PROOF OF THE SUFFICIENCY OF JACOBI'S CONDITION FOR A PERMANENT SIGN OF THE SECOND VARIATION IN THE SO-CALLED ISOPERIMETRIC PROBLEMS*

BY

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§ 1. *Formulation of the Problem.*

The discussion of the second variation for the simplest class of isoperimetric problems†, in parameter representation, leads to the following question:

Let H_1, H_2, T be three given functions ‡ of t, regular in an interval § (t_0, t_1); moreover it is supposed that $H_1 > 0$ and $T \not\equiv 0$ on (t_0, t_1). *Under what conditions will the definite integral*

$$\delta^2 I = \int_{t_0}^{t_1} \left[H_1 \left(\frac{dw}{dt} \right)^2 + H_2 w^2 \right] dt$$

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† The class treated in KNESER'S *Lehrbuch der Variationsrechnung* in Chap. IV, in which it is required to minimize an integral of the form

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt ,$$

while at the same time another integral of the same form

$$K = \int_{t_0}^{t_1} G(x, y, x', y')\, dt$$

has a prescribed value.

‡ H_1, H_2 are derived from $H = F + \lambda G$ in the same manner as F_1, F_2 from F in the theory of the unconditioned problem; for WEIERSTRASS' explicit expression of F_2 see BLISS, Transactions of the American Mathematical Society, vol. 3 (1902), p. 133. Further

$$T = G_{xy'} - G_{x'y} + G_1 (x'y'' - x''y'),$$

the literal subscripts denoting partial differentiation.

§ The notation implies that $t_0 < t_1$.

be positive for all functions w, not identically zero, satisfying the following conditions:

(a) $$w(t_0) = 0, \qquad w(t_1) = 0,$$

(b) $$\int_{t_0}^{t_1} wT dt = 0,$$

(c) *the functions w satisfy certain conditions concerning continuity and existence and continuity of the first derivative.*

With respect to (c) we make the assumption * that w itself shall be continuous on (t_0, t_1) and that the interval (t_0, t_1) can be divided into a finite number of subintervals such that on each subinterval the first derivative exists and is continuous,—with the understanding that in the lower (upper) endpoint of each subinterval "progressive (regressive) derivative" is substituted for "derivative."

The answer is as follows: Denote by $\Psi(w)$ the differential expression

$$\Psi(w) = H_2 w - \frac{d}{dt}\left(H_1 \frac{dw}{dt}\right),$$

and by U and V two integrals † of the differential equations

(1) $$\Psi(U) = 0, \qquad \Psi(V) = T$$

respectively, which vanish for $t = t_0$:

(2) $$U(t_0) = 0, \qquad V(t_0) = 0;$$

further let

$$M = \int_{t_0}^{t} UT dt, \qquad N = \int_{t_0}^{t} VT dt,$$

and put

$$\Delta(t, t_0) = MV - NU.$$

Then if t_0' denote the zero of $\Delta(t, t_0)$ next greater than t_0 (the "*conjugate to* t_0"), *the necessary and sufficient condition for the validity of the relation* $\delta^2 I > 0$ *is that*

(3) $$t_1 < t_0'.$$

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* This agreement means for the isoperimetric problem that we restrict ourselves to continuous curves made up of a finite number of arcs along each of which the curve has a continuously turning tangent.

† It is well known that the general integral of

$$\Psi(w) = \mu T,$$

where μ is any constant, can be derived from the general integral of the differential equation of the extremals by differentiation with respect to the constants of integration and the isoperimetric constant λ. See HORMANN, Dissertation, Göttingen, 1887, and KNESER, Mathematische Annalen, vol. 55 (1901), p. 93.

That this condition ("Jacobi's condition") is *necessary* is easily seen.* For the integral $\delta^2 I$ can be thrown, by an integration by parts, into the form

$$(4) \qquad \delta^2 I = \int_{t_0}^{t_1} w \Psi(w)\, dt,$$

so that, on account of (b),

$$(5) \qquad \delta^2 I = \int_{t_0}^{t_1} w \big(\Psi(w) - \mu T\big)\, dt,$$

μ being any constant. And if $t_0' \lesseqgtr t_1$, we could make $\delta^2 I = 0$ by choosing

$$\begin{aligned} \mu &= M(t_0'), \\ w &= M(t_0')V(t) - N(t_0')U(t) \quad \text{on} \quad (t_0, t_0'), \\ w &\equiv 0 \qquad\qquad\qquad\qquad\qquad \text{on} \quad (t_0', t_1); \end{aligned}$$

this function w fulfills the conditions (a), (b), (c) and besides it satisfies the differential equation

$$\Psi(w) = M(t_0')T,$$

and therefore it makes $\delta^2 I = 0$.†

It is less evident that the condition (3) is also *sufficient*, and to show this is the object of the present note.

§ 2. *Proof‡ of the Sufficiency of Jacobi's Condition.*

Our proof is based upon an extension of the lemma concerning the differential expression $\Psi(w)$ by which JACOBI proves the analogous theorem for the unconditioned problem, viz:

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* Compare HORMANN, l. c., and KNESER, l. c.; the proof had already been given, in slightly different form, in WEIERSTRASS' Lectures on the Calculus of Variations of 1872.

† $\delta^2 I$ can even be made < 0, as shown by KNESER, l. c.; compare also BOLZA, *Zur zweiten Variation bei isoperimetrischen Problemen*, in one of the forthcoming numbers of the Mathematische Annalen.

‡ Inasmuch as the isoperimetric problem here considered is a special case of the general problem : To minimize an integral of the form

$$I = \int_{t_0}^{t_1} F(t;\, x_1, x_2, \cdots, x_n;\, x_1', x_2', \cdots, x_n')\, dt,$$

the unknown functions $x_1, x_2, \cdots, x_n$ being connected by a number of relations of the form :

$$\phi_\alpha(t;\, x_1, x_2, \cdots, x_n;\, x_1', x_2', \cdots, x_n') = 0 \qquad (\alpha = 1, 2, \cdots, m),$$

the proof might be derived by specialization from MAYER'S researches on the second variation for the general problem (Crelle's Journal, vol. 69 (1868), p. 238, and Mathematische Annalen, vol. 13 (1878), p. 53; compare also C. JORDAN, *Cours d'Analyse*, III, Nos. 373–394 and v. ESCHERICH, Sitzungsberichte der Wiener Academie, vol. 107, II_a (1898), pp. 1191, 1268, 1384). A proof thus obtained would, however, be much more complicated than the direct proof given in the text.

If u be any integral of the differential equation

$$\Psi(u) = 0,$$

then for every function p admitting first and second derivatives the relation

$$(pu)\Psi(pu) = H_1(p'u)^2 - \frac{d}{dt} H_1 pp'u^2$$

holds, accents denoting differentiation with respect to t.

To obtain the desired extension of this lemma let u, v be integrals of the differential equations

$$\Psi(u) = 0, \qquad \Psi(v) = T \tag{6}$$

respectively, both vanishing in a point $t = \tau_0$:

$$u(\tau_0) = 0, \qquad v(\tau_0) = 0; \tag{7}$$

let further p, q be two arbitrary functions of t admitting first and second derivatives, and denote for shortness

$$\omega = pu + qv.$$

Then

$$\omega\Psi(\omega) = (pu + qv)qT - H_1(pu + qv)(p'u' + q'v') - (pu + qv)\frac{d}{dt} H_1(p'u + q'v),$$

which may easily be written

$$\omega\Psi(\omega) = H_1(p'u + q'v)^2 - H_1(pq' - p'q)(uv' - u'v) + (pu + qv)qT - \frac{d}{dt} H_1(pu + qv)(p'u + q'v).$$

Further if we introduce

$$m = \int_{\tau_0}^{t} uT dt, \qquad n = \int_{\tau_0}^{t} vT dt,$$

then

$$(pu + qv)qT = (pm' + qn')q = \frac{d}{dt}(pm + qn)q - (p'm + q'n)q - (pm + qn)q'.$$

But from (6) and (7) follows that *

$$H_1(uv' - u'v) = -m;$$

hence we obtain

$$(8)\qquad \begin{aligned} (pu+qv)\Psi(pu+qv) &= H_1(p'u+q'v)^2 - 2q(p'm+q'n) \\ &\quad - \frac{d}{dt}[H_1(pu+qv)(p'u+q'v)-(pm+qn)q], \end{aligned}$$

which is the desired extension of JACOBI's lemma.

Since

$$H_1\omega'^2 + H_2\omega^2 = \omega\Psi(\omega) + \frac{d}{dt}(H_1\omega\omega'),$$

we further derive from (8) the following relation:

$$(9)\qquad \begin{aligned} H_1\omega'^2 + H_2\omega^2 &= H_1(p'u+q'v)^2 - 2q(p'm+q'n) \\ &\quad + \frac{d}{dt}[H_1(pu+qv)(pu'+qv')+(pm+qn)q]. \end{aligned}$$

In the latter formula the second derivatives of p and q do not occur; hence it can be inferred that (9) *holds even for functions* p, q *admitting first but not second derivatives.*—

We now proceed to prove the sufficiency † of JACOBI's condition. We suppose then that

$$t_1 < t_0',$$

so that

$$\Delta(t, t_0) \neq 0 \qquad \text{for every } t, \qquad t_0 < t \leqq t_1.$$

Now choose $\tau_0 < t_0$ but so near to it that H_1, H_2, T remain regular in the enlarged interval (τ_0, t_1) and that

$$(10)\qquad \Delta(t, \tau_0) \neq 0,$$

for every t, $t_0 \leqq t \leqq t_1$; such a choice of τ_0 is always possible. ‡

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* Compare KRESER, l. c., equation (22).

† It is hardly necessary to say that we are always speaking of sufficiency *for a permanent sign of* $\delta^2 I$, not of sufficiency *for a minimum.*

‡ For a proof see C. JORDAN, *Cours d'Analyse*, vol. III, no. 393.

Let w be any function of t satisfying the conditions (a), (b), (c) for the interval (t_0, t_1) and $\equiv 0$ on (τ_0, t_0); then define the two functions p, q by the two equations

$$(11) \qquad \begin{aligned} pu + qv &= w, \\ p'm + q'n &= 0, \end{aligned}$$

with the initial condition

$$(12) \qquad p(t_0) = 0, \qquad q(t_0) = 0.$$

But from (11) it follows that

$$\frac{d}{dt}(pm + qn) = (pu + qv)\,T = wT,$$

hence integrating and remembering (12), we obtain

$$(13) \qquad pm + qn = \int_{t_0}^{t} wT\,dt.$$

We thus obtain for the determination of p, q two linear equations whose determinant is

$$mv - nu = \Delta(t, \tau_0),$$

and therefore $\neq 0$ on (t_0, t_1) according to (10); hence p and q are continuous on (t_0, t_1) and with respect to their derivatives of the same character as w.

For the same reason p and q, both vanish not only in t_0, but also in t_1, since according to (a) and (b), w and

$$\int_{t_0}^{t} wT\,dt$$

vanish in t_0 *and in* t_1. Using these functions p and q in the transformation (9) and integrating* between the limits t_0 and t_1 we obtain the *final result:*

$$(14) \qquad \delta^2 I = \int_{t_0}^{t_1} H_1 (p'u + q'v)^2\,dt.$$

This proves that indeed

$$\delta^2 I > 0$$

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* If w', and accordingly p' and q', have discontinuities of the kind admitted by our assumptions, the integral would have to be broken up into a sum of integrals taken over the subintervals; but as p and q themselves are continuous, formula (9) shows that (14) remains true also in this case.

for all functions w satisfying the conditions (a), (b), (c), *provided* $t_1 < t_0'$. For $\delta^2 I > 0$ unless $p'u + q'v$ were identically zero; but then it would follow from $p'm + q'n = 0$ and (10) that p' and q' must vanish identically and therefore also p and q themselves since they vanish in t_0 and are continuous; but this is against the assumption that w does not vanish identically.

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