ON THE GEOMETRY OF PLANES IN A PARABOLIC SPACE OF FOUR DIMENSIONS*

BY

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Literature.

Of the literature of the geometry of hyperspace that has accumulated in recent years the following papers are cited as having points of contact with the ideas here set forth:

CLIFFORD: *Preliminary Sketch of Biquaternions* in Proceedings of the London Mathematical Society, vol. 4 (1873), pp. 381–395. CLIFFORD'S theory of parallels in elliptic space is identical with the theory of isoclinal systems of planes in four-dimensional space; namely, planes that pass through a fixed point and make equal dihedral angles with any transversal plane through the same point. (See §§ 30–32 of this paper.)

CHARLES S. PEIRCE: Reprint of the *Linear Associative Algebra* of BENJAMIN PEIRCE in the American Journal of Mathematics, vol. 4 (1881). In the foot-note of page 132 attention is called to the fact that in four-dimensional space two planes may be so related to one another that every straight line in the one is perpendicular to every straight line in the other. (See § 28 (3) of this paper.)

I. STRINGHAM: (1) *On a Geometrical Interpretation of the Linear Bilateral Quaternion Equation;* (2) *On the Rotation of a Rigid System in Space of Four Dimensions;* (3) *On the Measure of Inclination of two Planes in Space of Four Dimensions.* Papers presented to Section A of the American Association for the Advancement of Science, the first two at the Philadelphia meeting of 1884, the third at the Cleveland meeting of 1888. Abstracts printed in Proceedings of the Association, 1884, pp. 54–56, and privately, 1888. These papers form the nucleus of the present investigation.

A. BUCHHEIM: *A Memoir on Biquaternions*, in the American Journal

* Presented to the Society (Chicago) December 27, 1900. Received for publication December 7, 1900.

of Mathematics, vol. 7 (1885). The geometrical part is devoted to the geometry of non-Euclidean space. The theory of parallels is discussed on pp. 301–306, 316–325.

W. KILLING: *Die nicht-Euklidischen Raumformen*, Leipzig, 1885. Section 8, pp. 148–160, discusses, under the title: "Die gegenseitige Lage zweier Ebene," the relations to one another of μ- and ν-dimensional planes in an n-dimensional space.

CAYLEY: *On the quaternion Equation* $qQ - Qq' = 0$, and *On the Matrical Equation* $qQ - Qq' = 0$; Messenger of Mathematics, vol. 14 (1885), pp. 108–112 and 300–304, or Mathematical Papers, vol. 12, pp. 300–304 and 311–313. The first paper discusses the character of the roots of the equation $qQ - Qq' = 0$; the second interprets q, q', Q as matrices.

KLEIN: *Vorlesungen über nicht-Euklidische Geometrie*, II., 1890 (Zweiter Abdruck, 1893). On pages 120–124 the author identifies quaternion multiplications with orthogonal substitutions in four variables. In particular the products $p \cdot q$ and $q \cdot p$ are called respectively "eine Schiebung erster Art" and "eine Schiebung zweiter Art"; these are the parallel and contra-parallel translations of Professor HATHAWAY's paper number (1), cited below. CLIFFORD's theory of parallels in elliptic space is explained in the Vorlesungen on pages 228–237. KLEIN's theory of Schiebungen was first presented to the members of his Seminar in January and February, 1880.

G. VERONESE: *Fondamenti di Geometria*, Padua, 1891, Part II, Book I, p. 455 et sq. Chapter 1 discusses elementary theorems in the geometry of four-dimensional space, some of which deal with the perpendicularity, parallelism, and intersections of planes.

M. BRÜCKNER; *Die Elemente der vierdimensionalen Geometrie mit besonderer Berücksichtigung der Polytrope;* Jahresber. d. Ver. f. Naturk., Zwickau, 1893. I have not been able to consult a copy of this paper, but SCHLEGEL, in Fortschritte der Mathematik, vol. 25, p. 1028, says: "Die ··· Arbeit giebt eine auf gründlicher Litteraturkenntnis beruhende und durch grosse Klarheit in der Darstellung sich empfehlende Zusammenstellung der in der elementaren vierdimensionalen Geometrie erzielten Resultate, und füllt dadurch in erwünschter Weise eine Lücke in der deutschen Litteratur aus."

P. CASSINI: *Sulla geometria pura Euclidiana ad n dimensioni;* Atti del Reale Instituto Veneto (7), vol. 5 (1894), p. 820 et sq., discusses metrical geometry in four-dimensional space; but see Fortschritte der Mathematik, vol. 25, p. 1035.

A. S. HATHAWAY: (1) *Quaternions as Numbers of Four-Dimensional Space*, in Bulletin of the American Mathematical Society, vol. 4 (1897), pp. 54–57; (2) *Alternate Processes*, in Proceedings of the Indiana Academy of Sciences, Indianapolis, 1897, pp. 1–10; (3) *Linear*

Transformations in Four-Dimensional Space, in Bulletin of the American Mathematical Society, vol. 5 (1898), pp. 93–94. The fundamental ideas and formulæ of these papers are the same as those that were used in my papers of 1884 and 1888. Specifically Professor HATHAWAY exemplifies the utility of the quaternion analysis for the following purposes: (1) To interpret CLIFFORD'S theory of parallels in elliptic space (KLEIN'S Schiebungen), there stated in terms of great circular arcs on the hypersphere of four-dimensional space (see this paper, §§ 30–32); (2) To determine certain angles, areas, and volumes in four-dimensional space; (3) To formulate the theory of certain four-dimensional space transformations, in particular rotations (cf. my paper numbered (2) above).

A. N. WHITEHEAD: *Universal Algebra*, vol. 1 (1898). At pp. 405–406, 409 the properties of parallels in elliptic space are explained by means of the Calculus of Extension (Ausdehnungslehre).

§§ 1–12. SOME FUNDAMENTAL CONSIDERATIONS.

1. *The Quaternionic Manifold.* The four-dimensional space here considered is a point-manifold whose point-elements are uniquely determined by the sets of real variable numbers w, x, y, z, regarded as rectangular coördinates. To real numbers shall correspond always and only real points.

The space may be defined as the domain of the continuous translational and rotational transformation groups expressed in terms of the coördinates. The translations are the linear transformations of the form:

$$w + g, \quad x + a, \quad y + b, \quad z + c;$$

and the rotations are the orthogonal transformations of the type:

$$g_i w + a_i x + b_i y + c_i z \qquad (i = 1, 2, 3, 4).$$

In the quaternion analysis these are respectively additions and multiplications, and the coördinates of the transformed point are, in every case, the resultant coefficients of the fundamental units 1, i, j, k. The quaternion terms, or factors, being

$$q = g + ai + bj + ck, \quad p = w + xi + yj + zk,$$

translations are represented by the sum

$$p + q = (w + g) + (x + a)i + (y + b)j + (z + c)k,$$

rotations by the products: pq, qp, qpq'^{-1}, etc., provided (in the latter case) the condition $Tq = Tq' =$ a constant be assigned. Thus, in particular,

$$qp = W + Xi + Yj + Zk,$$

where

$$W = gw - ax + by - cz,$$
$$X = aw + gx - cy + bz,$$
$$Y = bw + cx + gy - az,$$
$$Z = cw - bx + ay + gz,$$

and in these equations all the conditions of orthogonality are satisfied if

$$g^2 + a^2 + b^2 + c^2 = 1.$$

Such multiplications obviously constitute an orthogonal group, that is a group of rotations in the quaternionic manifold, rotations having here the same meaning as when expressed in terms of the Cartesian analysis.*

Now it is well known that the continuous translational and rotational transformation groups (expressed in the Cartesian form) constitute the totality of the possible real movements, without distortion, in a parabolic space defined by the variables w, x, y, z; and the quaternion operations do actually reproduce such transformation groups, are therefore competent to interpret the geometry of a parabolic space of four dimensions. But do quaternion additions and multiplications suffice to produce *all* the movements (without distortion) of such a space?

In order to answer this question I assume that a rigid body is fixed by four points that have no special relation to one another, e. g., do not lie in a space of two dimensions, and I then show that, by quaternion additions and multiplications, any one set of four points o_1, a_1, b_1, c_1 can be moved into any second set o', a', b', c' congruent to the first, but otherwise arbitrarily placed. The letters here used denote quaternions except where statement to the contrary is made.

Place the origin at the point defined by o', so that o', regarded as a quaternion, has a zero tensor.

By a first operation of addition applied to each of the points o_1, a_1, b_1, c_1, the additive term being $o' - o_1$, we transform (o_1, a_1, b_1, c_1) into (o', a, b, c) where

$$a, b, c = -o_1 + a_1, \; -o_1 + b_1, \; -o_1 + c_1,$$

and the application of the proper test shows that (o', a, b, c) is actually congruent to (o_1, a_1, b_1, c_1)

By a second operation of multiplication (o', a, b, c) can now be transformed into (o', a', b', c'). In fact, the origin being still at o', if it be defined in general that

$$r(a, b, \cdots)r' = (rar', rbr', \cdots),$$

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* Cf. KLEIN: loc. cit. and *Vorlesungen über das Ikosaeder* (1884), pp. 35, 36; also CAYLEY: *On Certain Results Relating to Quaternions* in Philosophical Magazine, vol. 26 (1845), pp. 141–145, or Mathematical Papers, vol. 1, pp. 123–126.

it is easily verified that

$$q(o', a, b, c)a^{-1}q^{-1}a' = (o', a', b', c'),$$

where

$$q = \beta'(\gamma' - \gamma) + (\gamma' - \gamma)\beta$$

and

$$\beta, \gamma, \beta', \gamma' = \mathrm{V}ba^{-1}, \mathrm{V}ca^{-1}, \mathrm{V}b'a'^{-1}, \mathrm{V}c'a'^{-1};$$

and again, the application of the proper test shows that (o', a', b', c') is actually congruent to (o', a, b, c).

By means of appropriate quaternion operations we may therefore transport without distortion the elements and configurations of our quaternionic space from any given position to any other.

The system of measurement is parabolic. But one should not forget that the quaternion analysis augmented into a calculus of biquaternions (CLIFFORD, BUCHHEIM), or of octonions (MCAULAY), may suffice to interpret both the elliptic and the hyperbolic forms of a four-dimensional space.

In concluding this preliminary discussion it is pertinent to remark that *the quaternion theory makes its interpretations in strict analogy with the vector interpretations of a parabolic space of three dimensions* and, so far as they are known, takes them for granted. The quaternionic four-dimensional space is therefore the analogue of that three-dimensional space which allows itself to be explained by the Hamiltonian vector analysis.

2. *Nomenclature.* The terms solid, surface, plane, sphere, curve, line, etc., will be used with their ordinary significations in Euclidean geometry.

By space, without qualification, is meant Euclidean space of three dimensions. When spoken of as a locus it may be conveniently called a Euclidean.

By the director of a point, or the quaternion of a point, is meant the directed straight line drawn from the origin to the point. Geometrically interpreted, a quaternion is a director.

The word perpendicular, when not qualified, is used in its ordinary sense; thus two planes are perpendicular to one another when a straight line can be found in one of them which is perpendicular to every straight line in the other.

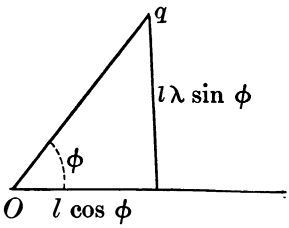
Two planes are said to be hyperperpendicular to one another when every straight line in the one plane is perpendicular to every straight line in the other.

In general, and unless specification to the contrary be made, the letters $a, b, c, d, e, p, q, r, s, t, u, v$ denote quaternions, $h, l, m, n, x, y, z, \theta, \phi, \psi, \chi$ scalars, $\alpha, \beta, \gamma, \delta, \rho, \sigma, \tau$ vectors. The four fundamental quaternion units are denoted by $1, i, j, k$ and their geometrical meaning is: four mutually perpendicular directors of unit length. The Hamiltonian notation is employed throughout.

3. *Amplitude.* When the quaternion is written in the form

$$q = l(\cos\phi + \lambda\sin\phi),$$

in which l is its tensor and λ is a unit vector, ϕ is called its amplitude. The scalar part, $l \cos \phi$, regarded as a director, lies along the scalar axis, and the vector part, $l\lambda \sin \phi$, is the director-perpendicular dropped from the extremity of q to the scalar axis; for this axis is, by definition, perpendicular to i, j, and k, and consequently also perpendicular to every vector. The amplitude is



therefore the angle (more strictly the arc-ratio of angle) between the quaternion, regarded as a director, and the scalar axis. When $\phi = \pi/2$, the quaternion director becomes a vector and is at right angles to the scalar axis; when $\phi = 0$, it is a part of the scalar axis itself.

4. *Geometric Addition.* As applied to a series of quaternions, interpreted as directors in a four-dimensional space, the law of geometric addition is: The sum of any two or more quaternion-directors (whether they be vectors, scalars, or any combinations of these) is the director that extends from the initial to the terminal extremity of the zig-zag formed by so disposing the several director-terms of the sum that all their intermediate extremities are conterminous.

By assuming the existence of a fourth independent direction in space, the geometrical interpretation of a quaternion (as given in § 2) makes this law of geometric addition a mere corollary of the law of vector addition in a three-dimensional domain. We require only to reiterate, for quadrinomials of the form $w + ix + jy + kz$, the statements that are valid for trinomials of the form $ix + jy + kz$.

Since quaternions obey the commutative law of addition, several geometrical steps lead to the same position in whatever order they may be taken, and two directors are identical when either one of them can be derived from the other by a simple translation unaccompanied by any rotation. If a rotation is required in order to bring the two directors into coincidence, they are distinguishable from one another by the fact that their versor parts are distinct. This commutative law in geometric addition implies the validity of the so-called parallel axiom and demands the existence of parallel elements (lines, planes, etc.), parallelism being determined by the usual Euclidean criteria.

5. *Relative Direction.* Let the quaternions q, q', which, for convenience and without loss of generality, may have unit tensors, be written in the binomial forms:

$$q = \cos \phi + \lambda \sin \phi,$$

$$q' = \cos \phi' + \lambda' \sin \phi',$$

and denote by θ the angle they form with one another. The ratio of q to q' is

$$q/q' = (\cos\phi + \lambda\sin\phi)(\cos\phi' - \lambda'\sin\phi'),$$

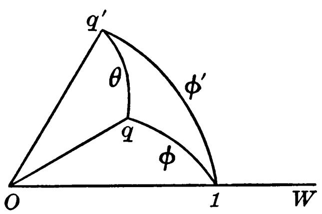
or, in its developed form,

$$q/q' = \cos\phi\cos\phi' - \sin\phi\sin\phi' S\lambda\lambda' + \lambda\sin\phi\cos\phi' - \lambda'\sin\phi'\cos\phi - \sin\phi\sin\phi' V\lambda\lambda',$$

and

$$Sq/q' = \cos\phi\cos\phi' - \sin\phi\sin\phi' S\lambda\lambda' = \cos\phi\cos\phi' + \sin\phi\sin\phi'\cos(\lambda, \lambda').$$

We may represent the arcs ϕ, ϕ', θ by the sides of a spherical triangle on the surface of a unit sphere. In such a figure the intersection of the planes of the



arcs ϕ, ϕ' is the scalar axis and the angle of the vectors λ, λ' is the angle between these two planes. Hence, by the cosine formula of spherical trigonometry,

$$\cos\theta = \cos\phi\cos\phi' + \sin\phi\sin\phi'\cos(\lambda, \lambda').$$

Thus the measure of the inclination of q to q' is

$$Sq/q' = \cos\theta .*$$

In order that q and q' may be perpendicular to one another it is necessary and sufficient that $\theta = \pi/2$ or an odd multiple of $\pi/2$. Hence:

The necessary and sufficient condition in order that two directors q, q' may be perpendicular to one another is that their ratio shall be a vector. Symbolically,

$$S\cdot qKq' = 0.$$

To obtain the condition of parallelism we note that the vector part of q/q' is

$$Vq/q' = \lambda\sin\phi\cos\phi' - \lambda'\sin\phi'\cos\phi - \sin\phi\sin\phi' V\lambda\lambda',$$

and that q and q' are parallel (or identical) if, and only if, their versor parts are either identical, or differ only in algebraic sign; that is, if

$$\phi' = \phi + n\pi, \quad \lambda' = \lambda, \quad V\lambda\lambda' = 0,$$

where $n = 0$, or a positive integer. But these conditions are fully expressed in the equation

* Cf. HATHAWAY: loc. cit. (1) p. 55.

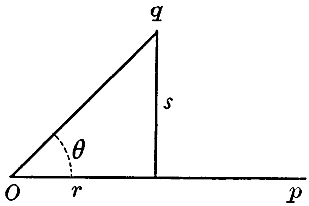
$$Vq/q' = \pm \lambda \sin(\phi - \phi') = 0 .$$

Hence, the necessary and sufficient condition in order that two directors q, q' *may be parallel is that their ratio shall be a scalar.* Symbolically,

$$V \cdot q K q' = 0 ,$$

In the conditions $S \cdot qKp = 0$, $V \cdot qKp = 0$, the requirement that p and q shall have unit tensors is obviously not necessary.

6. *Binomial Form of* q/p. Let θ be the angle between any two quaternion directors p, q, let s be the director-perpendicular dropped from the terminal ex-



tremity of q to p, and r the director from the origin (intersection of p, q) to the foot of this perpendicular. Then, by geometric addition,

$$q = r + s$$

and

$$q/p = r/p + s/p ;$$

and s/p, being the ratio of a pair of mutually perpendicular directors, is a vector. Hence, if ρ be defined as the ratio Us/Up, it is a unit vector and

$$q/p = \frac{Tq}{Tp}(\cos\theta + \rho \sin\theta) .$$

7. *Projection.* It is now evident that

$$S \cdot qKp = TqTp \cos(p, q) ,$$

and that

$$TV \cdot qKp = TqTp \sin(p, q) ;$$

and since obviously

$$S(V \cdot qKp)(Up \cdot Kp) = 0 ,$$

$(V \cdot qKp)Up$ is perpendicular to p. Hence the identity

$$q \equiv (S \cdot qKp)(Kp)^{-1} + (V \cdot qKp)(Kp)^{-1}$$

has the following evident interpretation:

$$\frac{S \cdot qKp}{Tp} Up = \text{projection of } q \text{ upon } p ,$$

$$\frac{V \cdot qKp}{Tp} Up = \text{projection of } q \text{ upon a director perpendicular to } p ,$$

and the sum of these two projections is q itself.

8. *Rectangular Coördinates.* The propositions of § 5 might have been proved in terms of rectangular coördinates. Suppose the two quaternions given in the quadrinomial form:

$$q = w + ix + jy + kz,$$

$$q' = w' + ix' + jy' + kz'.$$

Their direction cosines are

$$w/l, \quad x/l, \quad y/l, \quad z/l,$$

$$w'/l', \quad x'/l', \quad y'/l', \quad z'/l',$$

where

$$l = \sqrt{w^2 + x^2 + y^2 + z^2},$$

$$l' = \sqrt{w'^2 + x'^2 + y'^2 + z'^2},$$

and the formula for the cosine of the angle between them is

$$\cos\theta = \frac{ww' + xx' + yy' + zz'}{ll'}.$$

The ratio of q to q' may be written in the form

$$q/q' = \frac{(w + ix + jy + kz)(w' - ix' - jy' - kz')}{w'^2 + x'^2 + y'^2 + z'^2}.$$

In the numerator of this fraction the coefficients of 1, i, j, and k are respectively,

$$W = ww' + xx' + yy' + zz',$$

$$X = w'x - wx' + yz' - y'z,$$

$$Y = w'y - wy' + zx' - z'x,$$

$$Z = w'z - wz' + xy' - x'y;$$

and in terms of these q/q' takes the form

$$q/q' = \frac{W + iX + jY + kZ}{l'^2}.$$

In order that q and q' may be perpendicular to one another it is necessary and sufficient that

$$ww' + xx' + yy' + zz' = 0, \quad \text{i. e.,} \quad W = 0,$$

or otherwise expressed,

$$q/q' = \text{a vector.}$$

In order that q and q' may be parallel to one another it is necessary and sufficient that

$$\frac{w}{w'} = \frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'},$$

that is,

$$iX + jY + kZ = 0,$$

or otherwise expressed,

$$q/q' = \text{a scalar}.$$

9. *Equations of Loci.* In general it will be found that a quaternion equation in one variable has a definite, finite number of solutions and that, if it represent a locus, its coefficients must satisfy certain conditions, a remark in fact that is justified by the observation that the quaternion equation

$$f(q) = 0$$

is equivalent to a system of four scalar equations in four scalar variables. Thus a principal first step in the interpretation of the linear equation as a locus will be the assigning of the conditions necessary and sufficient in order that the equation may have an infinite number of solutions.

10. *Transference of Origin.* By virtue of the law of geometric addition, a locus, given in terms of a variable quaternion p, may be referred to a new origin, whose director is c, and be represented by an equation in terms of a new variable quaternion q, by substituting

$$p = c + q$$

in the given equation. Thus the equation

$$ap + pa^{-1}aa + 2a = 0 \qquad (a^2 = -1)$$

is satisfied by the value $p = aa$ and is referred to a point in the locus itself, as a new origin, by writing

$$p = aa + q;$$

and through this substitution it becomes

$$aq + qa^{-1}aa = 0.$$

In comparing two or more loci, advantage may be taken of this principle in order to simplify one or more of the equations by transferring the origin to a point within one of the loci, or to a point common to two or more of them.

11. *Solution of $a_1p + pa_2 = c$.* I indicate briefly HAMILTON'S solution of this equation.* Denote the conjugate of a_2 by $\bar{a}_2$† and multiply $a_1p + pa_2$ progressively by a_1, regressively by $\bar{a}_2$ and add; there results

* *Lectures on Quaternions*, p. 565, or TAIT'S *Treatise on Quaternions*, 3d ed., p. 136.

† This notation is used throughout the paper.

$$\{a_1^2 + a_1(a_2 + \bar{a}_2) + a_2\bar{a}_2\} p = a_1 c + c\bar{a}_2,$$

or

$$p = (a_1^2 + 2a_1 \mathrm{S}a_2 + a_2\bar{a}_1)^{-1}(a_1 c + c\bar{a}_2).$$

If nullitats (quaternions involving scalar $\sqrt{-1}$ and having zero tensors) be excluded the equation has in general but one root. For, denote any root by b; then

$$a_1 b + b a_2 = c,$$

and the original equation may be written in the form

$$a_1(p - b) + (p - b)a_2 = 0;$$

and if p be supposed to have any other value than b, for such value $p - b$ does not vanish and

$$a_1 = -(p - b)\,a_2(p - b)^{-1}$$

is not indeterminate; therefore

$$\mathrm{S}a_1 = -\mathrm{S}a_2, \quad \mathrm{T}a_1 = \mathrm{T}a_2.$$

Thus the coefficients are not independent and the equation may be written

$$\alpha_1 p + p\alpha_2 = c,$$

in which $\alpha_1 = \mathrm{V}a_1$, $\alpha_2 = \mathrm{V}a_2$, and $\mathrm{T}a_1 = \mathrm{T}a_2$.

It follows that unless the conditions: $\mathrm{S}a_1 = -\mathrm{S}a_2$, $\mathrm{T}a_1 = \mathrm{T}a_2$, be satisfied, the equation has but one root. But if it have more than one, it then has an infinite number of roots and represents a locus.

12. As an example of other equations in a variable quaternion p whose solution leads to this linear bilateral form, consider

$$\mathrm{S}ap = m, \quad \mathrm{S}bp = n,$$

which obviously represent a pair of Euclideans (ordinary spaces of three dimensions). They are equivalent to

$$ap + \bar{p}\bar{a} = 2m, \quad bp + \bar{p}\bar{b} = 2n,$$

from which is obtained, by a series of sufficiently simple operations,

$$\bar{b}ap - pb\bar{a} = 2(m\bar{b} - n\bar{a}),$$

and since $\mathrm{S}\bar{b}a = \mathrm{S}b\bar{a}$ and $\mathrm{T}\bar{b}a = \mathrm{T}b\bar{a} = \mathrm{T}ab$, this may be written

$$\mathrm{UV}\bar{b}a \cdot p - p \cdot \mathrm{UV}b\bar{a} + \frac{2(n\bar{a} - m\bar{b})}{\mathrm{TV}ab} = 0.$$

Here again is a single equation in a variable quaternion p representing a locus, viz., the intersection of a pair of Euclideans.

§§ 13–16. Loci of Linear Equations.

13. *Planes through the Origin.* If α_1 and α_2 be any two unit vectors and p a variable quaternion such that

$$\alpha_1 p + p\alpha_2 = 0\,,$$

then

$$S(\alpha_1 + \alpha_2)Vp = 0\,.$$

Thus, under the conditions imposed, Vp moves in a plane through the origin perpendicular to $\alpha_1 + \alpha_2$ and can therefore be written in the form

$$Vp = x(\alpha_1 - \alpha_2) + yV\alpha_1\alpha_2\,,$$

in which x and y are independent variable scalars; thus p has the form

$$p = x(\alpha_1 - \alpha_2) + yV\alpha_1\alpha_2 + Sp\,.$$

Subject this expression for p to the condition that it satisfy the equation $\alpha_1 p + p\alpha_2 = 0$ and note that the part involving $x(\alpha_1 - \alpha_2)$ vanishes identically. There remains

$$\alpha_1(yV\alpha_1\alpha_2 + Sp) + (yV\alpha_1\alpha_2 + Sp)\,\alpha_2 = 0\,,$$

and this solved for Sp gives

$$Sp = y(\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)^{-1}V\alpha_1\alpha_2 = y(1 + S\alpha_1\alpha_2)\,,$$

whence

$$p = x(\alpha_1 - \alpha_2) - y\alpha_1(\alpha_1 - \alpha_2)\,.$$

Thus, as x and y vary p moves in the plane of the two directors $\alpha_1 - \alpha_2$, $\alpha_1(\alpha_1 - \alpha_2)$ and the equation of this plane is

$$\alpha_1 p + p\alpha_2 = 0\,.$$

Or this equation may be written in the form

$$\alpha_1 p - pa^{-1}\alpha_1 a = 0\,,$$

where a is any quaternion satisfying the condition $\alpha_1 a + a\alpha_2 = 0$.

14. *Planes through any Point.* If q satisfy the equation

$$\alpha_1 q + q\alpha_2 + 2a = 0\,, \qquad (\alpha_1^2 = \alpha_2^2 = -1),$$

it is immediately evident that $\alpha_1 a = a\alpha_2$ and that therefore

$$\alpha_1 \cdot \alpha_1 a + \alpha_1 a \cdot \alpha_2 + 2a = 0\,.$$

Hence the values of q that satisfy this equation can be written in the form

(1) $$q = \alpha_1 a + x(\alpha_1 - \alpha_2) - y\alpha_1(\alpha_1 - \alpha_2)\,,\text{*}$$

where x and y are arbitrary scalars. The equation represents a plane through the extremity of $\alpha_1 a$.

* Stringham: loc. cit. (1) (1884), pp. 54–55.

These values of q may also be expressed in either of the forms:

$$q = a_1 a + l a_1^z (a_1 - a_2),$$

$$q = a_1 a + a_1 r - r a_2,$$

where l and z are arbitrary scalars and r is an arbitrary quaternion. The verifications are easily made by substitution in the equation.

It is important to observe that since $a_1 a = a a_2$, only two constants, a unit vector and a quaternion, are necessary for the complete determination of a plane. Thus, these constants being a_1 and a, the equation is

$$a_1 \cdot p + p \cdot a^{-1} a_1 a + 2a = 0.$$

15. *Planes containing given Elements.* Such an equation as the one last written represents any plane in four-dimensional space. For, suppose the plane to be determined by a point a_0 and two straight lines c, e through this point. We may find two unit vectors a_1, a_2 such that

$$a_1 c + c a_2 = 0, \quad a_1 e + e a_2 = 0.$$

In fact the solution of these equations for a_1, a_2 is

$$a_1 = \pm \mathrm{UV} e\bar{c}, \quad a_2 = \pm \mathrm{UV} \bar{e} c,$$

and by substitution in the equations it is found that these versors must have like signs. The vectors a_1, a_2 being thus determined, let $a = - a_1 a_0$ (equivalent to $a_0 = a_1 a$) and consider the equation

$$a_1 p + p a_2 + 2a = 0.$$

It is the equation of a plane and it is satisfied by a_0, by $a_0 + c$ and by $a_0 + e$, and thus, as required, it represents the plane that contains the point a_0 and the lines c, e.

If the plane be determined by three points, c, d, e, the equations of condition for a_1, a_2 may be written in the form

$$a_1(c - d) + (c - d)a_2 = 0, \quad a_1(c - e) + (c - e)a_2 = 0,$$

and then

$$a_1 = \mathrm{UV}(c\bar{d} + d\bar{e} + e\bar{c}), \quad a_2 = \mathrm{UV}(\bar{c}d + \bar{d}e + \bar{e}c),$$

$$a = -\tfrac{1}{2}(a_1 c + c a_2) = -\tfrac{1}{2}(a_1 d + d a_2) = -\tfrac{1}{2}(a_1 e + e a_2).$$

16. *The director $a_1 a$ is perpendicular to the plane.* For since $a_1 a = a a_2$,

$$\mathrm{S} a_1 a \{x(a_2 - a_1) - y(a_1 - a_2)a_1\} = 0,$$

the condition for perpendicularity (§ 5). Thus, the equation of a plane through the extremity of, and perpendicular to, a given director a_0 is

$$a_1 q + q a_2 - 2 a_1 a_0 = 0,$$

in which α_1, α_2 are any two unit vectors, satisfying the condition $\alpha_1 \alpha_0 = \alpha_0 \alpha_2$.

It will be convenient to write $(\alpha_1, \alpha_2, 2a)$ as an abbreviation meaning: the plane whose equation is $\alpha_1 q + q\alpha_2 + 2a = 0$ $(\alpha_2 = a^{-1}\alpha_1 a)$.

§§ 17–22. Intersections.

17. *Director of Meeting-point.* It is here shown that two planes,

$$\alpha_1 p + p\alpha_2 + 2a = 0, \quad \beta_1 p + p\beta_2 + 2b = 0,$$

always meet in at least one point, and the director to this point is found in terms of the constants of the equation. Operate progressively and regressively on the first equation with β_1, β_2, on the second with α_1, α_2 and in each case take the difference of the products and then add these differences together; the result is

$$2p\mathrm{S}(\alpha_1\beta_1 - \alpha_2\beta_2) + 2(\beta_1 a - a\beta_2 + \alpha_1 b - b\alpha_2) = 0.$$

Hence, in general

$$p = \frac{\beta_1 a - a\beta_2 + \alpha_1 b - b\alpha_2}{\mathrm{S}(\alpha_2\beta_2 - \alpha_1\beta_1)}.$$

If the planes have two points in common they have a line of intersection and a relation between the coefficients is obviously necessary. Hence, in general, two planes meet in one and only one point.

18. *Intersection Lines at the Origin.—In order that the two planes* $(\alpha_1, \alpha_2, 0)$, $(\beta_1, \beta_2, 0)$, *which meet at the origin, may meet in a straight line, it is necessary and sufficient that* $\mathrm{S}\alpha_1\beta_1 = \mathrm{S}\alpha_2\beta_2$.

It is necessary; for, if there be some value of p not zero that satisfies the two equations $\alpha_1 p + p\alpha_2 = 0$, $\beta_1 p + p\beta_2 = 0$, then

$$\alpha_1 = -p\alpha_2 p^{-1}, \quad \beta_1 = -p\beta_2 p^{-1},$$

and therefore

$$\mathrm{S}\alpha_1\beta_1 = \mathrm{S}\alpha_2\beta_2.$$

It is sufficient; for, if

$$c = \beta_1(x - y\alpha_1)(\alpha_1 + \alpha_2) - (x - y\alpha_1)(\alpha_1 + \alpha_2)\beta_2,$$

where x and y are arbitrary scalars, then the condition $\mathrm{S}\alpha_1\beta_1 = \mathrm{S}\alpha_2\beta_2$ suffices to make $\alpha_1 c + c\alpha_2 = 0$, $\beta_1 c + c\beta_2 = 0$, and thus the two planes meet in c.

19. *Cosine of a Dihedral Angle.* Two planes are supposed to meet in a straight line at the origin. Let u, v be two unit directors drawn, u in the plane $(\alpha_1, \alpha_2, 0)$, v in the plane $(\beta_1, \beta_2, 0)$ perpendicular to their line of intersection. Then $\alpha_1 u$ lies in $(\alpha_1, \alpha_2, 0)$ and $\beta_1 v$ in $(\beta_1, \beta_2, 0)$, and identically

$$\mathrm{S}\alpha_1 u\bar{u} = \mathrm{S}\beta_1 v\bar{v} = 0,$$

conditions sufficient to make $\alpha_1 u$ and $\beta_1 v$ perpendicular respectively to u and v.

Hence a_1u and β_1v are two expressions for the intersection line of the two planes and $a_1u = \pm\beta_1v$, whence

$$Su\bar{v} = \pm Sa_1\beta_1 = \pm Sa_2\beta_2. \qquad (\S 18)$$

These are the cosines of the dihedral angles formed by the two planes.

The planes are perpendicular to one another if $Sa_1\beta_1 = Sa_2\beta_2 = 0$.

20. *Intersection Lines in General.—In order that the two planes* $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ *may meet in a straight line it is necessary and sufficient that*

$$a_1b - ba_2 + \beta_1a - a\beta_2 = 0.$$

If the two planes meet in a straight line at a finite distance it will be possible to transfer the origin to a point in this line and thus reduce the equations to the form

$$a_1q + qa_2 = 0, \quad \beta_1q + q\beta_2 = 0.$$

Hence (§18) a necessary condition for a straight-line intersection is $Sa_1\beta_1 = Sa_2\beta_2$. But when this condition is satisfied the expression (§ 17)

$$\frac{a_1b - ba_2 + \beta_1a - a\beta_2}{S(a_2\beta_2 - a_1\beta_1)},$$

for a *point* of intersection of the two planes, is determinate and infinite unless the numerator vanish. Hence the condition

$$a_1b - ba_2 + \beta_1a - a\beta_2 = 0$$

is necessary.

This condition requires that $Sa_1\beta_1 = Sa_2\beta_2$, but the converse of this statement is not true; for if

$$f = a_1b - ba_2, \quad g = \beta_1a - a\beta_2, \quad m = S(a_2\beta_2 - a_1\beta_1),$$

then the identical relations

$$a_1f + fa_2 = 0, \quad a_1g + ga_2 + 2am = 0,$$

$$\beta_1g + g\beta_2 = 0, \quad \beta_1f + f\beta_2 + 2bm = 0,$$

put in immediate evidence the fact that, x being any scalar, $f = xg$ makes $m = 0$, and *vice versa*, but $m = 0$ does not involve $f + g = 0$.

The sufficiency of the condition $f + g = 0$ will now be proved if it be shown that the vanishing of $f + g$ makes it possible to assign a singly infinite series of quaternion directors which satisfy the equations of the two planes. But identically,

$$a_1Va\bar{b}\cdot f + Va\bar{b}\cdot fa_2 = af\bar{f},$$

$$\beta_1Vb\bar{a}\cdot g + Vb\bar{a}\cdot g\beta_2 = bg\bar{g},$$

and, if $f + g = 0$, also

$$\beta_1Va\bar{b}\cdot f + Va\bar{b}\cdot f\beta_2 = bf\bar{f},$$

$$a_1xf + xfa_2 = 0, \quad \beta_1xf + xf\beta_2 = 0,$$

where x is an arbitrary scalar. Hence, *provided f and g be not separately zero*, $(x - 2\mathrm{V}a\bar{b})/\bar{f}$ represents a singly infinite series of quaternions possessing the required property.

Thus the condition $f = -g \neq 0$ determines that the two planes meet in a straight line and enables us to assign as its equation

$$p = (x - 2\mathrm{V}a\bar{b})/\bar{f}.$$

Here neither f nor $\mathrm{V}\bar{a}b$ can vanish separately; for if $b = ya$, then $f = y(a_1 a - a a_2) = 0$, and if $f = 0$ both a and b represent the same director, namely the intersection of the two planes $(a_1, -a_2, 0)$, $(\beta_1, -\beta_2, 0)$, which can only happen when b is a numerical multiple of a.

Suppose then that $f = g = 0$; there are two cases:

(1) $a_1 b = b a_2 \neq \beta_1 a = a\beta_2$; (2) $a_1 b = b a_2 = \beta_1 a = a\beta_2$.

(1) If $a_1 b = b a_2 \neq \beta_1 a = a\beta_2$, then the above expression for p may be eplaced by

$$p = (\mathrm{V}a_1\beta_1)^{-1}(\beta_1 a - a_1 b + xa)$$

which, as may be easily verified, reduces the equations of $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ to identities for all (scalar) values of x. This is again the equation of a straight line. (I have not proved that $\mathrm{V}a_1\beta_1$ is not zero.)

(2) If $a_1 b = b a_2 = \beta_1 a = a\beta_2$, a formal solution is

$$p = (\mathrm{V}a_1\beta_1)^{-1}(\beta_1 + x)a,$$

an expression which again reduces the equations of the two planes to identities. But here, since b is a multiple of a and since now $a_1\beta_1 = -ab^{-1}$, we have $\mathrm{V}a_1\beta_1 = 0$ and therefore

$$\beta_1 = \pm a_1, \quad \beta_2 = \pm a_2,$$

and the signs must be $+\,+$ or $-\,-$. Hence $p = \infty$. This is the case of parallel planes, as will be shown in the sequel (§ 29).

Thus, in whatever way $f + g$ becomes zero the two planes meet in a straight line, either at a finite or at an infinite distance.

Incidentally, either $-2\mathrm{V}a\bar{b}/\bar{f}$ or $(\mathrm{V}a_1\beta_1)^{-1}(\beta_1 a - a_1 b)$, the former in the general case, the latter when $a_1 b = b a_2 \neq \beta_1 a = a\beta_2$, is determined as the director-perpendicular from the origin to the intersection-line of the two planes. This is made evident by applying the test of (§ 5).

21. *Intersections at Infinity.* If the two planes meet only once at an infinite distance, we must have $(f + g)/m = 0$, and $f + g \neq 0$ (§ 20), and these conditions are obviously sufficient. Hence, taking account of § 20 (2):

For the two planes $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ to meet at infinity, the necessary and sufficient conditions are:

if at a point, $\quad \mathrm{S}a_1\beta_1 = \mathrm{S}a_2\beta_2, \quad f + g \neq 0;$

if in a line, $\quad \beta_1 = \pm a_1, \quad \beta_2 = \pm a_2.$

22. *Normals.* It is easily verified that:

(1) The perpendicular distance (director-normal) between the two planes $(\alpha_1, \alpha_2, 2a)$ and $(\alpha_1, \alpha_2, 2b)$ is, in both magnitude and direction, $\alpha_1(a - b)$.

(2) The director normal from the extremity of c to the plane $(\alpha_1, \alpha_2, 2b)$ is

$$\tfrac{1}{2}\alpha_1(2a + \alpha_1 c + c\alpha_2).$$

(3) It has been shown in § 21 that the director-normal from the origin to the intersection of $(\alpha_1, \alpha_2, 2a)$ and $(\beta_1, \beta_2, 2b)$ is

$$(a\bar{b} - b\bar{a})/(\bar{b}\dot{a}_1 - a_2\bar{b})$$

the conditions $f + g = 0$, $f \neq g$ being here essential.

§§ 23–27. Divergence of Two Planes and its Measure.

23. *Isoclinal Angles.* Two planes meet, in general, in a point or in a straight line (§§ 17, 20). It is always possible, as will presently appear, to pass through any point common to two given planes other (transversal) planes which meet the former in straight lines and form with them equal opposite interior dihedral angles. The plane angle formed by the edges of the two dihedrals, whose plane as viewed from a point between the two given planes is equally inclined to them, will be called their *isoclinal angle.* This angle is an appropriate index (measure) of the amount (rapidity) of the divergence of the two planes under a special aspect, namely, from the vertex and in the direction of the sides of the isoclinal angle. Its variations afford the means for determining certain important relations of planes to one another.

The given planes being $(\alpha_1, \alpha_2, 2a)$, $(\beta_1, \beta_2, 2b)$, we transfer the origin to the point of, or to a point in, their intersection. Their equations become (§ 13)

$$\alpha_1 p + p\alpha_2 = 0, \quad \beta_1 p + p\beta_2 = 0;$$

and x, y being continuous scalar variables,

$$u = \alpha_1^x(\alpha_1 - \alpha_2), \quad v = \beta_1^y(\beta_1 - \beta_2)$$

represent two directors lying in them, u in the first, v in the second (§ 14). I enquire whether and under what conditions the plane angle formed by u and v may have a maximal or a minimal value. Without loss of generality we may suppose that $\mathrm{T}u = \mathrm{T}v = 1$.

The first and second partial derivatives of $\mathrm{S}u\bar{v}$ with respect to x and y are

$$\frac{\partial\,\mathrm{S}u\bar{v}}{\partial x} = \frac{\pi}{2}\,\mathrm{S}\alpha_1 u\bar{v}, \quad \frac{\partial\,\mathrm{S}u\bar{v}}{\partial y} = -\frac{\pi}{2}\,\mathrm{S}\beta_1 u\bar{v},$$

$$\frac{\partial^2\,\mathrm{S}u\bar{v}}{\partial x^2} = -\frac{\pi^2}{4}\,\mathrm{S}u\bar{v}, \quad \frac{\partial^2\,\mathrm{S}u\bar{v}}{\partial y^2} = -\frac{\pi^2}{4}\,\mathrm{S}u\bar{v},$$

and

$$\frac{\partial^2\,\mathrm{S}u\bar{v}}{\partial x\,\partial y} = -\frac{\pi^2}{4}\,\mathrm{S}\beta_1\alpha_1 u\bar{v}.$$

The two second partial derivatives, with respect to x, and with respect to y, being negative as compared with the function itself, the conditions for a maximum or a minimum of $\mathrm{S}u\bar{v}$ are

$$\mathrm{S}\alpha_1 u\bar{v} = 0, \quad \mathrm{S}\beta_1 u\bar{v} = 0,$$

and

$$\mathrm{S}^2 u\bar{v} > \mathrm{S}^2\beta_1\alpha_1 u\bar{v}, \quad \text{or} \quad \mathrm{S}^2 u\bar{v} < \mathrm{S}^2\beta_1\alpha_1 u\bar{v}.$$

Thus $\mathrm{S}u\bar{v}$, *and therefore also the angle of* u, v, *will have maxima and minima if there exist an inequality of the form:*

$$|\mathrm{S}u\bar{v}| \mp |\mathrm{S}\alpha_1 u\overline{\beta_1 v}|.$$

Let it be supposed, for the moment, that the directors u, v are fixed by the conditions thus imposed (it will be shown presently that this is actually the case) and let u and v be turned in their respective planes about the origin through the same angle $\frac{1}{2}\pi\theta$ by multiplying the first by α_1^θ, and the second by β_1^θ, where θ is a scalar variable.

(1) The essential preliminary condition: *The plane of* $\alpha_1^\theta u$ *and* $\beta_1^\theta v$ *shall intersect both* $(\alpha_1, \alpha_2, 0)$ *and* $(\beta_1, \beta_2, 0)$ *in straight lines*, is here fulfilled; for γ_1, γ_2 being $= \mathrm{UV}\alpha_1^\theta u\bar{v}\beta_1^{-\theta}$, $\mathrm{UV}\bar{u}\alpha_1^{-\theta}\beta_1^\theta v$, the equation of this plane is, by § 15 (1),

$$\gamma_1 p + p\gamma_2 = 0,$$

and the sufficient conditions (§ 18)

$$\mathrm{S}\alpha_1\gamma_1 = \mathrm{S}\alpha_2\gamma_2, \quad \mathrm{S}\beta_1\gamma_1 = \mathrm{S}\beta_2\gamma_2,$$

are here satisfied.

(2) *The angle of* $\alpha_1^\theta u$ *and* $\beta_1^\theta v$ *is isoclinal to* $(\alpha_1, \alpha_2, 0)$ *and* $(\beta_1, \beta_2, 0)$ *for all values of* θ. For we may write

$$\alpha_1^\theta = x + y\alpha_1, \quad \beta_1^\theta = x + y\beta_1 \qquad (x^2 + y^2 = 1),$$

and then, the conditions $\mathrm{S}\alpha_1 u\bar{v} = \mathrm{S}\beta_1 u\bar{v} = 0$ being satisfied,

$$\begin{aligned}\mathrm{S}\alpha_1\gamma_1 &= xy\mathrm{S}(\beta_1\alpha_1 u\bar{v} - u\bar{v}) = -\mathrm{S}\beta_1\gamma_1 \\ &= \mathrm{S}\alpha_2\gamma_2 = -\mathrm{S}\beta_2\gamma_2,\end{aligned}$$

that is, the corresponding exterior-interior dihedral angles on the same side of the transversal are equal. This isoclinal angle may thus be treated as a function of θ.

(3) Let the values of θ that make the angle of $\alpha_1^\theta u$ and $\beta_1^\theta v$ a maximum or a minimum be sought. The first and second derivatives of $\mathrm{S}\alpha_1^\theta u\bar{v}\beta_1^{-\theta}$ with respect to θ are

$$\frac{d\,\mathrm{S}w}{d\theta} = \frac{\pi}{2}\mathrm{S}(\alpha_1 - \beta_1)w \qquad (w = \alpha_1^\theta u\bar{v}\beta_1^{-\theta}),$$

$$\frac{d^2\mathrm{S}w}{d\theta^2} = -\frac{\pi^2}{2}\mathrm{S}(1 + \beta_1\alpha_1)w,$$

and by virtue of the previously assumed conditions $S\alpha_1 u\bar{v} = S\beta_1 u\bar{v} = 0$, the first derivative vanishes for all integral values of θ.

When $\theta = 0$ or an even number, the second derivative is a negative quantity if $Su\bar{v} > S\alpha_1 u\overline{\beta_1 v}$, and these (as previously found) are the conditions that make $Su\bar{v}$ a maximum.

When $\theta =$ an odd number, the second derivative is a positive quantity if $Su\bar{v} > S\alpha_1 u\overline{\beta_1 v}$, and these are the conditions that make $Su\bar{v}$ a minimum.

It is immediately evident that if there be maxima and minima, there are two of each, occurring alternately at intervals of ninety degrees; for the successive angles are:

$$\angle(\alpha_1^\theta u\,,\ \beta_1^\theta v) \qquad (\theta = 0,\ 1,\ 2,\ 3),$$

and $S\alpha_1^\theta u\bar{u} = S\beta_1^\theta v\bar{v} = 0$ if θ be an odd integer.

It will be shown presently (§ 24) that non-integral values of θ do not give rise to maximal or minimal isoclinal angles. The above enumeration is therefore exhaustive.

(4) *The two planes of the maximal and minimal isoclinal angles, and these only, are orthogonal to both of the given planes.* For, in order that the conditions for perpendicularity (§ 19) may be satisfied, namely:

$$S\alpha_1\gamma_1 = S\alpha_2\gamma_2 = S\beta_1\gamma_1 = S\beta_2\gamma_2 = 0\,,$$

it is necessary and sufficient that x or $y = 0$ (in $\alpha_1^\theta = x + y\alpha_1$, $\beta_1^\theta = x + y\beta_1$, of (2)), and these are precisely the condition $\theta =$ an integer.

(5) If the terms of $Su\bar{v} > S\alpha_1 u\overline{\beta_1 v}$ be interchanged, $Su\bar{v}$ becomes a minimum and $S\alpha_1 u\overline{\beta_1 v}$ a maximum. Such an interchange, however, is merely equivalent to assigning $u' = \alpha_1 u$, $v' = \beta_1 v$ as the two director boundaries of the minimal and $\alpha_1 u'$, $\beta_1 v'$ as the director boundaries of the maximal angle. And unless $Su\bar{v} = S\alpha_1 u\overline{\beta_1 v}$, an inequality of the form $Su\bar{v} > S\alpha_1 u\overline{\beta_1 v}$ may be assumed to exist. Hence if the conditions

$$S\alpha_1 u\bar{v} = 0\,, \quad S\beta_1 u\bar{v} = 0\,, \quad Su\bar{v} \neq S\alpha_1 u\overline{\beta_1 v}$$

be satisfied, either $Su\bar{v}$ is a maximum and $S\alpha_1 u\overline{\beta_1 v}$ a minimum, or *vice versa*.

(6) It remains to show that the conditions $S\alpha_1 u\bar{v} = S\beta_1 u\bar{v} = 0$ suffice to determine a convenient measure of the divergence of the two planes. In another form, $S\alpha_1 u\bar{v} = 0$ is

$$\alpha_1 u\bar{v} - v\bar{u}\alpha_1 = 0\,,$$

which is also

$$\beta_1\alpha_1 u\bar{v} - \beta_1 v\bar{u}\alpha_1 = 0\,.$$

But

$$\beta_1\alpha_1 u\bar{v} + v\bar{u}\alpha_1\beta_1 = 2S\beta_1\alpha_1 u\bar{v} = \beta_1 v\bar{u}\alpha_1 + v\bar{u}\alpha_1\beta_1\,.$$

Multiplied by $u\bar{v}$ this last equation is

$$2u\bar{v}S\beta_1\alpha_1u\bar{v} = u\bar{v}v\beta_2\alpha_2\bar{u} + v\bar{v}u\bar{u}\alpha_1\beta_1,$$

whence, by equating scalar parts and dividing by 2,

$$Su\bar{v}\cdot S\alpha_1u\overline{\beta_1v} = -\tfrac{1}{2}S(\alpha_1\beta_1 + \alpha_2\beta_2).*$$

There is thus obtained, as the measure of divergence of the two planes, the product of the cosines of their numerical maximal and minimal isoclinal angles.

24. *The Ultimate Criteria of Maxima and Minima.* The higher odd-order derivatives of Sw, $= S\alpha_1^\theta u\bar{v}\beta_1^{-\theta}$ are

$$\frac{d^{2n-1}Sw}{d\theta^{2n-1}} = (-1)^{n-1}\frac{\pi^{2n-1}}{2}S(\alpha_1 - \beta_1)w,$$

and the even-order derivatives are

$$\frac{d^{2n}Sw}{d\theta^{2n}} = (-1)^n\frac{\pi^{2n}}{2}S(1 + \beta_1\alpha_1)w.$$

If we write

$$(\alpha_1^\theta,\ \beta_1^\theta) = (x + y\alpha_1,\ x + y\beta_1) \qquad (x^2 + y^2 = 1),$$

and maintain intact the conditions $S\alpha_1u\bar{v} = S\beta_1u\bar{v} = 0$, these higher derivatives become

$$\frac{d^{2n-1}Sw}{d\theta^{2n-1}} = (-1)^{n-1}\pi^{2n-1}xyS(\alpha_1u\overline{\beta_1v} - u\bar{v}),$$

$$\frac{d^{2n}Sw}{d\theta^{2n}} = (-1)^n\frac{\pi^{2n}}{2}(y^2 - x^2)S(\alpha_1u\overline{\beta_1v} - u\bar{v}).$$

Hence the derivatives of odd order vanish if, and only if, $x = 0$, or $y = 0$, or $S\alpha_1u\overline{\beta_1v} = Su\bar{v}$; and if $S\alpha_1u\overline{\beta_1v} = Su\bar{v}$ all the derivatives of even order vanish. The immediate consequence is:

Non-integral values of θ (corresponding to $x \neq 0$, $y \neq 0$) give rise to neither maxima nor minima of the isoclinal angle $(\alpha_1^\theta u,\ \beta_1^\theta v)$ and the enumeration of § 23 (3) *is complete.*

If the equations of the two planes be in the general form, referred to an arbitrary origin, the common vertex of their isoclinal angles is their point of meeting, or a point in their line of intersection, and c being the director to this point, we may assign as general solutions of their equations: $c + \alpha_1^\theta u$ and $c + \beta_1^\theta v$ respectively, where u and v are determined by the foregoing conditions for maxima and minima. The final form of our criterion then is:

The conditions necessary and sufficient in order that the two planes

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* STRINGHAM: loc. cit. (3) (1888), p. 64. Of course the formulæ leading up to this result had been determined prior to the date August, 1888. Cf. HATHAWAY: loc. cit. (2) (1897), § 45, p. 10.

$(a_1, \beta_1, 2a)$, $(\beta_1, \beta_2, 2b)$ *may have maximal and minimal isoclinal angles are: that there exist solutions* $c + a_1^\theta u$ *and* $c + \beta_1^\theta v$, *of their respective equations, such that*

$$Sa_1 u\bar{v} = 0\,, \quad S\beta_1 u\bar{v} = 0\,, \quad Su\bar{v} \neq Sa_1 u\overline{\beta_1 v}\,.$$

25. *The Maximal and Minimal Angles Determined.* It has been seen (§ 23 (4)) that the planes of the maximal and minimal isoclinal angles, and these only, cut both the given planes orthogonally. But the conditions of perpendicularity will be here satisfied by writing as the equations of these two orthogonal transversals,

$$\gamma_1 p + p\gamma_2 = 0, \quad \gamma_1 p - p\gamma_2 = 0\,,$$

where

$$\gamma_1 = UVa_1\beta_1, \quad \gamma_2 = UVa_2\beta_2.$$

Hence the four intersections of these two planes with $(a_1, a_2, 0)$ and $(\beta_1, \beta_2, 0)$ must be the directors hitherto designated by u, v, $a_1 u$, $\beta_1 v$; and we may write (§ 18)

$$u = a_1(\gamma_1 + \gamma_2) - (\gamma_1 + \gamma_2)a_2\,,$$

$$v = \beta_1(\gamma_1 + \gamma_2) - (\gamma_1 + \gamma_2)\beta_2\,,$$

$$a_1 u = u' = a_1(\gamma_1 - \gamma_2) - (\gamma_1 - \gamma_2)a_2\,,$$

$$\beta_1 v = v' = \beta_1(\gamma_1 - \gamma_2) - (\gamma_1 - \gamma_2)\beta_2\,;$$

and we may verify independently the necessary conditions

$$Sa_1 u\bar{v} = 0, \quad S\beta_1 u\bar{v} = 0, \quad Sa_1 u'\bar{v}' = 0, \quad S\beta_1 u'\bar{v}' = 0\,,$$

which are in fact identities if u, v, u', v' have the values above assigned.

The further conditions that u, v shall be perpendicular to u', v' (§ 23 (3)) are here also satisfied, for the planes $(\gamma_1, \gamma_2, 0)$, $(\gamma_1, -\gamma_2, 0)$ are hyperperpendicular to one another (§ 28); or it may be verified independently that

$$Su\bar{u}' = 0, \quad Sv\bar{v}' = 0\,.$$

This determination fails if either $\beta_1 = \pm a_1$ or $\beta_2 = \pm a_2$, for then either $\gamma_1 = 0$, or $\gamma_2 = 0$; but these are the conditions for the failure of maxima and minima (§ 26).

26. *Failure of the Condition* $Su\bar{v} \neq Sa_1 u\overline{\beta_1 v}$. From the conditions:

$$Sa_1 u\bar{v} = Sa_2\bar{v}u = 0\,, \quad S\beta_1 u\bar{v} = S\beta_2\bar{v}u = 0\,,$$

it follows, provided $a_1 \neq \beta_1$ and $a_2 \neq \beta_2$, that $Vu\bar{v}$ and $V\bar{v}u$ are numerical multiples of $V\beta_1 a_1$ and $Va_2\beta_2$ respectively, and (assuming $Tu = Tv = 1$) we may write

$$u\bar{v} = \cos\phi \pm \delta_1 \sin\phi \qquad (\delta_1 = \mathrm{UV}\beta_1 a_1),$$

$$\bar{v}u = \cos\psi \pm \delta_2 \sin\psi \qquad (\delta_2 = \mathrm{UV}a_2\beta_2).$$

We may also assume

$$\beta_1 a_1 = -\cos\epsilon_1 + \delta_1 \sin\epsilon_1,$$

$$a_2\beta_2 = -\cos\epsilon_2 + \delta_2 \sin\epsilon_2.$$

Then

$$\mathrm{S}(1 + \beta_1 a_1)u\bar{v} = 2\sin\frac{\epsilon_1}{2}\sin\left(\frac{\epsilon_1}{2} \pm \phi\right),$$

$$\mathrm{S}(1 + a_2\beta_2)\bar{v}u = 2\sin\frac{\epsilon_2}{2}\sin\left(\frac{\epsilon_2}{2} \pm \psi\right),$$

and, therefore, in order that the condition $\mathrm{S}u\bar{v} \neq \mathrm{S}a_1 u\beta_1 v$ may fail it is necessary and sufficient that

$$\epsilon_1 = 2n_1\pi \text{ or } 2n_1\pi \pm \phi,$$

and

$$\epsilon_2 = 2n_2\pi \text{ or } 2n_2\pi \pm \psi,$$

or, in equivalent terms,

$$\beta_1 a_1 = -1 \text{ or } -(v\bar{u})^2 \text{ or } -(u\bar{v})^2,$$

and

$$a_2\beta_2 = -1 \text{ or } -(\bar{u}v)^2 \text{ or } -(\bar{v}u)^2,$$

and there are nine combinations. Any combination of the type $\beta_1 a_1 = -(v\bar{u})^2$, $a_2\beta_2 = -(\bar{u}v)^2$, however, leads to results identical with those obtained from $\beta_1 a_1 = -1$, $a_2\beta_2 = -1$; for, from any of the pairs of equations of this type and from § 23 (6) follows

$$\mathrm{S}(u\bar{v})^2 = -\tfrac{1}{2}\mathrm{S}(a_1\beta_1 + a_2\beta_2) = \mathrm{S}^2 u\bar{v},$$

an interchange of factors under S being here permissible; thus $\mathrm{V}u\bar{v} = 0$, $u = \pm v$, and in fact

$$\beta_1 a_1 = -1, \quad a_2\beta_2 = -1.$$

But also, this combination is a particular case (obtained by making $u = \pm v$) of either of the remaining two and thus the two alternative conditions

$$\beta_1 = a_1, \quad \beta_2 = a_2(\bar{u}v)^2 \text{ or } a_2(\bar{v}u)^2,$$

and

$$\beta_2 = a_2, \quad \beta_1 = (v\bar{u})^2 a_1 \text{ or } (u\bar{v})^2 a_1,$$

account for all the cases that can arise. The specific value of $u\bar{v}$ is here undetermined and may vary from a unit scalar to a unit vector value.

It has been tacitly assumed that a_1, β_1 in the one case, and a_2, β_2 in the other, are both essentially positive, but it is evident that the algebraic signs in the equations of the two planes can always be so disposed that this shall be the fact. The conclusion, in its general form, is:

(1) *In order that the two planes* $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ *may cease to have maximal and minimal isoclinal angles it is necessary that* $\beta_1 = \pm a_1$, *or* $\beta_2 = \pm a_2$ *and any one of these conditions is sufficient.*

(2) *The isoclinal angle* $(a_1 u, \beta_1 v)$ *is constant under all variations of* θ; this is apparent in the equations

$$Sa_1^{\theta} u \bar{v} \beta_1^{-\theta} = Su\bar{v}, \quad Sua_2^{-\theta}\beta_2^{\theta}\bar{v} = Su\bar{v}, *$$

the first of which is a consequence of $\beta_1 = a_1$, the second of $\beta_2 = a_2$.

26. *An Example.* The following example illustrates the failure of the conditions for maximal and minimal isoclinal angles. Suppose the equations of the two planes to be

$$ap + pa = 0, \quad ap + p\beta = 0.$$

The first is equivalent to

$$2aSp + 2SaVp = 0,$$

which involves the two equations

$$Sp = 0, \quad SaVp = 0.$$

Hence the values of p that satisfy the first equation represent vectors perpendicular to a and may be written in the form

$$p = xa^{\theta}\gamma,$$

in which x and θ are scalar variables and γ is a vector constant, and the condition $Sa\gamma = 0$ is necessary.

The general values of p that satisfy the second equation may have the form

$$p = xa^{\theta}(a - \beta).$$

If now $u = \gamma$ and $v = a - \beta$, then

$$Sau\bar{v} = Sa\gamma(\beta - a),$$

which vanishes if $Sa\gamma\beta = 0$, that is, if γ be chosen coplanar with a and β. Hence, with this choice of γ made, the directors γ and $a - \beta$ satisfy the first two conditions for a maximal or a minimal isoclinal angle. But

$$Sa^{\theta}\gamma(\beta - a)a^{-\theta} = S\gamma(\beta - a) = Su\bar{v}$$

for all values of θ and the third condition fails. The isoclinal angle $(a^{\theta}u, a^{\theta}v)$ remains unchanged during any variations of θ. In other words, every line in either plane has the same inclination to the other plane.

---

* Note that $a_1^{\theta}u = ua_2^{-\theta}$, $\beta_1^{\theta}v = v\beta_2^{-\theta}$, and therefore $Sa_1^{\theta}u\bar{v}\beta_1^{-\theta} = Sua_2^{-\theta}\beta_2^{\theta}\bar{v}$.

The angle of this inclination (constant isoclinal angle) is represented at C, in the accompanying figure. The equation

$$\alpha\rho + \rho\alpha = 0$$

represents a plane through O perpendicular to α, lying wholly within vector space (a space containing only vectors), and

$$\alpha\rho + \rho\beta = 0$$

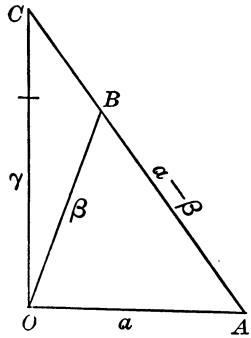
is the equation of a plane that intersects this vector space in a straight line through O parallel to $\alpha - \beta$.

§§ 28, 29. Some Special Cases.

28. *Perpendicularity.* If the maximal isoclinal angle, say $\angle(\alpha_1 u, \beta_1 v)$, be a right angle, then not only $S\alpha_1 u\bar{v} = 0$, but also $S\alpha_1 u\overline{\beta_1 v} = 0$, and

$$S\alpha_1 u\bar{v}\beta_1^{-\theta} = 0$$

for all values of θ; that is, $\alpha_1 u$, a director in one of the planes, is perpendicular to $\beta_1^\theta v$, any director (through their meeting point) in the other. This is per-



pendicularity according to the ordinary definition, though not of the ordinary sort along a line of intersection, for the planes may not meet in a line.

Conversely, if the two planes be perpendicular to one another it must be possible to assign in either of them a director that shall be perpendicular to every director in the other, say $\alpha_1 u$ and $\beta_1 v$, such that

$$S\alpha_1 u\bar{v}\beta_1^{-y} = 0, \quad S\alpha_1^x u\bar{v}\beta_1 = 0,$$

where x and y are arbitrary scalars; and it follows (by assuming in succession $x, y = 0, 1$) that

$$S\alpha_1 u\bar{v} = S\beta_1 u\bar{v} = S\beta_1\alpha_1 u\bar{v} = 0 .$$

These conditions make $\angle(\alpha_1 u, \beta_1 v)$ a maximal isoclinal angle if $Su\bar{v} \neq 0$.

But $S\beta_1 a_1 u\bar{v}$ is a factor in $S(a_1\beta_1 + a_2\beta_2)$ and therefore:

In order that the planes $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ *may satisfy the ordinary definition of perpendicularity it is necessary and sufficient that*

$$S(a_1\beta_1 + a_2\beta_2) = 0.$$

There are three further distinct criteria to be considered, giving rise to three kinds of perpendicularity.

(1) $Sa_1 u\overline{\beta_1 v} = 0$, $Su\bar{v} \neq 0$ or 1. Resume the equations of § 26 for $u\bar{v}$ and $\beta_1 a_1$, omitting the ambiguous sign not here needed; we have

$$u\bar{v} = \cos\phi + \delta_1 \sin\phi, \quad \beta_1 a_1 = -\cos\epsilon_1 + \delta_1 \sin\epsilon_1,$$

and therefore

$$Sa_1 u\overline{\beta_1 v} = -\cos(\epsilon_1 + \phi) = 0,$$

$$\epsilon_1 + \phi = \frac{\pi}{2}.$$

Hence $Sa_1\beta_1 \neq Sa_2\beta_2$ *and the planes meet only in a point.* According as $\angle(\beta_1, a_1)$ is large or small the minimal isoclinal angle is small or large. The condition may be stated in the form:

$$Sa_1\beta_1 = -Sa_2\beta_2 \neq 0, \text{ or } \pm 1.$$

(2) $Sa_1 u\overline{\beta_1 v} = 0$, $Su\bar{v} = 1$. Here $\phi = 0$, $\epsilon_1 = \pi/2$, and $\beta_1 a_1$ is a vector; hence

$$Sa_1\beta_1 = Sa_2\beta_2 = 0,$$

is the condition necessary and sufficient for simple perpendicularity along a line of intersection (the ordinary kind of perpendicularity); provided also, the planes being $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$, the further condition

$$a_1 b - b a_2 + \beta_1 a - a\beta_2 = 0$$

is satisfied (§§ 20, 21).

(3) $Sa_1 u\beta_1 v = Su\bar{v} = 0$. Here $u\bar{v} + v\bar{u} = 0$, $(u\bar{v})^2 = -1$, and therefore

$$\beta_1 a_1 = \pm 1, \quad a_2\beta_2 = \mp 1,$$

that is, $\beta_1 = \mp a_1$ and $\beta_2 = \pm a_2$, the combinations of signs being either $-+$ or $+-$.

If either of these pairs of conditions be assigned, that is, if u' and v' satisfy the equations

$$a_1 u' + u' a_2 = 0, \quad a_1 v' - v' a_2 = 0,$$

we may write (§ 14)

$$u' = a_1^x(a_1 - a_2), \quad v' = a_1^y(a_1 + a_2),$$

in which x and y are arbitrary scalars, and then

$$Su'\bar{v}' = -Sa_1^{x-y}(a_1 a_2 - a_2 a_1),$$

and this is zero for all values of x and y. The geometrical interpretation of this result is that *every line in either plane is perpendicular to every line in the other*. In other words:

The necessary and sufficient conditions for the hyperperpendicularity of the planes $(\alpha_1, \alpha_2, 2a)$, $(\beta_1, \beta_2, 2b)$ *are*

$$\beta_1 = \pm \alpha_1, \quad \beta_2 = \mp \alpha_2;$$

or, expressed in another form,

$$S\alpha_1\beta_1 = - S\alpha_2\beta_2 = \pm 1.$$

The conditions for maximal and minimal isoclinal angles fail.

29. *Parallelism.* If two planes $(\alpha_1, \alpha_2, 2a)$, $(\beta_1, \beta_2, 2b)$ be parallel, the pairs of directors $\alpha_1^\theta u$, $\beta_1^\theta v$* that form their isoclinal angles are parallel for all values of θ, and in particular, if $Tu = Tv = 1$,

$$Su\bar{v} = S\alpha_1 u \overline{\beta_1 v} = 1,$$

whence

$$S(\alpha_1\beta_1 + \alpha_2\beta_2) = -2.$$

But also, they will have no point of intersection except at an infinite distance, and therefore

$$\frac{\alpha_1 b - b\alpha_2 + \beta_1 a - a\beta_2}{S(\alpha_2\beta_2 - \alpha_1\beta_1)} = \infty,$$

and the further condition $S\alpha_2\beta_2 = S\alpha_1\beta_1$ is necessary. Hence, for the two planes to be parallel it is necessary that

$$S\alpha_2\beta_2 = S\alpha_1\beta_1 = -1.$$

This condition is also sufficient, for, if it is satisfied, it immediately follows that $S(\alpha_1\beta_1 + \alpha_2\beta_2) = -2$ and $Su\bar{v} = \overline{S\alpha_1 u}\beta_1 v = 1$. Therefore:

(1) *In order that* $(\alpha_1, \alpha_2, 2a)$ *and* $(\beta_1, \beta_2, 2b)$ *may be parallel it is necessary and sufficient that*

$$\alpha_1 = \beta_1, \quad \alpha_2 = \beta_2.$$

(It is understood that the signs before α_1, α_2 in their respective equations are both positive.)

It is immediately evident that the condition

$$\alpha_1 b - b\alpha_2 + \beta_1 a - a\beta_2 = 0$$

is satisfied. This I interpret as meaning that two parallel planes meet in a straight line at infinity.

* These directors are supposed to be placed in their respective planes without reference to a particular origin.

(2) When the conditions for parallelism are satisfied,

$$a_1 a - a a_2 = 0, \quad a_1 b - b a_2 = 0,$$

and we may write

$$a = m a^{\theta}(a_1 + a_2), \quad b = n a_1^{\phi} a,$$

where m, n, θ, ϕ are arbitrary scalars, and the equation of $(\beta_1, \beta_2, 2b)$ becomes

$$a_1 p + p a_2 + 2n a_1^{\phi} a = 0.$$

It represents a doubly infinite series of planes parallel to $(a_1, a_2, 2a)$, one series being obtained by varying n, the other by varying ϕ.

(3) When the point of intersection of a pair of planes is moved to an infinite distance, they have in general at the limit a point of intersection at infinity. But if the conditions for maximal and minimal isoclinal angles fail these angles become ultimately zero and the conditions for parallelism are satisfied. *Thus planes that have a constant isoclinal angle and meet at infinity are parallel and meet in a straight line.*

§§ 29–32. Isoclinal Systems.

30. *Two-dimensional Systems.* The following group of theorems (§§ 30–32) restate for planes through a point in four-dimensional space the fundamental parts of Clifford's theory of parallels in elliptic space of three dimensions (Hathaway's theory of parallel and contra-parallel great circles on the hypersphere).* The planes and lines here considered are, without exception, supposed to pass through the origin; and by any plane, or any line, is meant a plane or a line satisfying this condition.

Consider a series of planes meeting in a point no two of which have maximal and minimal isoclinal angles, and place the origin at their point of meeting. Corresponding to the two conditions $\beta_1 = \pm a_1$, $\beta_2 = \pm a_2$ of § 26, there are two systems of planes: an a-system whose equations may be written in the form (§ 13)

$$ap - pa^{-1}aa = 0,$$

and a β-system whose equations similarly constructed are

$$b\beta b^{-1}p - p\beta = 0,$$

where a and b are arbitrary versors whose different values produce the several

* It was Professor Hathaway's paper on *Quaternions as Numbers of Four-Dimensional Space* that forced upon my attention the essential identity of the two theories. It should be remarked that here, as also in Hathaway's paper, the theory is stated as applying to elliptic space in its antipodal (spherical) form.

The references to Clifford, Buchheim, Klein, Whitehead are here important.

individual planes of the systems. Any system is thus determined by a characteristic vector a, or β, and by the corresponding form of its equations.

(1) *Any two planes of a system have a unique (constant) isoclinal angle* (§ 23), a consequence of the failure of the conditions for maximal and minimal isoclinal angles (§ 26 (2)).

(2) *A transversal meeting any two planes of a system in straight lines makes equal dihedral angles with them.* For, the two planes being respectively $(a, -a^{-1}aa, 0)$, $(a, -a'^{-1}aa', 0)$ and their transversal being $(\gamma, -\gamma', 0)$, the conditions for straight-line intersections are

$$Sa\gamma = Sa^{-1}aa\gamma' = Sa'^{-1}aa'\gamma',$$

and in either case the cosine of the dihedral angle is $-Sa\gamma$ (§ 19). Here, the signs of the scalar functions being alike, the equal dihedrals are the alternate exterior-interior angles on the same side of the transversal.

Thus all the planes of a system determined by any unit vector are equally inclined to any transversal meeting them in straight lines. They constitute what may be called an *isoclinal system*, or a *system of isoclines.* Two planes may be said to be mutually isoclinal when their isoclinal angle is constant; and from this definition it immediately follows that *any two planes that are isoclinal to a third plane are isoclinal to each other.*

(3) *To a given plane through a given straight line there exist always two isoclines.* For, given a plane $(a, -\beta, 0)$ and a straight line c, both of the planes whose equations are

$$ap - pc^{-1}ac = 0, \quad c\beta c^{-1}p - p\beta = 0$$

are isoclinal to $(a, -\beta, 0)$, and they intersect in c.

(4) Given a and a, the equation $ap - pa^{-1}aa = 0$ is uniquely determined; but since we have identically

$$a^{-1}aa = (a^{\psi}a)^{-1}a(a^{\psi}a),$$

any pair of values a, $a^{\psi}a$ determines this same plane. Let $a = \beta^{\phi}$; the equation of the planes of the a-system has then the form

$$ap - p(a^{\psi}\beta^{\phi})^{-1}a(a^{\psi}\beta^{\phi}) = 0.$$

Changes in ψ give no new planes, but there is a distinct plane for each value of ϕ (within the limits 0, 4), and some (not all) of the differing values of β correspond to different planes.

If a change from β to β_1 makes no effective change in the equation, then

$$\beta_1^{-\phi}a\beta_1^{\phi} = \beta^{-\phi}a\beta^{\phi}, \quad a\beta_1^{\phi}\beta^{-\phi} = \beta_1^{\phi}\beta^{-\phi}a,$$

from which it follows that $V\beta_1^{\phi}\beta^{-\phi}$ is parallel to a; then

$$\beta_1^\phi = a^z \beta^\phi .$$

Hence, corresponding to the varying scalar values of z, there is a singly infinite series of changes in β which give rise to no new planes.

But any unit vector is determined by two independent scalar parameters and thus one parameter in β is at our disposal for producing new planes of the system. Therefore:

A system of isoclines of given type (an a-system or a β-system), corresponding to a given vector, consists of a doubly infinite series of planes. It is a two-dimensional system.

31. *Cardinal and Ordinal Systems.** For brevity write $a_\phi = \beta^{-\phi} a \beta^\phi$ and define the two-dimensional system of a-isoclines by the equation

$$ap - pa_\phi = 0 .$$

Changes in ϕ produce a singly infinite series of planes whose isoclinal angles, formed by the successive members of the series with a fixed member of it, are all different; for the squared cosine of the isoclinal angle formed by $(a, -a_{\phi'}, 0)$, regarded as fixed, and $(a, -a_\phi, 0)$, any other plane of the series, is by § 23 (6)

$$S^2 u\bar{v} = \tfrac{1}{2}(1 - Sa_\phi a_{\phi'}),$$

and this varies with ϕ.

Now an infinite series of a-planes which form with a fixed member of the a-series the same (constant) isoclinal angle may be determined in the following manner:

Subject the variable p in the equation $ap - pa_\phi = 0$ to the rotational operation $a^\theta(\ )a_{\phi'}^{-\theta}$. This leaves undisturbed all the points of the fixed plane $(a, -a_{\phi'}, 0)$ and rotates, without distortion, all other configurations in our four-dimensional space through the angle $\pi\theta$.† The equation is transformed into

$$aa^\theta p a_{\phi'}^{-\theta} - a^\theta p a_{\phi'}^{-\theta} a_\phi = 0 ,$$

and p now satisfies the equation

$$ap - pa_{\phi'}^{-\theta} a_\phi a_{\phi'}^\theta = 0 ,$$

which belongs to the a-system; and the isoclinal angle formed by the plane represented by this equation, whose variations depend on θ, with the fixed plane $(a, -a_{\phi'}, 0)$ is the same for all values of θ; for its squared cosine is

$$S^2 u\bar{v} = \tfrac{1}{2}(1 - Sa_\phi a_{\phi'}) \qquad (\S 23\ (6)) .$$

Each new value of θ (within the limits 0, 4) produces a new plane.

* So far as I know the characterization of one-dimensional systems of isoclines (parallels in elliptic space) as of two types, here called cardinal and ordinal, has been made in no previous investigation.

† STRINGHAM: loc. cit. (2) (1884), pp. 55–56. Compare also §1 of this paper and HATHAWAY: loc. cit. (3) (1898), pp. 93–94.

The doubly infinite series of a-isoclines is hereby determined. The planes of the system produced by the variations of ϕ are transmuted into one another by successive changes of the isoclinal angles (all different) which they form with a fixed member of the series; I call them *ordinals.* The planes of the system produced by the variations of θ pass into one another by rotations about the fixed plane $(a, -a_{\phi'}, 0)$, with which they form the same isoclinal angle; I call them *cardinals.* Thus the two-dimensional a-system is made up of one-dimensional ordinal and cardinal systems; and the same remark is also obviously true of a β-system.

Evidently any a-isocline belongs both to an ordinal system and to a cardinal system, but two planes in a system of either type belong to different systems of the other type. We may exhaust the doubly infinite series of a-isoclines by setting up an infinite number of systems of either type; for example, a fixed value of ϕ determines a cardinal system and the variations of θ produce the planes of this system; then a second value of ϕ determines a second cardinal system, and so on. Briefly expressed:

A complete system of isoclines consists indifferently of a simply infinite set of ordinal systems, or of a simply infinite set of cardinal systems.

I call attention to the forms of the expressions for p that satisfy respectively the two equations: namely, for the ordinal system $a^x\beta^\phi$, and for the cardinal system $a^y(a + a_\phi)a^\theta_{\phi'}$, where x, y are arbitrary scalar variables.*

32. *Conjugate Systems.* In general, a plane arbitrarily placed does not meet the isoclines of a system in straight lines.

(1) *If a plane have straight-line intersections with three members of a one-dimensional system (ordinal or cardinal) it meets them all in straight lines.*

We may assign as the mono-parametric equation of the system

$$ap - p\lambda_\phi = 0 \qquad (\lambda_\phi = \mu^{-\phi}\lambda\mu^\phi),$$

where λ and μ are constant unit vectors which may be so determined as to fit the case of either an ordinal or a cardinal system. The cutting plane being $(\gamma, -\gamma', 0)$, the condition is that $S\gamma a = S\gamma'\lambda_\phi$ for three different values of ϕ. This makes γ' the axis of a cone on whose surface the three vectors λ_{ϕ_i} $(i = 1, 2, 3)$ lie. But μ is also the axis of this cone and therefore $\gamma' = \mu$ for all values of ϕ, and the condition becomes

$$S\gamma a = S\gamma'\lambda_\phi = S\lambda\mu;$$

and to make $S\gamma a = S\lambda\mu$ it suffices to determine γ by the condition that its angle with a shall be equal to the angle of λ, μ. This condition being assigned write $\gamma = a^\psi\beta a^{-\psi}$, so that $S\gamma a = Sa\beta = S\lambda\mu$. The transversal plane is now an isocline of the ψ-μ-system

* Cf. HATHAWAY: loc. cit. (1) (1897), p. 55.

$$\beta_\psi p - p\mu = 0 \qquad (\beta_\psi = a^\psi \beta a^{-\psi}),$$

every member of which meets in straight lines all the planes of the a-ϕ-system. Thus (1) is proved not merely for one plane but for an entire one-dimensional system.

(2) *No two planes of the same system intersect in a straight line;* for $Sa^2 \neq S\lambda_\phi \lambda_{\phi'}$, unless $\phi = \phi'$.

The two systems are, in the proper sense, *conjugate* to one another.

Note that one plane, $(a, -\mu, 0)$ is isoclinal to all the planes of both systems.

(3) *The corresponding exterior-interior dihedral angles formed at their intersections by the planes of two conjugate systems are all equal to one another;* for (see also § 30 (2)), independently of the values of ϕ and ψ,

$$Sa\beta_\psi = Sa\beta = S\lambda\mu = S\mu\lambda_\phi .$$

(4) *Conjugate Systems of Ordinals.* By assigning $\lambda = a$ and $\mu = \beta$ we obtain at once, as the equations of the conjugate systems of ordinals

$$ap - pa_\phi = 0 \qquad (a_\phi = \beta^{-\phi} a \beta^\phi),$$

$$\beta_\psi p - p\beta = 0 \qquad (\beta_\psi = a^\psi \beta a^{-\psi}).$$

(5) *Conjugate Systems of Cardinals.* If any three planes of the cardinal system,

$$ap - pa_{\phi'}^{-\theta} a_\phi a_{\phi'}^\theta = 0,$$

be met by $(\gamma, -\gamma', 0)$ in straight lines we must have

$$S\gamma a = S\gamma' a_{\phi'}^{-\theta} a_\phi a_{\phi'}^\theta$$

for three distinct values of θ; whence follows $\gamma' = a_{\phi'}$ and then

$$S\gamma a = Sa_\phi a_{\phi'} = Saa_{\phi - \phi'}.$$

For the determination of γ it suffices to write $\gamma = a^\epsilon a_{\phi-\phi'} a^{-\epsilon}$. The equation of the conjugate system of cardinals then takes the form

$$a^\epsilon a_{\phi - \phi'} a^{-\epsilon} p - pa_{\phi'} = 0.$$

If the equation of the initial system be

$$\beta_{\psi'}^\epsilon \beta_\psi \beta_{\psi'}^{-\epsilon} p - p\beta = 0,$$

the corresponding equation of the conjugate system is

$$\beta_{\psi'} p - p\beta^\theta \beta_{\psi - \psi'} \beta^{-\theta} = 0.$$

32. *The Transition to Elliptic Space.* The foregoing theory of isoclinal systems is clearly a three-dimensional geometry with planes as elements; it is, in fact, the geometry of the sheaf of planes in four dimensional space. All of these planes meet the hypersphere in great circles, the straight lines of an elliptic space (antipodal) whose aggregate is a "space of lines." Hence in order to

translate the propositions of isoclinal systems into their equivalents in elliptic space we have merely to take note of the following dualistic correspondences :

In parabolic four-dimensional space :		In elliptic three-dimensional space :
straight lines	~	points,
planes	~	straight lines,
isoclines	~	parallels,
dihedral angles	~	plane angles,
isoclinal angle of two planes	~	perpendicular distance between two straight lines.

UNIVERSITY OF CALIFORNIA,
November, 1900.