ON HYPERCOMPLEX NUMBER SYSTEMS*

BY

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Introduction.

The theories of hypercomplex numbers and of continuous groups were first explicitly connected by POINCARÉ† in 1884 with the statement that the problem of complex numbers was reduced to that of finding all the linear continuous group of substitutions in n variables of which the coefficients are linear functions of n arbitrary parameters, and since that time the advance in the theory of hypercomplex numbers has been largely suggested by the theory of continuous groups. In 1889 STUDY‡ and SCHEFFERS§ developed the relation between these theories to a considerable extent, and the latter|| in 1891 arrived at a complete enumeration of systems in less than six units which are inequivalent (of different "Typus"), non-reciprocal, irreducible, and which possess moduli. Previously (1889) STUDY¶ had enumerated all inequivalent systems with moduli in less than five units without direct use of the theory of continuous groups, and in 1890 ROHR** continued the work through systems in five units.

The problem of enumerating hypercomplex number systems had been attacked by BENJAMIN PEIRCE about twenty years before the investigations of STUDY and SCHEFFERS. His results were not printed, however, until after his death.††

With the methods of the European investigators in mind, I have else-

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† Comptes Rendus, vol. 99, p. 740, 1884.

‡ Leipziger Berichte, p. 177, 1889.

§ Leipziger Berichte, p. 400, 1889.

|| Mathematische Annalen, vol. 39, p. 293, 1891.

¶ Göttinger Nachrichten, p. 237, 1889; Monatshefte für Mathematik und Physik, vol. 1, p. 283, 1890.

** Dissertation, Marburg, *Ueber die aus 5 Haupteinheiten gebildeten komplexen Zahlensysteme*, 1890.—To afford a basis for a rough estimate of the comparative directness of the various methods of enumeration it may be noted that SCHEFFERS' memoir contains 87 pages, while the portion of STUDY'S memoir (Monatshefte) concerned in the enumeration and ROHR'S dissertation together contain 81 pages.

†† American Journal of Mathematics, vol. 4, p. 97, 1881.—In 1870 PEIRCE published his work in lithographed form. This publication was reviewed by SPOTTISWOODE in 1872 in the Proceedings of the London Mathematical Society, vol. 4, p. 147. However, since this lithographed edition was small, PEIRCE'S work was not easily accessible to mathematicians until 1881.

where * closely scrutinized PEIRCE'S memoir, which for many years has been subject to neglect or adverse criticism. In this paper I propose to show that by using PEIRCE'S principles as a foundation we can deduce a method more powerful than those hitherto given for enumerating all number systems of the types SCHEFFERS has considered. Theorems V, VI, X, XI are, I believe, new. It is in the first two of these theorems that the ease and directness of this method appear. Those of PEIRCE'S theorems which I have needed I have not hesitated to prove. This has been done partly to make the paper complete in itself, and partly to place PEIRCE'S work on a clear and rigorous basis.

Definitions.†

Let a and β be any two numbers of an hypercomplex number system.

Def. 1. If $a\beta = \beta a = a$, a is an *idemfactor* with respect to β.

Def. 2. If $\beta a = a$, a is *idemfaciend* with respect to β.

Def. 3. If $a\beta = a$, a is *idemfacient* with respect to β.

Def. 4. If $\beta a = a\beta = 0$, a is a *nilfactor* with respect to β.

Def. 5. If $\beta a = 0$, a is *nilfaciend* with respect to β.

Def. 6. If $a\beta = 0$, a is *nilfacient* with respect to β.

Def. 7. If for the number a a positive integer n exists so that $a^n = 0$, a is *nilpotent.*

Def. 8. If the non-zero number a is equal to its square: $a^2 = a$, a is *idempotent.*

I.

IDEMPOTENT SYSTEMS.

THEOREM I. ‡ *In every n-tuple number system, in which not every number is a divisor of zero, there exists an idempotent number.*

Let a be any number not a divisor of zero. Then there exists a non-zero number γ such that

$$a\gamma = a. \tag{1}$$

This number γ is an idempotent number, for we have:

$$a\gamma^2 = a\gamma,$$

and so indeed:

$$\gamma^2 = \gamma. \tag{2}$$

Here we have used the theorem § that any two numbers a, β of which a is not a divisor of zero serve to determine a definite number $x = \xi$ of the number system, which satisfies the equation:

* American Journal of Mathematics, vol. 24, 1902.

† PEIRCE, loc cit., p. 104.

‡ This theorem is stated by PEIRCE, loc. cit., p. 109.

§ Cf., e. g., BERLOTY'S thesis, *Théorie des Quantités Complexes à n Unités Principales*, p. 19, 1886, Paris.

$$ax = \beta.$$

Def. 9. A number system which contains an idempotent number is an *idempotent system.*

Def. 10. A number system which contains no idempotent number is a *nilpotent system.*

For the present we deal with idempotent systems.

The units of any idempotent system may be transformed by the equations

$$e'_k = \sum_{i=1}^{n} a_{ik} e_i, \quad e'_n = \gamma \qquad (k=1, \cdots, n-1),$$

so that the new unit e_n *is idempotent. We assume in every case that this transformation has been made.* The coefficients a_{ik} may have any values such that the determinant of the equations of transformation does not vanish.

THEOREM II.* *All the units of an n-tuple system with one idempotent unit* e_n *may be chosen so as to fall into the following groups :*

GROUP I *contains units* e_k *which are idemfactors with respect to* e_n. *Thus* $e_k e_n = e_n e_k = e_k$. *This group is symbolized by* (dd).

GROUP II *contains units* e_k *which are idemfaciend and nilfacient with respect to* e_n. *Thus* $e_n e_k = e_k$, $e_k e_n = 0$. *This group is symbolized by* (dn).

GROUP III *contains units* e_k *which are nilfaciend and idemfacient with respect to* e_n. Thus $e_k e_n = e_k$, $e_n e_k = 0$. *This group is symbolized by* (nd).

GROUP IV *contains units* e_k *which are nilfactors with respect to* e_n. *Thus* $e_k e_n = e_n e_k = 0$. *This group is symbolized by* (nn).

The proof falls into two parts.

[A] *The units of an idempotent number system* S *may be so chosen as to be either idemfaciend or nilfaciend with respect to* e_n.

Any number a of S is in one of two classes :

(*a*) A number β exists such that $e_n \beta = a$.
(*b*) No such number β exists.

Consider numbers a in class (*a*):

$$e_n a = e_n^2 \beta = e_n \beta = a.$$

Thus all numbers of (*a*) are idemfaciend with respect to e_n. Suppose there are $n - r$ independent numbers of this class. Transform the system so that these numbers are the units $e_{r+1} e_{r+2} \cdots e_n$ of which we still assume that e_n is idempotent. Any number of (*a*) contains only those units or else we should have numbers in (*a*) independent of $e_{r+1} e_{r+2} \cdots e_n$. Thus $e_1 \cdots e_r$ may be any properly chosen numbers independent of $e_{r+1} e_{r+2} \cdots e_n$. These units may be taken nilfaciend with respect to e_n. For since they are not of (*a*),

* PEIRCE, loc. cit., p. 109.

$$e_n e_k = \gamma + \delta \qquad (k \leqq r),$$

where

$$\gamma = \sum_{i=1}^{r} a_i e_i, \quad \delta = \sum_{i=r+1}^{n} a_i e_i.$$

If $\gamma \neq 0$, then $e_n(e_k - \delta) = \delta + \gamma - \delta = \gamma$, and γ is in (a) which is impossible.

We have, therefore, $\gamma = 0$.

Let now

$$e_i' = e_i - \sum_{s=r+1}^{n} \gamma_{nis} e_s \qquad (i = 1, 2, \cdots, r),$$

$$e_i' = e_i \qquad (i = r+1, \cdots, n).$$

where γ_{nis} are some of the constants of multiplication * of the system.

Then

$$e_n' e_i' = e_n e_i - \sum_{s=r+1}^{n} \gamma_{nis} e_s = \sum_{s=r+1}^{n} \gamma_{nis} e_s - \sum_{s=r+1}^{n} \gamma_{nis} e_s = 0 \qquad (i = 1, 2, \cdots r),$$

$$e_n' e_k' = e_n e_k = e_k = e_k' \qquad (k = r+1, \cdots, n).$$

It is clear that the new units $e_1', \cdots e_n'$ are independent.

Thus [A] is established.

[B] *Without disturbing the property of the units just established we can so transform our system that the new units are either nilfacient or idemfacient with respect to* e_n.

We observe that

(3)
$$e_\lambda e_n = \sum_{s=1}^{r} \gamma_{\lambda ns} e_s \qquad (\lambda = 1, \cdots, r),$$

(4)
$$e_\nu e_n = \sum_{s=r+1}^{n} \gamma_{\nu ns} e_s \qquad (\nu = r+1, \cdots, n).$$

For let

$$e_\lambda e_n = \sum_{s=1}^{n} \gamma_{\lambda ns} e_s \qquad (\lambda = 1, \cdots, r).$$

By the associative law,

$$e_n e_\lambda \cdot e_n = 0 = e_n \cdot e_\lambda e_n = \sum_{s=r+1}^{n} \gamma_{\lambda ns} e_s \qquad (\lambda = 1, \cdots, r).$$

Thus

$$\gamma_{\lambda ns} = 0 \qquad (\lambda = 1, \cdots, r;\ s = r+1, \cdots, n),$$

which establishes (3). By similar use of the associative law on the product $e_n e_\nu e_n\ (\nu = r+1, \cdots, n)$ we verify (4).

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* The n^3 constants γ_{iks} which occur in the equations

$$e_i e_k = \sum_{s=1}^{n} \gamma_{iks} e_s \qquad (i, k = 1, \cdots, n)$$

we call (after HAMILTON) constants of multiplication.

The units of our system are now in two groups, $e_1 \cdots e_r$ and $e_{r+1} \cdots e_n$, and are subject to [A] and equations (3) and (4). We can now use the proof of [A], with interchange of the order of multiplication of e_n and e_k, *inside* the group $e_1 \cdots e_r$ to show that the units of that group may be so transformed as to be idemfacient or nilfacient with respect to e_n. The same process may be applied to the set of units $e_{r+1} \cdots e_n$ with a similar result. Thus the units of our system fall into the four required groups. Evidently e_n is in group I.

THEOREM III.* *The group of the non-vanishing product of any two units of the various groups is shown by the following table:*

	nn	nd	dn	dd
nn	nn	nd	0	0
nd	0	0	nn	nd
dn	dn	dd	0	0
dd	0	0	dn	dd

The multiplier is in the left hand column of the above table.

For example: take e_k from group III or (nd), and e_j from group II or (dn). Then

$$e_n e_k e_j = 0 = e_k e_j e_n,$$

since e_k is nilfaciend, and e_j is nilfacient. Thus the product is in group IV or (nn), provided it does not vanish. Similarly each product of the table may be verified.

Def. 11. When the units of a system satisfy the conditions indicated by the table we call the system *regular* with respect to the unit e_n.

From an inspection of the regular table it appears that idempotent units can exist only in groups I or IV. If e_n is the modulus, † the system can contain only units of group I.

THEOREM IV.‡ *If there are two or more independent idempotent numbers in group* I, *the system may be transformed without disturbing its regularity so that e_n is the only idempotent number in group* I.

Let S denote the system $e_1, \cdots, e_n$, regular with respect to e_n, of which $e_1, \cdots, e_r, e_n$ are in group I thus constituting a system S_1 by themselves. Let a be a number in S_1 independent of e_n such that $a^2 = a$. Transform S by the equations

$$e'_i = \sum a_{ij} e_j, \qquad (i = 1, 2, \cdots, n-1);$$

$$e'_n = e_n - a,$$

* PEIRCE, loc. cit., p. 111.

† See def. 13.

‡ PEIRCE, loc. cit., p. 112.

where the a's are so chosen that the determinant of the system does not vanish. By theorem II we may so choose e_1', e_2', $\cdots$, e_{n-1}' that each falls into one of the four groups I, II, III, IV, with respect to e_n'.

But from the table it appears that no expression linear in the old units can be idemfactorial with respect to e_n' except an expression linear in $e_1, e_2, \cdots, e_r, e_n$, alone. Thus the units of S in group I with respect to e_n' are the same, or may be so chosen, as the units of S_1 in group I with respect to e_n'. At least one of the units of S_1 is in group IV with respect to e_n', since there is a number linear in $e_1, \cdots, e_r, e_n$, namely, a nilfactorial with respect to e_n'. Therefore the first group of S_1 and hence the first group of S with respect to e_n' contains at most only r units, whereas group I with respect to e_n contained $r+1$ units. In this way whenever group I contains two independent idempotent numbers we may transform the system so as to reduce by at least one the number of units in that group.

Theorem V. *If there is an idempotent number in group* IV *it may be taken as a unit, and the whole system made regular with respect to it without disturbing the regularity with respect to* e_n.

Let e_{n-1} be the idempotent unit of group IV. Let e_k ($k = r+1, \cdots, s$) be units of group II. Then one has

$$e_{n-1} e_k = 0, \qquad e_k e_{n-1} = \sum_{i=r+1}^{s} \gamma_{k\,n-1\,i} e_i,$$

by the table of theorem III. We must now transform our system so that the new units e_k', are in group II with respect to e_n but in group III or IV with respect to e_{n-1}, that is, so that $e_k' e_{n-1} = 0$ or e_k'. But since the units in the original group II together with e_{n-1} form a system by themselves, we can, in fact, use the method of proof [A] to show that the units of group II with respect to e_n can be so transformed that the new units are either idemfacient or nilfacient with respect to e_{n-1}, and consequently fall in group III or IV with respect to that unit. Since the transformations involve only units in the original group II the regularity with respect to e_n is not affected. We can proceed similarly to show that group III with respect to e_n may be so transformed as to contain only units of group II or IV with respect to e_{n-1}. Group I with respect to e_n is evidently in group IV with respect to e_{n-1}. Theorem II may be repeated to show that the units in group IV with respect to e_n may be so chosen as to fall into the four groups with respect to e_{n-1}. This again does not affect the regularity of the system since the equations of transformation involve only units of IV. We denote the four groups with respect to e_{n-1} by I_1, II_1, III_1, IV_1. We also see that all units of I_1 are in IV, all units of II_1 are in III or IV, units of III_1 are in II or IV, and units of IV_1 are in I, II, III, or IV.

Let the idempotent units be represented by $e_{n-t+1}, \cdots, e_n$.

THEOREM VI. *The system remaining regular with respect to* e_n *and* e_{n-1} *can be so transformed that only one idempotent unit remains in* I_1 *and* I *respectively. All the other idempotent numbers pass to* IV_1. *The system retaining its regularity can then be transformed so as to be regular with respect to any one of these idempotent units of* IV_1 *as* e_{n-2}.

The first part of this theorem is proved by the method used in theorem IV. The second part is shown by an essential repetition of theorem V.

This process may be continued as long as two independent units remain in any group IV. We finally have our system regular with respect to each of the idempotent units. Let the four groups into which the units fall with respect to e_{n-k} be denoted by I_k, II_k, III_k, IV_k. Each group I_k contains one and only one idempotent number, namely e_{n-k}. The product of any two distinct idempotent units vanishes since any such unit is in group IV with respect to every other idempotent unit. We are now able to form a general table of the relations of the various groups to one another. Since I_2 consists of units which are idemfactors with respect to a unit which is in IV_1 and IV, all units in I_2 must be in IV_1 and IV. Units in II_2 may be in III_1 or IV_1. Units in III_1 may be in II or IV, but only those units in III_1 which are also in IV can be in II_2 by the table in theorem III. Further only the units in IV_1 which are also in III or IV can be in II_2. Similarly units in III_2, which may contain units in II_1 and IV_1, can contain only units common to II_1 and IV, and units common to IV_1 and II or IV. In this way we arrive at the following tables. Units in a certain group with respect to e_{n-k} are also in the group found on the same line reading toward the right.

I_k	IV_{k-1}	IV_{k-2}	$\cdots$	IV
II_k	$\mathrm{III}_{k-1}\,\mathrm{IV}_{k-1}$	$\mathrm{III}_{k-2}\,\mathrm{IV}_{k-2}$	$\cdots$	III IV
III_k	$\mathrm{II}_{k-1}\,\mathrm{IV}_{k-1}$	$\mathrm{II}_{k-2}\,\mathrm{IV}_{k-2}$	$\cdots$	II IV
IV_k	$\mathrm{I}_{k-1}\,\mathrm{II}_{k-1}\,\mathrm{III}_{k-1}\,\mathrm{IV}_{k-1}$	$\mathrm{I}_{k-2}\,\mathrm{II}_{k-2}\,\mathrm{III}_{k-2}\,\mathrm{IV}_{k-2}$	$\cdots$	I II III IV

I_k	IV_{k+1}	IV_{k+2}	$\cdots$	IV_{n-t+1}
II_k	$\mathrm{III}_{k+1}\,\mathrm{IV}_{k+1}$	$\mathrm{III}_{k+2}\,\mathrm{IV}_{k+2}$	$\cdots$	$\mathrm{III}_{n-t+1}\,\mathrm{IV}_{n-t+1}$
III_k	$\mathrm{II}_{k+1}\,\mathrm{IV}_{k+1}$	$\mathrm{II}_{k+2}\,\mathrm{IV}_{k+2}$	$\cdots$	$\mathrm{II}_{n-t+1}\,\mathrm{IV}_{n-t+1}$
IV_k	$\mathrm{I}_{k+1}\,\mathrm{II}_{k+1}\,\mathrm{III}_{k+1}\,\mathrm{IV}_{k+1}$	$\mathrm{I}_{k+2}\,\mathrm{II}_{k+2}\,\mathrm{III}_{k+2}\,\mathrm{IV}_{k+2}$	$\cdots$	$\mathrm{I}_{n-t+1}\,\mathrm{II}_{n-t+1}\,\mathrm{III}_{n-t+1}\,\mathrm{IV}_{n-t+1}$

LIMITATION TO SYSTEMS WITH MODULI.—From the two preceding tables we see that *in a system thus regularized, which moreover contains for no index* k $(k = 0, \cdots, t-1)$ *units in a group* II_k, III_k, *or* IV_k *and at the same time in every group* IV_σ $(\sigma = 0, \cdots, t-1;\ \sigma \neq k)$, *there exists for every* $e_\rho\,(\rho = 1, \cdots, n-t)$ *one and only one pair of integers* $(\lambda\lambda_1)$, *where* $t-1 \geqq \lambda$, $\lambda_1 \geqq 0$, *such that*

$$e_\rho e_{n-\lambda_1} = e_\rho ; \quad e_{n-\lambda} e_\rho = e_\rho .$$

If e_ρ is in I_k, evidently $\lambda_1 = \lambda = k$.

Def. 12. We call the pair of integers $(\lambda \lambda_1)$ which corresponds in this way to e_ρ the *character* * of e_ρ.

We consider henceforth only idempotent systems satisfying the condition italicized above, and proceed to show that these are exactly the systems with moduli.

Def. 13. A number μ of a system S is *a modulus* if for every a of the system

$$a\mu = \mu a = a . \tag{5}$$

Note. If μ and μ' are moduli of a system S, then by definition (5)

$$\mu\mu' = \mu , \quad \mu\mu' = \mu' , \tag{6}$$

and so $\mu = \mu'$. Thus one speaks of *the modulus* of a system.

Now one sees immediately that in the systems under consideration the number

$$\mu = \sum_{k=1}^{t} e_{n-t+k}$$

is a modulus. (One speaks of a system with a modulus in t idempotent units.)

Further no system so far excluded has a modulus. For a system with a modulus is an idempotent system, and no idempotent system excluded by the condition in question has a modulus. For in such a system let, for example, e_λ be a unit which is simultaneously in the groups $\mathrm{III}_k (0 \leqq k \leqq t-1)$ and in the groups IV_σ $(\sigma = 0, \cdots, t-1; \sigma \neq k)$. A modulus if existent is idempotent and a linear combination of $e_{n-t+1}, \cdots, e_n$. For the modulus if existent must contain all these idempotent units, and can contain no other number. But since e_λ is nilfaciend with respect to each of these units, it is nilfaciend with respect to every idempotent number linear in $e_{n-t+1}, \cdots, e_n$. Hence no modulus exists.

Lemma. *If in any group* I *the idempotent unit does not occur in any product of non-idempotent units, every number in the group not involving the idempotent unit is nilpotent.*

For convenience of notation let the group I comprise the entire system, and let the idempotent unit be e_n. (This is permissible since the product of two units of a group I involves only units of that group.)

Let a be any number in group I not involving e_n. Then some least integer $m \leqq n$ must exist such that

* Scheffers, *Zurückführung complexer Zahlensysteme auf typische Formen*, Mathematische Annalen, vol. 39, p. 313.

$$\sum_{k=1}^{m} a_k a^k = 0 \qquad (a_m \neq 0). \tag{7}$$

If $m = 1$, then a vanishes and is nilpotent. If $m > 1$, two cases arise.

1°) $$a_1 \neq 0.$$

Setting
$$\beta = \sum_{k=2}^{m} a_k a^{k-1},$$
we write (7) in the form:
$$0 = a_1 a + \beta a = (a_1 e_n + \beta) a,$$
from which we easily obtain
$$(a_1 e_n + \beta)\beta = 0,$$
or
$$\left(-\frac{\beta}{a_1}\right)^2 = -\frac{\beta}{a_1}.$$

But β does not contain e_n and hence is independent of e_n. Thus, since $\beta \neq 0$, we have in $-\beta/a_1$ a second idempotent number in the group. But this is impossible. Thus the case 1°) does not occur.

2°) $$a_k = 0, \quad a_\lambda \neq 0 \qquad (k = 1, \cdots, \lambda - 1;\ \lambda \geqq 2).$$
Then
$$\sum_{i=\lambda}^{m} a_i a^i = 0. \tag{8}$$
We set $m = \lambda + \nu$.

If $\nu = 0$, then $a^m = 0$ and a is nilpotent.

If $\nu > 0$, there is an equation of degree m':
$$\sum_{k=1}^{m'} b_k \beta^k = 0 \qquad \left(\begin{matrix} 1 + \nu \geqq m' \geqq \mu \\ b_{m'} \neq 0;\ \beta = a^\lambda \end{matrix}\right); \tag{9}$$
here μ denotes the least integer such that $\mu > 1$, $\lambda\mu \geqq \lambda + \nu$. When this statement is proved, it is apparent that a is nilpotent, for β is a power of a and the equation (9) is of the type of equation (7) but of lower degree, and so in the sequential application of the reasoning leading from (7) to (9) the equations analogous to (9) occur only a finite number of times and when no such equation occurs a certain power of a is recognized as a nilpotent number.

From (8) we have the equations
$$\sum_{i=\lambda}^{\lambda+\nu} a_i a^{i+j} = 0 \qquad (j = 0, 1, 2, \cdots). \tag{8_j}$$

We think of an equation (8_j) as written with powers of $\beta\,(\beta = a^\lambda)$ thrown to the right and with the other powers of a retained on the left and written in order of decreasing exponents. The lowest power of a in (8_j) is $a^{\lambda+j}$ and the highest power is $a^{\lambda+\nu+j}$. The first equation (8_j) involving $\beta^\kappa = a^{\lambda\kappa} (\kappa \geqq \mu)$ is the equation $(8_{\lambda\kappa-\lambda-\nu})$.

The system $\{8\}_\kappa$ of $\lambda\kappa - \lambda - \nu + 1 = \rho_\kappa$ equations (8_j), $j = 0, 1, \cdots, \rho_\kappa - 1$, arranged according to ascending values of j, is taken for consideration. This system of equations involves on the right $\beta^1, \cdots, \beta^\kappa$ and on the left homogeneously certain $\rho_\kappa + \nu - \kappa$ powers of a not powers of β.

For this system $\{8\}_\kappa$ we have an eliminant

$$(9_\kappa) \qquad \sum_{k=1}^{\kappa} b_{k\kappa} \beta^k = 0$$

in case every determinant D_κ of order ρ_κ of the array A_κ of the system $\{8\}_\kappa$ vanishes, and indeed one of the type (9), i. e., with $b_{\kappa\kappa} \neq 0$, in case $\rho_\kappa = 1$ or in case of the array A'_κ obtained by omitting the last line of A_κ not every determinant D'_κ of order $\rho_\kappa - 1$ vanishes. The latter condition if fulfilled implies that $\kappa \leqq 1 + \nu$. If $\kappa = 1 + \nu$ the former condition is surely satisfied.

We shall prove that both conditions are satisfied for some value of κ,

$$\kappa = m', \quad \mu \leqq m' \leqq 1 + \nu.$$

The corresponding eliminant (9_κ) is the desired equation (9).

Unless both conditions are satisfied for $\kappa = 1 + \nu$, every (the single) determinant $D'_{1+\nu}$ vanishes. Remembering that $a_{\lambda+\nu} \neq 0$ we see then that every (the single) determinant D_ν vanishes. And if every D'_ν vanishes, then every $D_{\nu-1}$ vanishes. Thus proceeding we see that for some $\kappa = m' (\mu < m' \leqq 1 + \nu)$ every $D_{m'}$ vanishes and not every $D'_{m'}$ vanishes, or else every D_μ vanishes and either $\rho_\mu = 1$ or not every D'_μ vanishes; for if $\rho_\mu > 1$ one determinant D'_μ has the non-zero value $(-1)^{\frac{1}{2}\rho_\mu(\rho_\mu - 1)} a_{\lambda+\nu}^{\rho_\mu - 1}$.

Def. 14. The exponent in the highest non-vanishing power of a nilpotent number is called its *degree*.

Def. 15.* If a unit e_k is erased from every position it occupies in a multiplication table, the system is said to be *deleted* by that unit.

THEOREM VII.† *Every number in a nilpotent system is nilpotent.*

Let S be any nilpotent system in $n - 1$ units. Border this system on the right and below by a unit e_n such that $e_k e_n = e_n e_k = e_k$ $(k = 1, \cdots, n)$. We have then a system with no idempotent number independent of e_n, and so of the nature considered in the preceding lemma. Thus the multiplicative properties of the nilpotent system must conform to the condition there established and all its numbers must be nilpotent.

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* SCHEFFERS, loc. cit., p. 307

† PEIRCE, loc. cit., p. 113.

THEOREM VIII.* *The non-vanishing powers of a nilpotent number are independent.*

Let a be a nilpotent number of degree m. Hence $a^m \neq 0$, $a^t = 0\ (t > m)$. Then if a, a^2, $\cdots$, a^m are not independent they satisfy a linear relation of the form,

$$\sum_{k=r}^{m} a_k a^k = 0 \qquad (a_r \neq 0,\ r \geqq 1). \tag{10}$$

Multiply (10) by a^{m-r} and we get $a_r a^m = 0$; hence $a_r = 0$. Thus no such linear relation exists and the powers of a are seen to be independent. It is often convenient to take for units these independent powers of the unit of highest degree in a system.

THEOREM IX. *The multiplication table of any number system which has a modulus in one idempotent unit, can be transformed so that e_n occurs only in the product e_n^2.*

To any number system with a modulus corresponds a pair of reciprocal, simply transitive projective groups. A system is called non-quaternion or quaternion according as the corresponding groups are integrable or not. The simplest quaternion system is HAMILTON'S system of quaternions, which occurs as a subsystem in every quaternion system.† Since HAMILTON'S system contains two idempotent numbers no quaternion system is considered under the present theorem.

The groups which correspond to non-quaternion systems have the following well known property. Consider the parameters in the equations defining the group to be coördinates of points in space of n dimensions. Then in this space there is one point, one line through the point, one plane through the line, etc., which remain invariant under the group. Considering now a point in space as affording a complex number of our system we can state the corresponding theorem for complex numbers, as follows. There is in our system one number (which we will take for the unit e_1) which, when multiplied by any number in the system remains unchanged except for a constant of multiplication. Thus

$$e_k e_1 = \gamma_{k11} e_1 \qquad (k = 1, 2, \cdots, n).$$

There is also a number e_2 such that the product of any number on it (in the same order as in the previous product) is a linear combination of e_1 and e_2. Thus

$$e_k e_2 = \gamma_{k21} e_1 + \gamma_{k22} e_2, \qquad (k = 2, 3, \cdots, n).$$

Similarly

$$e_k e_l = \gamma_{kl1} e_1 + \gamma_{kl2} e_2 + \cdots + \gamma_{kll} e_l \quad (k, l = 1, 2, \cdots, n;\ k \geqq l)$$

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* PEIRCE, loc. cit., p. 114.

† Encyclopädie der Mathematischen Wissenschaften, vol. I, page 181. See also CARTAN, Comptes Rendus, 1897, page 1220.

But to any given system corresponds also a second group reciprocal to the one just considered and from the same reasoning we see that

$$e_l e_k = \gamma_{lk1} e_1 + \gamma_{lk2} e_2 + \cdots + \gamma_{lkl} e_l \quad (k, l = 1, 2, \cdots, n;\ k \geqq l).$$

Thus

$$e_i e_k = \sum_{s=1}^{l} \gamma_{iks} e_s$$

where l is the less of i and k.* The single idempotent number of our system is the modulus μ, and must contain e_n. If we now transform our system by the equations

$$e'_k = e_k, \qquad e'_n = \mu \qquad (k = 1, 2, \cdots, n-1),$$

the new units are independent of each other and the theorem is established.

THEOREM X. *Two systems containing different numbers of independent idempotent numbers are inequivalent.*

Let S and S_1 be two systems containing r and $r+s$ independent idempotent numbers respectively which we assume are taken as the units of highest indices. If they were equivalent, we should have

$$e'_{n-k} = \sum_{\lambda=1}^{n} a_{n-k,\lambda} e_\lambda \qquad (k = 0, \cdots, r+s-1),$$

where e'_{n-k} are units in S_1. But since these numbers are idempotent and independent the right-hand members must be the same, which would give us $r+s$ independent idempotent numbers in S; and this is impossible.

Similarly we can show that systems which have been regularized (according o theorem VI) and in which the units fall into groups in different arrangements are inequivalent.

We are now in a position to write down all systems in less than six units which have moduli in more than one idempotent unit. Tables of such systems have been given by SCHEFFERS,† and in part by STUDY ‡ and ROHR. § As an illustration of the method let us derive all systems in five units with two idempotent units. This case is selected because it is the only one in the least complicated.

By the tables on pages 316 and 318 we see that the following six systems embrace all the irreducible, inequivalent, non-reciprocal systems in question. The group of the nilpotent units e_k ($k = 1, 2, 3$) with respect to e_4 or e_5 is found in the row of e_k and the column of e_4 or e_5.

* The fact here stated is proved in substantially the same manner by SCHEFFERS, loc. cit., page 306.

† Mathematische Annalen, vol. 39.

‡ Göttinger Nachrichten, 1889.

§ Dissertation, loc. cit.

	1		2		3		4		5		6	
	e_4	e_5	e_4	e_5	e_4	e_5	e_4	e_5	e_4	e_5	e_4	e_5
e_1	II	III	II	III	IV	I	II	III	II	III	II	III
e_2	IV	I	II	III	II	III	I	IV	II	III	II	III
e_3	IV	I	IV	I	III	II	IV	I	II	III	III	II

1. By the associative law on the products $e_1e_3^k$ and $e_1e_2^t$ and by theorem IX we see that $e_1e_3 = e_1e_2 = 0$. The sub-systems formed by e_2e_3 must be nilpotent systems in two units, and hence must be in one of the forms given in the tables below (pages 327 and 328 for $n = 2$). (The Roman numerals with subscripts indicate the respectively equivalent systems in SCHEFFERS' classification.)

TABLE 1_1 (V_2)

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	e_1
e_2	0	0	0	0	e_2
e_3	0	0	e_2	0	e_3
e_4	e_1	0	0	e_4	0
e_5	0	e_2	e_3	0	e_5

TABLE 1_2 (V_{14})

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	e_1
e_2	0	0	0	0	e_2
e_3	0	0	0	0	e_3
e_4	e_1	0	0	e_4	0
e_5	0	e_2	e_3	0	e_5

2. From the above general table we have immediately the table following.

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	$a_{13}e_1 + b_{13}e_2$	0	e_1
e_2	0	0	$a_{23}e_1 + b_{23}e_2$	0	e_2
e_3	0	0	$c_3e_3 + f_3e_5$	0	e_3
e_4	e_1	e_2	0	e_4	0
e_5	0	0	e_3	0	e_5

By the transformation $e_3' = e_3 + i\sqrt{f_3}\,e_5$, f_3 vanishes, and since we can have only two independent idempotent numbers $c_3 = 0$.

Two cases arise:

2_1) $b_{13} = 0$;

2_2) $b_{13} \neq 0$.

2_1. By the associative law on $e_1e_3^2$ and $e_2e_3^2$ we get $a_{13} = b_{23} = 0$. We have then the two following systems according as $a_{23} \neq 0$ or $a_{23} = 0$.

TABLE 2_{11} (V_{10})

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	e_1
e_2	0	0	e_1	0	e_2
e_3	0	0	0	0	e_3
e_4	e_1	e_2	0	e_4	0
e_5	0	0	e_3	0	e_5

TABLE 2_{12} (V_{11})

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	e_1
e_2	0	0	0	0	e_2
e_3	0	0	0	0	e_3
e_4	e_1	e_2	0	e_4	0
e_5	0	0	e_3	0	e_5

2_2. If $b_{13} \neq 0$ we transform by

$$e_1' = e_2 + \frac{a_{13}}{b_{13}} e_1$$
$$e_2' = e_1,$$

use the associative law as before and arrive at table 2_{11}.

3. We have immediately from the general table and theorem IX this table:

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	$c_{13}e_3$	0	e_1
e_2	$b_{21}e_2$	0	$d_{23}e_4$	0	e_2
e_3	0	$a_{32}e_1 + f_{32}e_5$	0	e_3	0
e_4	0	e_2	0	e_4	0
e_5	e_1	0	e_3	0	e_5

By the associative law on $e_1^2e_3 = 0$ and $e_2e_1^2 = 0$ we get $c_{13} = b_{21} = 0$. From the product $e_2e_3e_2$ we see that $f_{32} = d_{23}$, and its product $e_3e_2e_1$ shows that $f_{32} = 0$.

Thus we get tables 3_1 and 3_2 according as $a_{23} \neq 0$ or $a_{23} = 0$.

TABLE 3_1 (V_{12})

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	e_1
e_2	0	0	0	0	e_2
e_3	0	e_1	0	e_3	0
e_4	0	e_2	0	e_4	0
e_5	e_1	0	e_3	0	e_5

TABLE 3_2 (V_{13})

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	e_1
e_2	0	0	0	0	e_2
e_3	0	0	0	e_3	0
e_4	0	e_2	0	e_4	0
e_5	e_1	0	e_3	0	e_5

4. From the table on p. 324, theorem IX and by use of the associative law on the products $e_1 e_3^2$ and $e_2^2 e_1$ we have the table 4.

5. Table 5 comes immediately from the table on p. 324.

TABLE 4 (V_3)

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	e_1
e_2	0	0	0	e_2	0
e_3	0	0	0	0	e_3
e_4	e_1	e_2	0	e_4	0
e_5	0	0	e_3	0	e_5

TABLE 5 (V_{30})

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	e_1
e_2	0	0	0	0	e_2
e_3	0	0	0	0	e_3
e_4	e_1	e_2	e_3	e_4	0
e_5	0	0	0	0	e_5

6. From the table on p. 324, by use of the associative law on the products $e_1 e_3 e_1$, $e_2 e_3 e_2$ and $e_1 e_3 e_2$ we get the table 6.

TABLE 6 (V_{31})

	e_1	e_2	e_3	e_4	e_5
e_1	0	0	0	0	e_1
e_2	0	0	0	0	e_2
e_3	0	0	0	e_3	0
e_4	e_1	e_2	0	e_4	0
e^5	0	0	e_3	0	e_5

II.

NILPOTENT SYSTEMS.

In order to derive all systems with a modulus in one idempotent unit it is convenient to prove certain theorems regarding nilpotent systems and their relation to the desired systems.

It is plain from theorem IX *that if any system in n units with a modulus with one idempotent unit is deleted by the modulus, a nilpotent system in* $n-1$ *units remains. If we border a nilpotent system on the right and below with a unit* e_n *such that* $e_k e_n = e_n e_k = e_k$ $(k = 1, \cdots, n)$ *we have a system in group* I *with modulus* e_n.

Def. 16. We call the idempotent and nilpotent systems bearing the above relation to each other *corresponding* systems.

THEOREM XI. *The necessary and sufficient condition that two systems in n units with moduli in one idempotent unit are equivalent is that their corresponding nilpotent systems are equivalent.*

(a) If two idempotent systems are equivalent their corresponding systems are equivalent. Let I and I' be idempotent systems with the units $e_1 e_2 \cdots e_n$ and $e_1' e_2' \cdots e_n'$ respectively. Let the equations of transformation be

$$e_k' = \sum_{i=1}^{n} a_{ki} e_i \qquad (k = 1, \cdots, n; \ |a_{ki}| \neq 0).$$

The equation

$$e_n' = \sum_{i=1}^{n} a_{ni} e_i,$$

must reduce to $e_n' = e_n$ or else we should have two independent idempotent units in I. The remaining equations have $a_{kn} = 0$, otherwise the units $e_1', e_2', \cdots, e_{n-1}'$ would not be nilpotent. Thus the equations of transformation become

$$e_k' = \sum_{i=1}^{n-1} a_{ki} e_i, \quad e_n' = e_n \qquad (k = 1, \cdots, n-1; \ |a_{ik}| \neq 0)$$

and the condition is necessary.

(b) The condition is plainly sufficient, for if we adjoin to the equations which connect the two nilpotent systems in $n - 1$ units the equation $e_n' = e_n$, we have a set of transformations connecting I and I'.

Def. 17.* The *degree of a nilpotent system* is the same as that of the number of highest degree in that system.

By the same method as used in theorem X we see that nilpotent systems of different degrees are inequivalent.

General forms of nilpotent tables.

Degree n.

We have $e_n^n \neq 0$, $e_n^{n+1} = 0$. Then by theorem VIII we have the table

	e_1	e_2	e_3	$\cdots$	e_{n-2}	e_{n-1}	e_n
e_1	0	0	0	$\cdots$	0	0	0
e_2	0	0	0	$\cdots$	0	0	e_1
e_3	0	0	0	$\cdots$	0	e_1	e_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
e_{n-2}	0	0	0	$\cdots$	e_{n-5}	e_{n-4}	e_{n-3}
e_{n-1}	0	0	e_1	$\cdots$	e_{n-4}	e_{n-3}	e_{n-2}
e_n	0	e_1	e_2	$\cdots$	e_{n-3}	e_{n-2}	e_{n-1}

* PEIRCE, loc. cit., p. 115.

Degree $n-1$.

We have $e_n^{n-1} \neq 0$, $e_n^n = 0$.

By use of the associative law on the product $e_n^n e_1$ and the transformation

$$e_1' = e_1 - \gamma_{n12} e_3 - \gamma_{n13} e_4 - \cdots - \gamma_{n\,1\,n-1} e_n$$

we get

$$e_n e_1 = 0.$$

Hence

$$e_k e_1 = 0 \qquad (k = 2, \cdots, n),$$

since we take the units $e_k = e_n^{n-k+1}$. By the use of the product $e_n e_1 e_n$ we get

$$\gamma_{1n3} = \gamma_{1n4} = \cdots = \gamma_{1nn} = 0.$$

Thus

$$e_1 e_n = \gamma_{1n1} e_1 + \gamma_{1n2} e_2.$$

Similarly using $e_n e_1^2$ we have

$$e_1^2 = \gamma_{111} e_1 + \gamma_{112} e_2.$$

By the use of the product $e_1 e_n^n$ we get $\gamma_{1n1} = 0$, from which we see that $e_1 e_k = 0$ $(k = 2, \cdots, n-1)$. Similarly by $e_1^2 e_2$ we get $\gamma_{111} = 0$. If $\gamma_{112} \neq 0$, $\gamma_{1n2} \neq 0$ we observe by the transformation $e_1' = e_1/\sqrt{\gamma_{112}}$ that $e_1'^2 = e_2$. By the transformation

$$e_1' = \gamma_{1n2}^{\frac{n-1}{n-3}} e_1, \qquad e_{n-k}' = \gamma_{1n2}^{\frac{2(k+1)}{n-3}} e_{n-k} \qquad (k = 0, 1, \cdots, n-2), \tag{11}$$

we get

$$e_1' e_n' = e_2, \quad e_1' e_k' = 0 \qquad (k = 2, \cdots, n-1),$$

while $e_1'^2 = e_2$. If $\gamma_{112} = 0$ we easily transform the system so that $\gamma_{1n2} = 1$ if it does not vanish, and vice versa. It is necessary to note that transformations (11) fail when $n = 3$, so that a special investigation must be made when deriving systems in three units.

The following table gives the four possible systems of degree $n-1$.

	e_1	e_2	e_3	e_4	$\cdots$	e_{n-1}	e_n
e_1	0 or e_2	0	0	0	$\cdots$	0	0 or e_2
e_2	0	0	0	0	$\cdots$	0	0
e_3	0	0	0	0	$\cdots$	0	e_2
e_4	0	0	0	0	$\cdots$	e_2	e_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
e_{n-1}	0	0	0	e_2	$\cdots$	e_{n-3}	e_{n-2}
e_n	0	0	e_2	e_3	$\cdots$	e_{n-2}	e_{n-1}

Degree $n-2$.

We have $e_n^{n-2} \neq 0$, $e_n^{n-1} = 0$. Thus

$$e_{n-j}e_{n-k} = e_{n-(j+k+1)}{}^{*} \qquad (j+k+1<n-2),$$

$$e_{n-j}e_{n-k} = 0^{*} \qquad (j,\ k<n-2;\ j+k+1 \geqq n-2).$$

If $\gamma_{n21} \neq 0$ we may transform so that $e_n e_2 = e_1$.* If $\gamma_{n21} = 0$ we easily get $e_n e_2 = 0$. In either case $e_n e_1 = 0$.* Thus

$$e_k e_2 = e_k e_1 = 0^{*} \qquad (k = n-1, \cdots, 3).$$

(*Case* 1°) $\qquad e_n e_2 = e_1.$

By the associative law on the products $e_n e_1 e_n$ and $e_1 e_n^{n-2}$ we get

$$e_1 e_n = \gamma_{1n3} e_3.^{*}$$

Thus

$$e_1 e_{n-k} = 0^{*} \qquad (k=1, \cdots, n-3).$$

By the product $e_1 e_n e_2$ we get $e_1^2 = 0$.* By the products $e_n e_1 e_2$, $e_n e_2^2$ and e_2^k ($e_2^k = 0$) we get

$$e_2^2 = \gamma_{221} e_1 + \gamma_{223} e_3 + \gamma_{224} e_4.$$

By the product $e_2^3 = 0$ we see that if $n=4$, $\gamma_{221}\gamma_{123} = 0$ * and $\gamma_{224} = 0$. By $e_n e_2^2$ we get

$$e_1 e_2 = \gamma_{224} e_3.^{*}$$

Similarly

$$e_2 e_1 = \gamma_{213} e_3.^{*}$$

By $e_n e_2 e_n$ we have

$$e_2 e_n = \gamma_{2n1} e_1 + \gamma_{2n3} e_3 + \gamma_{1n3} e_4,^{*}$$

and

$$e_2 e_{n-1} = \gamma_{1n3}(\gamma_{2n1}+1) e_3.^{*}$$

Thus

$$e_2 e_{n-k} = 0^{*} \qquad (k=2, \cdots, n-3).$$

For $n=4$ by $e_2 e_n e_2$ we get $\gamma_{213} = \gamma_{1n3} = 0$. For any n we find $\gamma_{213} = \gamma_{2n1}\gamma_{123}$. Thus we have the following table.

	e_1	e_2	e_3	e_4	$\cdots$	e_{n-2}	e_{n-1}	e_n
e_1	0	$\gamma_{224}e_3$	0	0	$\cdots$	0	0	$\gamma_{1n3}e_3$
e_2	$\gamma_{213}e_3$	$\gamma_{221}e_1 + \gamma_{223}e_3 + \gamma_{224}e_4$	0	0	$\cdots$	0	$\gamma_{1n3}(\gamma_{2n1}+1)e_3$	$\gamma_{2n1}e_1 + \gamma_{2n3}e_3 + \gamma_{1n3}e_4$
e_3	0	0	0	0	$\cdots$	0	0	0
e_4	0	0	0	0	$\cdots$	0	0	e_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
e_{n-2}	0	0	0	0	$\cdots$	e_{n-5}	e_{n-4}	e_{n-3}
e_{n-1}	0	0	0	0	$\cdots$	e_{n-4}	e_{n-3}	e_{n-2}
e_n	0	e_1	0	e_3	$\cdots$	e_{n-3}	e_{n-2}	e_{n-1}

* Results to be used directly in the table following are marked with an asterisk.

(*Case* 2°) $$e_n e_2 = 0.$$

In this case by evident use of the associative law we get

$$e_{n-k} e_1 = e_{n-k} e_2 = 0 \qquad (k = 0, \cdots, n-3),$$

$$e_i e_k = \sum_{s=1}^{3} \gamma_{iks} e_s \qquad (i = 1, 2;\ k = 1, \cdots, n).$$

Degree 1.*

Since the square of every number vanishes, if α and β be any two numbers by developing $(\alpha + \beta)^2$ we see that $\alpha\beta = -\beta\alpha$.

From these general tables by considering the various values that the constants in our tables may assume, after some reductions which it is not necessary to reproduce here, we find all the systems desired, the tables so obtained being identical with those in one idempotent unit enumerated by SCHEFFERS, loc. cit., p. 355 ff.

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* PEIRCE, loc. cit., p. 117.