu}\frac{a_2}{a_5} + 4\frac{a_2}{a_5}\frac{a_3}{a_5}}{4\frac{a_2}{a_5}\frac{a_3}{a_5} - 2\frac{\partial}{\partial \nu}\frac{a_2}{a_5}},$$

$$2F_1^2(\tfrac{1}{2}F_1^{-2} + V_2) = 1 + 2F_1^2 V_2 = \Delta_2 = \frac{\frac{4}{3}\frac{\partial}{\partial\mu}\frac{a_3}{a_5} - \frac{2}{3}\frac{\partial}{\partial\nu}\frac{a_2}{a_5} + 4\frac{a_2}{a_5}\frac{a_3}{a_5}}{4\frac{a_2}{a_5}\frac{a_3}{a_5} + 2\frac{\partial}{\partial\mu}\frac{a_3}{a_5}}.$$

We consider the first of equations (3), i. e., $\Delta_1\Delta_2 = 1$.

We write

$$\Delta_1 = \frac{\frac{2}{3}a - \frac{4}{3}b + c}{c - 2b} = 1 + \frac{\frac{2}{3}(a+b)}{c-2b},$$

and also

$$\Delta_2 = \frac{\frac{4}{3}a - \frac{2}{3}b + c}{c + 2a} = 1 - \frac{\frac{2}{3}(a+b)}{c+2a}.$$

Therefore

$$\Delta_1\Delta_2 = 1 + \frac{\frac{2}{3}(a+b)}{c-2b} - \frac{\frac{2}{3}(a+b)}{c+2a} - \frac{\frac{4}{9}(a+b)^2}{(c+2a)(c-2b)}$$

$$= 1 + \frac{\frac{2}{3}(a+b)}{(c+2a)(c-2b)}\left[c + 2a - c + 2b - \tfrac{2}{3}(a+b)\right]$$

$$= 1 + \frac{\frac{8}{9}(a+b)}{(c+2a)(c-2b)}.$$

Hence we must have $a + b = 0$ and therefore

$$\frac{\partial}{\partial\mu}\frac{a_3}{a_5} + \frac{\partial}{\partial\nu}\frac{a_2}{a_5} = 0.$$

Hence $\Delta_1 = 1$ and $\Delta_2 = 1$. Then the other two equations of (32) are satisfied and we may conclude that

$$(44)\qquad \begin{aligned} a_1 &= a_3, \\ a_2 &= a_4, \\ \frac{\partial}{\partial\mu}\frac{a_3}{a_5} + \frac{\partial}{\partial\nu}\frac{a_2}{a_5} &= 0, \end{aligned}$$

are the necessary and sufficient conditions that $E = G$.

The next question will be the study of these surfaces, their existence and properties. In particular I have already finished part of the work for the curve $y^2 = ax^3 + bx^2 + cx + d$, i. e., for surfaces whose geodetic lines go into cubics upon the plane by a conformal transformation. The straight line and circle lead to the surfaces of constant curvature with their interesting properties. Here is a much wider and more interesting field.

GÖTTINGEN, *November* 3, 1900.