CONJUGATE RECTILINEAR CONGRUENCES*

BY

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Introduction.

CIFARELLI † has established certain formulæ showing the relation existing between the Kummer functions of a rectilinear congruence referred to a general double family of parametric ruled surfaces of the congruence. In § 1 we have applied these formulæ to several special cases—afforded by taking in turn for the double family of parametric ruled surfaces the principal surfaces, the mean ruled surfaces, and one family of the developables and their orthogonal trajectories—for the determination of all congruences having a given spherical representation of any one of these three double families. In each case we find that the abscissa, measured from the surface of reference, of the point where the line of shortest distance between the lines (u, v), $(u + du, v)$ meets the former line satisfies a partial differential equation of the second order whose coefficients involve the coefficients of the fundamental quadratic differential form of the sphere. When this abscissa has been found, the further determination reduces to the solution of a Riccati equation and quadratures.

In § 2 we restrict ourselves to systems upon the sphere consisting of a family of great circles, $u =$ const., and their orthogonal trajectories, $v =$ const. Having shown that the line of shortest distance between the lines (u, v), $(u + du, v)$ has a different direction from the line of shortest distance between the latter and the consecutive line in the ruled surface $v =$ const. which passes through the line (u, v), we consider the congruence of lines upon which the shortest distances are measured and call it a *conjugate* of the original congruence. Evidently for each choice of a family of great circles on the sphere there is a conjugate of a given congruence, hence the conjugate is not determinate until the parametric system of curves is given. The above definition fails when the ruled surfaces $v =$ const. are developables, but another definition, consistent with the former, removes this exception.

In § 3, certain relations are found to hold between the Kummer functions of a congruence and any of its conjugates, from which one finds the equation of

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† *Le Congruence*, Annali di Matematica, ser. 3, vol. 2 (1899), p. 139.

condition that a conjugate congruence of a normal congruence be normal. In order that the lines of a conjugate congruence of a normal congruence be tangent to one of the orthogonal surfaces of the latter, they must be tangents to asymptotic lines upon this surface.

It is shown in § 5 that when a family of great circles and their orthogonal trajectories are given, the determination of the direction cosines is the same problem as the finding of a skew curve from its intrinsic equations.

In §§ 6, 7 three cases are discussed, according as a given system of great circles and their orthogonal trajectories are the spherical representation of the principal ruled surfaces, of the mean ruled surfaces, or of one family of developables and the ruled surfaces of the congruence cutting them orthogonally. In particular, those congruences are considered for which the corresponding conjugate congruences have ruled surfaces of like character in correspondence. Of special interest is the result that, when a congruence and the conjugate which corresponds to its developables are normal congruences, the developables correspond and the determination of all such congruences with a given spherical representation of their developables reduces to quadratures.

§ 1. *Determination of congruences with a given spherical representation of particular ruled surfaces.*

Consider a rectilinear congruence referred to a general system of parameters, u, v. Let X, Y, Z denote the direction cosines of any line Γ of the congruence, and x, y, z, the cartesian coördinates of the point where the line meets the surface of reference. Write

$$(1)\qquad E=\sum\left(\frac{\partial X}{\partial u}\right)^2,\quad F=\sum\frac{\partial X}{\partial u}\frac{\partial X}{\partial v},\quad G=\sum\left(\frac{\partial X}{\partial v}\right)^2,$$

$$e=\sum\frac{\partial x}{\partial u}\frac{\partial X}{\partial u},\quad f=\sum\frac{\partial x}{\partial v}\frac{\partial X}{\partial u},\quad f'=\sum\frac{\partial x}{\partial u}\frac{\partial X}{\partial v},\quad g=\sum\frac{\partial x}{\partial v}\frac{\partial X}{\partial v}.$$

These are the KUMMER functions, and CIFARELLI* has shown that they are connected by the following relations:

$$(2)\qquad \begin{aligned}\frac{\partial x}{\partial u}&=\frac{eG-f'F}{EG-F^2}\frac{\partial X}{\partial u}+\frac{f'E-eF}{EG-F^2}\frac{\partial X}{\partial v}+AX,\\ \frac{\partial x}{\partial v}&=\frac{fG-gF}{EG-F^2}\frac{\partial X}{\partial u}+\frac{gE-fF}{EG-F^2}\frac{\partial X}{\partial v}+BX,\end{aligned}$$

where

$$(3)\qquad A=\sum X\frac{\partial x}{\partial u},\quad B=\sum X\frac{\partial X}{\partial v}.$$

* l. c., p. 141.

The expressions in y and z follow at once by analogy. The condition of integrability of (2) leads to the following relations:

$$\frac{\partial e}{\partial v}-\frac{\partial f}{\partial u}-\left\{\begin{matrix}12\\1\end{matrix}\right\}'e+\left\{\begin{matrix}11\\1\end{matrix}\right\}'f-\left\{\begin{matrix}12\\2\end{matrix}\right\}'f' +\left\{\begin{matrix}11\\2\end{matrix}\right\}'g+FA-EB=0,$$

$$(4)\qquad \frac{\partial f'}{\partial v}-\frac{\partial g}{\partial u}-\left\{\begin{matrix}22\\1\end{matrix}\right\}'e+\left\{\begin{matrix}12\\1\end{matrix}\right\}'f-\left\{\begin{matrix}22\\2\end{matrix}\right\}'f' +\left\{\begin{matrix}12\\2\end{matrix}\right\}'g+GA-FB=0,$$

$$\frac{\partial A}{\partial v}-\frac{\partial B}{\partial u}+f-f'=0,$$

where the Christoffel symbols $\left\{\begin{smallmatrix}rs\\t\end{smallmatrix}\right\}'$ are formed with respect to the quadratic form

$$(5)\qquad E du^2+2F du\, dv+G dv^2.$$

We now apply these general results to several particular cases.

Consider first the case where the spherical representation is that of the *principal* ruled surfaces of the congruence, and take for the surface of reference the middle surface of the congruence. From this hypothesis we have *

$$(6)\qquad F=0,\qquad f+f'=0,\qquad eG+gE=0.$$

Indicating by r the abscissa of the limit point corresponding to the lines (u, v), $(u+du, v)$, we have in consequence of the last equation of (6) †

$$(7)\qquad r=-\frac{e}{E}=\frac{g}{G}.$$

In conformity with the second equation of (6) we introduce with Bianchi ‡ a new function ϕ defined by

$$(8)\qquad f=\phi\sqrt{EG},\qquad f'=-\phi\sqrt{EG}.$$

Substituting these expressions for e, f, f', g, F in (2) and replacing A and B by their values given in the first two of equations (4), we have for the mean surface

* Bianchi, *Lezioni*, p. 258.

† l. c., p. 247.

‡ l. c., p. 275.

$$
(9) \qquad
\begin{aligned}
\frac{\partial x}{\partial u} &= -r\frac{\partial X}{\partial u} - \sqrt{\frac{E}{G}}\phi\frac{\partial X}{\partial v} + \left(\sqrt{\frac{E}{G}}\frac{\partial \phi}{\partial v} + \frac{1}{G}\frac{\partial (rG)}{\partial u}\right)X, \\
\frac{\partial x}{\partial v} &= r\frac{\partial X}{\partial v} + \sqrt{\frac{G}{E}}\phi\frac{\partial X}{\partial u} - \left(\frac{1}{E}\frac{\partial (rE)}{\partial v} + \sqrt{\frac{G}{E}}\frac{\partial \phi}{\partial u}\right)X,
\end{aligned}
$$

with analogous expressions in y and z. And the third of equations (4) reduces to

$$
(10) \qquad
\begin{aligned}
&2\frac{\partial^2 r}{\partial u\,\partial v} + \frac{\partial \log E}{\partial v}\frac{\partial r}{\partial u} + \frac{\partial \log G}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial^2 \log EG}{\partial u\,\partial v}r \\
&\qquad = -\left[\frac{\partial}{\partial u}\left(\sqrt{\frac{G}{E}}\frac{\partial \phi}{\partial v}\right) + \frac{\partial}{\partial v}\left(\sqrt{\frac{E}{G}}\frac{\partial \phi}{\partial u}\right)\right] - 2\sqrt{EG}\,\phi,
\end{aligned}
$$

which is the necessary and sufficient condition which r and ϕ must satisfy in order that the congruence shall have the given representation of its principal surfaces. Evidently one of these functions can be chosen arbitrarily and the other is determined by the integration of an equation of Laplace.

In the second case we consider the determination of congruences with a given spherical representation of their *mean ruled surfaces*. As defined by CIFARELLI,* the mean ruled surfaces form a double system, in general unique, for which the surfaces of one family cut those of the other family orthogonally, and the lines of striction of the two surfaces which pass through any line meet the latter in its middle point. When the congruence is referred to a general parametric system, the directions of the spherical representation of the mean ruled surfaces are given by the equation obtained by equating to zero the Jacobian of equation (5) and of the left-hand member of the general equation for the directions of the principal ruled surfaces.† This gives

$$
(11) \qquad
\begin{aligned}
&\{E(gE - eG) - F[(f+f')E - 2eF]\}du^2 \\
&\qquad + 2[F(gE+eG) - (f+f')EG]\,du\,dv \\
&\qquad\qquad + \{F[2gF - (f+f')G] - G(gE - eG)\}dv^2 = 0.
\end{aligned}
$$

In order that the parametric curves on the sphere be the images of the mean surfaces, it is necessary and sufficient that

$$
(12) \qquad gE - eG = 0, \qquad F = 0;
$$

and if we take one of the limit surfaces for the surface of reference, we must have ‡

* Annali di Matematica, ser. 3, vol. 2 (1899), p. 151.

† BIANCHI, *Lezioni*, p. 251.

‡ l. c., p. 251.

$$eg - \left(\frac{f+f'}{2}\right)^2 = 0, \tag{13}$$

and conversely. From (12) we get

$$r = -\frac{e}{E} = -\frac{g}{G}; \tag{14}$$

and in conformity with (13) we introduce a new function ϕ, defined by

$$\frac{f}{\sqrt{EG}} = -r + \phi, \qquad \frac{f'}{\sqrt{EG}} = -(r + \phi). \tag{15}$$

Proceeding as in the former case we obtain

$$\begin{aligned} \frac{\partial x}{\partial u} &= -r\frac{\partial X}{\partial u} - \sqrt{\frac{E}{G}}(r+\phi)\frac{\partial X}{\partial v} \\ &\qquad - \left(\frac{\partial r}{\partial u} - \sqrt{\frac{E}{G}}\frac{\partial}{\partial v}(r+\phi) - \frac{1}{\sqrt{EG}}\frac{\partial E}{\partial v}r\right)X, \\ \frac{\partial x}{\partial v} &= -\sqrt{\frac{G}{E}}(r-\phi)\frac{\partial X}{\partial u} - r\frac{\partial X}{\partial v} \\ &\qquad - \left(\frac{\partial r}{\partial v} - \sqrt{\frac{G}{E}}\frac{\partial}{\partial u}(r-\phi) - \frac{1}{\sqrt{EG}}\frac{\partial G}{\partial u}r\right)X, \end{aligned} \tag{16}$$

and similar equations in y and z. Again we find that the functions r and ϕ satisfy a condition of the second order; it is

$$\begin{aligned} &\sqrt{\frac{G}{E}}\frac{\partial^2 r}{\partial u^2} - \sqrt{\frac{E}{G}}\frac{\partial^2 r}{\partial v^2} + \left(\frac{\partial}{\partial u}\sqrt{\frac{G}{E}} + \frac{1}{\sqrt{EG}}\frac{\partial G}{\partial u}\right)\frac{\partial r}{\partial u} \\ &- \left(\frac{\partial}{\partial v}\sqrt{\frac{E}{G}} + \frac{1}{\sqrt{EG}}\frac{\partial E}{\partial v}\right)\frac{\partial r}{\partial v} + \left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial u}\right) - \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial v}\right)\right]r \\ &\qquad = \frac{\partial}{\partial u}\left(\sqrt{\frac{G}{E}}\frac{\partial \phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\sqrt{\frac{E}{G}}\frac{\partial \phi}{\partial v}\right) + 2\sqrt{EG}\,\phi. \end{aligned} \tag{17}$$

In the third case, we seek to determine the congruences for which one family of developables is represented by a given family of curves upon the sphere. Let $v =$ const. represent these curves and let $u =$ const. be their orthogonal trajectories; then $F = 0$. From the general equation for the direction of the developables * we find for this choice of parameters

$$f' = 0. \tag{18}$$

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* l. c., p. 252.

Take for the surface of reference the locus of the points half way between the focal point given by the lines (u, v), $(u + du, v)$, and the point where the line of shortest distance between the lines (u, v), $(u, v + dv)$ meets the former line; then

$$r = -\frac{e}{E} = \frac{g}{G}. \tag{19}$$

If we put

$$\frac{f}{E} = \phi, \tag{20}$$

the surface of reference is given by

$$\begin{aligned} \frac{\partial x}{\partial u} &= -r\frac{\partial X}{\partial u} + \left(\frac{\partial r}{\partial u} + r\frac{\partial \log G}{\partial u} - \frac{1}{2G}\frac{\partial E}{\partial v}\phi\right)X, \\ \frac{\partial x}{\partial v} &= \phi\frac{\partial X}{\partial u} + r\frac{\partial X}{\partial v} - \left(\frac{\partial r}{\partial v} + r\frac{\partial \log E}{\partial v} + \frac{\partial \phi}{\partial u} + \frac{\partial \log \sqrt{E}}{\partial u}\phi\right)X, \end{aligned} \tag{21}$$

and similar equations in y and z. And the functions ϕ, r, satisfy the condition

$$\begin{aligned} 2\frac{\partial^2 r}{\partial u\,\partial v} &+ \frac{\partial \log E}{\partial v}\frac{\partial r}{\partial u} + \frac{\partial \log G}{\partial u}\frac{\partial r}{\partial v} + \frac{\partial^2 \log EG}{\partial u\,\partial v}r \\ &= -\left[\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial \log \sqrt{E}}{\partial u}\frac{\partial \phi}{\partial u} - \frac{1}{2G}\frac{\partial E}{\partial v}\frac{\partial \phi}{\partial v}\right. \\ &\qquad \left. + \phi\left(\frac{1}{2G^2}\frac{\partial E}{\partial v}\frac{\partial G}{\partial v} - \frac{1}{2G}\frac{\partial^2 E}{\partial v^2} + \frac{\partial^2 \log \sqrt{E}}{\partial u^2} + E\right)\right]. \end{aligned} \tag{22}$$

In all three cases the congruence is normal when $\phi = 0$. Furthermore, when this condition is satisfied, the first and third cases are the same.

§ 2. *Definition of conjugate congruences.*

Let λ, μ, ν denote the direction-cosines of the line of shortest distance between the lines (u, v), $(u + du, v + dv)$ of a congruence referred to a general parametric system. It can be shown that *

$$\lambda = \frac{\left(E\frac{\partial X}{\partial v} - F\frac{\partial X}{\partial u}\right)du + \left(F\frac{\partial X}{\partial v} - G\frac{\partial X}{\partial u}\right)dv}{\sqrt{EG - F^2}\sqrt{E du^2 + 2F du\,dv + G dv^2}},$$

and similarly for μ and ν. If in particular we denote by X_1, Y_1, Z_1; X_2, Y_2 Z_2

* BIANCHI, l. c., p. 246.

the direction cosines of the line of shortest distance from the line (u, v) to $(u + du, v)$ and $(u, v + dv)$ respectively, we have from the preceding formula

$$
(23)\qquad X_1 = \frac{E\frac{\partial X}{\partial v} - F\frac{\partial X}{\partial u}}{\sqrt{E}\sqrt{EG - F^2}}, \qquad Y_1 = \frac{E\frac{\partial Y}{\partial v} - F\frac{\partial Y}{\partial u}}{\sqrt{E}\sqrt{EG - F^2}},
$$
$$
Z_1 = \frac{E\frac{\partial Z}{\partial v} - F\frac{\partial Z}{\partial u}}{\sqrt{E}\sqrt{EG - F^2}};
$$

$$
(24)\qquad X_2 = \frac{F\frac{\partial X}{\partial v} - G\frac{\partial X}{\partial u}}{\sqrt{G}\sqrt{EG - F^2}}, \qquad Y_2 = \frac{F\frac{\partial Y}{\partial v} - G\frac{\partial Y}{\partial u}}{\sqrt{G}\sqrt{EG - F^2}},
$$
$$
Z_2 = \frac{F\frac{\partial Z}{\partial v} - G\frac{\partial Z}{\partial u}}{\sqrt{G}\sqrt{EG - F^2}}.
$$

From these we find upon differentiation and reduction

$$
(25)\qquad \frac{\partial X_1}{\partial u} = -\frac{\sqrt{EG - F^2}}{E^{\frac{3}{2}}}\left\{\begin{matrix}11\\2\end{matrix}\right\}'\frac{\partial X}{\partial u},
$$
$$
\frac{\partial X_1}{\partial v} = -\frac{\sqrt{EG - F^2}}{E^{\frac{3}{2}}}\left(\left\{\begin{matrix}12\\2\end{matrix}\right\}'\frac{\partial X}{\partial u} + EX\right),
$$

$$
(26)\qquad \frac{\partial X_2}{\partial u} = \frac{\sqrt{EG - F^2}}{G^{\frac{3}{2}}}\left(\left\{\begin{matrix}12\\1\end{matrix}\right\}'\frac{\partial X}{\partial v} + GX\right),
$$
$$
\frac{\partial X_2}{\partial v} = \frac{\sqrt{EG - F^2}}{G^{\frac{3}{2}}}\left\{\begin{matrix}22\\1\end{matrix}\right\}'X,
$$

and similar equations in the Y's and Z's.

The necessary and sufficient condition that the elements of any ruled surface $u = \text{const.}$ of a congruence be parallel to a plane, whose direction varies with the value of this constant, is that the surface be represented on the sphere by a great circle. The analytical expression of this condition is *

$$
(27)\qquad \left\{\begin{matrix}22\\1\end{matrix}\right\}' = 0.
$$

In this case, as is seen from (26), X_2, Y_2, Z_2 are functions of u alone, and conse-

* BIANCHI, l. c., p. 146.

quently all the lines of shortest length between consecutive lines of a ruled surface $u = \text{const.}$ have the same direction. Conversely, when this property is possessed by all of the surfaces $u = \text{const.}$, it is necessary that

$$\left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' \frac{\partial X}{\partial v} = \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' \frac{\partial Y}{\partial v} = \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\}' \frac{\partial Z}{\partial v} = 0.$$

Since one at least of the functions X, Y, Z must be a function of v, the above equations are satisfied only by the condition (27). Hence the

THEOREM.— *When all the lines of shortest length between consecutive lines of any surface in a family of ruled surfaces of a congruence have the same direction, the generatrices are parallel to a plane which is different for different surfaces of the family.*

We assume that the sphere is referred to a family of great circles and their orthogonal trajectories. Then (27) is satisfied, and by a proper choice of parameters we have *

$$F = 0, \qquad G = 1. \tag{28}$$

Consider in connection with the given congruence C the system of lines upon which is measured the shortest distances between the lines (u, v), $(u + du, v)$ of C. In order that this system of lines be a congruence C_1, it is sufficient that to each line Γ of C there corresponds a unique line Γ_1 of C_1. This would fail to occur only when the lines of shortest distance between three consecutive generatrices in a ruled surface $v = \text{const.}$ coincide, in which case the three generatrices would be parallel to the same plane. Hence from what has preceded we see that the sufficient condition that there be a unique line Γ_1 for each line Γ is given by

$$\frac{\partial X_1}{\partial u} \neq 0, \qquad \frac{\partial Y_1}{\partial u} \neq 0, \qquad \frac{\partial Z_1}{\partial u} \neq 0.$$

For from (25) it is clear that if these conditions were not satisfied we would have

$$\left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\}' = 0,$$

that is,† the lines $v = \text{const.}$ on the sphere would be great circles. Since there cannot exist upon a sphere an orthogonal system of great circles we are brought to the

THEOREM.— *When a family of great circles $u = \text{const.}$ and their orthogonal trajectories are the spherical representation of a congruence, the lines upon*

* BIANCHI, l. c., p. 154.

† l. c., p. 146.

which are measured the shortest distances between the lines (u, v), $(u + du, v)$ *from a congruence.*

We say that the latter congruence is *conjugate* to the former.

In consequence of the choice of parameters giving (28) the formulæ (23), (24) take the forms

$$(29) \qquad X_1 = \frac{\partial X}{\partial v}, \qquad \frac{\partial X_1}{\partial u} = \frac{\partial \log \sqrt{E}}{\partial v} \frac{\partial X}{\partial u}, \qquad \frac{\partial X_1}{\partial v} = -X.$$

These forms lead to an extension of the definition to an exceptional case.

Since consecutive generatrices of a developable surface intersect, the line of shortest distance between them is indeterminate. Consequently, if the ruled surfaces $v =$ const. of the congruence C are developables, the conjugate C_1, as above defined, would have no meaning. In accordance with the first equation of (29) we define the corresponding congruence C_1 in this case as the double system of lines with direction cosines equal to $\partial X/\partial v$, $\partial Y/\partial v$, $\partial Z/\partial v$ which pass through the corresponding focal points of the lines Γ.

Denote by E_1, F_1, G_1 the coefficients of the spherical representation of C_1; then from (29) we have

$$(30) \qquad E_1 = \left(\frac{\partial \sqrt{E}}{\partial v}\right)^2, \qquad F_1 = 0, \qquad G_1 = 1.$$

From this it follows that the curves $u =$ const. on the sphere for C_1 are great circles, and consequently the generatrices of any one of the ruled surfaces $u =$ const. of C_1 are parallel to a plane. It follows from analogy that the direction-cosines of the conjugate of C_1 are $\partial X_1/\partial v$, $\partial Y_1/\partial v$, $\partial Z_1/\partial v$; but by (29) these are equal to $-X$, $-Y$, $-Z$; that is, the image of this line on the sphere is diametrically opposite the corresponding point for the original congruence. It can be seen geometrically that this line not only has the same direction as the corresponding line of C but actually coincides with it. Hence the

THEOREM.—*The conjugate of any conjugate of a congruence is this congruence itself with the positive direction of the lines reversed.*

§ 3. *Relations between a congruence and any conjugate.*

Let x_1, y_1, z_1 denote the cartesian coördinates of the point on the surface of reference S_1 of C_1 in which it is met by the line $\Gamma_1(u, v)$. If r and r_1 denote the distances from S and S_1 respectively of the point of intersection of Γ and Γ_1, we have

$$x + rX = x_1 + r_1 X_1, \qquad y + rY = y_1 + r_1 Y_1, \qquad z + rZ = z_1 + r_1 Z_1.$$

With the aid of (29) we find

$$(31)\qquad \begin{cases} \dfrac{\partial \log \sqrt{E}}{\partial v}(e + rE) = e_1 + r_1 E_1, & f\dfrac{\partial \log \sqrt{E}}{\partial v} = f_1, \\ -\left(\sum X \dfrac{\partial x}{\partial u} + \dfrac{\partial r}{\partial u}\right) = f_1', & -\left(\sum X \dfrac{\partial x}{\partial v} + \dfrac{\partial r}{\partial v}\right) = g_1 + r_1, \end{cases}$$

where the subscripts denote that the functions belong to C_1.

When C is a normal congruence, $f = f'$, and we find from (31) that the necessary and sufficient condition that C_1 also be normal is

$$(32)\qquad \sum X\frac{\partial x}{\partial u} + \frac{\partial r}{\partial u} + \frac{\partial \log \sqrt{E}}{\partial v} f' = 0.$$

The equation $f = f'$ is obtained from the condition that $\sum X dx$ shall be an exact differential, that is,

$$\sum X dx + d\rho = 0.$$

From this equation it follows also that ρ is the abscissa of the point where the line (u, v) of the congruence is met by one of the orthogonal surfaces. We take this surface for the surface of reference S; then $\rho = 0$ and

$$(33)\qquad e = -D, \qquad f = f' = -D', \qquad g = -D'',$$

where D, D', D'' are the coefficients of the second fundamental quadratic form of S. In order that the functions ρ and r may be the same and the tangents to the curves $v =$ const. on S form a congruence C_1, it is necessary and sufficient that $r = 0$, or, from (33),

$$D = 0;$$

that is, the curves $v =$ const. on S must be asymptotic. Hence the

THEOREM.—*The necessary and sufficient condition that the tangents to a family of curves on a surface form a conjugate to the congruence of normals is that the curves be asymptotic lines and be represented upon the sphere by the orthogonal trajectories of a family of great circles.*

It is evident that there is only a particular class of surfaces possessing this latter property. Since the normals to a ruled surface along a generatrix are parallel to a plane it follows that for ruled surfaces of this class the non-linear asymptotic lines are the ones referred to. For future purposes we will determine all minimal surfaces of this category.

It is well known that minimal surfaces are characterized by the property that their asymptotic lines are orthogonal, and that when these lines are taken as parametric the square of the linear element of the surface and spherical representation are * respectively

* BIANCHI, l. c., p. 126.

$$ds^2 = \rho\,(du^2 + dv^2), \qquad ds'^2 = \frac{1}{\rho}(du^2 + dv^2),$$

where $-1/\rho^2$ is equal to the total curvature of the surface. If the curves $u =$ const. on the sphere are to be great circles, ρ must be a function of v alone, and consequently the asymptotic lines $u =$ const. on the surface are geodesics, that is, straight lines. As the helicoid with plane director is the only ruled minimal surface, we have the

THEOREM.—*The only minimal surface for which one family of asymptotic lines is represented by a family of geodesic parallels on the sphere is the helicoid with plane director.*

We return now to the consideration of the congruences of normals to any of the particular class of surfaces referred to above. From (31) we have

$$r_1 = -g_1.$$

Hence the ruled surfaces $u =$ const., $v =$ const. of C, are such that

$$\frac{e_1}{E_1} = \frac{g_1}{G_1}, \qquad F = 0,$$

and therefore by (14) they are the mean ruled surfaces of C_1. Hence *the point of tangency of any tangent to the curves $v = const.$ is the mean point.* Were this point to be the mean point of Γ also, the surface S would be minimal. These results may be stated as follows:

The tangents to the helices on a helicoid with plane director form a congruence conjugate to the congruence of normals, and the helicoid is the mean surface for each congruence. Moreover, this is the only normal congruence possessing this double property.

Since the lines $v =$ const. on any of these surfaces are curved asymptotic lines and consequently not geodesics, we have the

THEOREM.—*When C is a normal congruence and C_1 is composed of the tangents to the asymptotic lines on one of the orthogonal surfaces, C_1 is not a normal congruence.*

§ 4. *Determination of a conjugate congruence when the original congruence is defined in a particular way.*

A congruence for which all the generatrices of any ruled surface $u =$ const. are parallel to a plane may be defined by equations of the form

$$lx + my + nz = f_1(u, v), \qquad \lambda x + \mu y + \nu z = f_2(u, v), \tag{33'}$$

where l, m, n are functions of u alone and λ, μ, ν are functions of u and v. If X, Y, Z denote the direction cosines, we have

$$lX + mY + nZ = 0.$$

By differentiation with respect to v, we have in view of (29),

$$lX_1 + mY_1 + nZ_1 = 0.$$

Hence the ruled surfaces $u = c$ of C and $u = c$ of C_1 have their generatrices parallel to the same plane. From this it follows that the great circles $u =$ const. for C_1 are the same as for C, and from the definition of conjugate congruences it follows that corresponding points are at the distance of a quadrant.

Denote by Γ and Γ' the lines (u, v), $(u + du, v)$ of C and by Γ_1 their line of shortest distance. From (33′) we have

$$X,\ Y,\ Z = \frac{m\nu - \mu n,\ \ n\lambda - \nu l,\ \ l\mu - m\lambda}{[\sum(m\nu - \mu n)^2]^{\frac{1}{2}}},$$

and if X', Y', Z' denote the direction-cosines of Γ', they are given by

$$X',\ Y',\ Z' = X + \frac{\partial X}{\partial u} du,\quad Y + \frac{\partial Y}{\partial u} du,\quad Z + \frac{\partial Z}{\partial u} du.$$

Hence, when (33′) is given, the functions $X, \cdots, Z'$ can be calculated at once. The equation of the plane determined by Γ' and Γ_1 is

$$\text{(34)}\qquad \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ X' & Y' & Z' \\ YZ' - ZY' & ZX' - XZ' & XY' - YX' \end{vmatrix} = 0,$$

where x, y, z are current coördinates and x_1, y_1, z_1 are the coördinates of the intersection of Γ' and Γ_1. These coördinates are given by the equations

$$\text{(35)}\qquad \begin{cases} lx_1 + my_1 + nz_1 = f_1(u, v), \\ \dfrac{\partial l}{\partial u} x_1 + \dfrac{\partial m}{\partial u} y_1 + \dfrac{\partial n}{\partial u} z_1 = \dfrac{\partial f_1}{\partial u}, \\ \left(\lambda + \dfrac{\partial \lambda}{\partial u} du\right) x_1 + \left(\mu + \dfrac{\partial \mu}{\partial u} du\right) y_1 + \left(\nu + \dfrac{\partial \nu}{\partial u} du\right) z_1 = 0. \end{cases}$$

Since the plane (34) passes through the intersection of Γ and Γ_1, we see that if the expressions for x_1, y_1, z_1 obtained from (35) are put in (34) and this equation is combined with (33′), we can find the coördinates of the point where Γ meets Γ_1.

From the definition of C_1 and the results found above, we find that C_1 is given by the equations

$$\text{(33″)}\qquad lx + my + nz = f_1(u, v), \qquad \lambda_1 x + \mu_1 y + \nu_1 z = f_3(u, v),$$

where λ_1, μ_1, ν_1 are functions of u and v satisfying the condition

$$\sum (m\nu - \mu n)(m\nu_1 - \mu_1 n) = 0,$$

and where $f_3(u, v)$ is determined by the condition that the plane defined by the second of equations (33″) passes through the intersection of Γ and Γ_1. Hence:

When a congruence is defined in the form (33′) *the corresponding conjugate congruence can be found without integration.*

§ 5. *Determination of the direction cosines when the system on the sphere is given.*

Consider the sphere referred to a parametric system such that

$$F = 0, \qquad G = 1. \tag{28}$$

The Gauss equation * reduces to

$$\frac{\partial^2 \sqrt{E}}{\partial v^2} + \sqrt{E} = 0,$$

whence

$$\sqrt{E} = U_1 \cos v + U_2 \sin v, \tag{36}$$

where U_1, U_2 are functions of u alone.

Denoting by λ_1, μ_1, ν_1; λ_2, μ_2, ν_2 the direction cosines of the tangents to the curves $v =$ const. and $u =$ const. respectively on the sphere, we have †

$$d\lambda_2 = \frac{\partial \sqrt{E}}{\partial v} \lambda_1 du - X dv, \qquad dX = \sqrt{E}\, \lambda_1 du + \lambda_2 dv, \tag{37}$$

and similar relations between Y, μ_1, μ_2 and Z, ν_1, ν_2. From these equations we find that

$$\frac{\partial^2 X}{\partial v^2} + X = 0,$$

from which it follows that

$$X = U_{11} \cos v + U_{12} \sin v, \tag{37′}$$

where U_{11}, U_{12} are functions of u alone; similarly for Y, Z. The functions of u appearing in these expressions must satisfy the following conditions:

$$U_{11}^2 + U_{21}^2 + U_{31}^2 = 1, \quad U_{12}^2 + U_{22}^2 + U_{32}^2 = 1, \quad U_{11} U_{12} + U_{21} U_{22} + U_{31} U_{32} = 0. \tag{38}$$

Again from equations (37) we find

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*BIANCHI, l. c., p. 67.

† l. c., p. 94.

$$\frac{\partial \lambda_2}{\partial u} = \frac{\partial \log \sqrt{E}}{\partial v} \frac{\partial X}{\partial u},$$

from which it follows that

$$U_1 U'_{11} - U_2 U'_{12} = 0,$$

where the accents denote differentiation; similarly for the functions $U_{21}, \cdots, U_{32}$.

Substituting the above expressions for E, X, Y, Z in

$$E = \sum \left(\frac{\partial X}{\partial u} \right)^2,$$

we find the following relations:

$$\sum U'^2_{11} = U^2_1, \qquad \sum U'^2_{12} = U^2_2, \qquad \sum U'_{11} U'_{12} = U_1 U_2.$$

By means of the preceding systems of equations we find readily the following relations:

$$(39)\quad \begin{aligned} \frac{U'_{11}}{U_{21}U_{32} - U_{31}U_{22}} &= \frac{U'_{21}}{U_{31}U_{12} - U_{11}U_{32}} = \frac{U'_{31}}{U_{11}U_{22} - U_{21}U_{12}} = U_1, \\ \frac{U'_{12}}{U_{21}U_{32} - U_{31}U_{22}} &= \frac{U'_{22}}{U_{31}U_{12} - U_{11}U_{32}} = \frac{U'_{32}}{U_{11}U_{22} - U_{21}U_{12}} = U_2. \end{aligned}$$

The equations of condition (38), (39) are those which would obtain if

$$U_{11}, \quad U_{21}, \quad U_{31}; \quad U_{12}, \quad U_{22}, \quad U_{32}$$

were the direction cosines of the tangent and binormal respectively of a curve with curvature U_1 and torsion U_2. Hence we have the following

THEOREM.—*The problem of finding the direction cosines of the lines of a congruence for which the parametric ruled surfaces in one family are represented by great circles is equivalent to the complete determination of a skew curve from its intrinsic equations.*

As is well known, the latter requires the solution of a Riccati equation.

Conversely, when the direction cosines of the lines of a congruence can be expressed in the form (37′) where $U_{11}, \cdots, U_{32}$ satisfy conditions (38) and

$$\frac{U'_{11}}{U'_{12}} = \frac{U'_{21}}{U'_{22}} = \frac{U'_{31}}{U'_{32}},$$

the ruled surfaces $u =$ const. have a family of great circles for their spherical representation.

§ 6. *When the system on the sphere is the image of the principal surfaces.*

Let the great circles $u =$ const. on the sphere be the spherical representation of one family of principal ruled surfaces. The equation (10) reduces to the form:

$$
\begin{aligned}
&\frac{\partial}{\partial u}\left(2\frac{\partial r}{\partial v}+\frac{\partial \log E}{\partial v}r\right)\\
(40)\qquad &\qquad = -\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial \phi}{\partial v}\right)+\frac{\partial}{\partial v}\left(\sqrt{E}\frac{\partial \phi}{\partial u}\right)+2\phi\sqrt{E}\right].
\end{aligned}
$$

When ϕ is given an arbitrary form, r is obtained by quadratures. Hence the

THEOREM.—*The determination of all congruences for which one family of the principal ruled surfaces is represented on the sphere by a family of great circles reduces to quadratures, after the direction cosines of the lines of the congruence have been found.*

With every congruence C, whose principal ruled surfaces have a given spherical representation, there is associated a unique conjugate congruence C_1, determined by this parametric system. From (31) we find that the necessary and sufficient condition that the principal surfaces of C_1 correspond to those of C is given by the equation

$$
\sum X\frac{\partial x}{\partial u}+\frac{\partial r}{\partial u}+\frac{\partial \log \sqrt{E}}{\partial v}f'=0, \tag{41}
$$

or, by (9),

$$
\sqrt{E}\frac{\partial \phi}{\partial v}+2\frac{\partial r}{\partial u}-\frac{\partial \sqrt{E}}{\partial v}\phi=0. \tag{41'}
$$

The elimination of r between this equation and (40) leads to an equation in ϕ of the third order. Hence the

THEOREM.—*The determination of all congruences whose principal ruled surfaces are represented by a given family of great circles and their orthogonal trajectories and correspond to the principal surfaces of the conjugate congruence requires the solution of a partial differential equation of the third order.*

The condition (41) is the same as that which expresses that C_1 is normal when C is normal. Since in the latter case the principal surfaces are developable we have the

THEOREM.—*When a congruence and the conjugate corresponding to one family of developables are both normal, their developable surfaces correspond.*

When C is a normal congruence the equation (40) reduces to

$$
\frac{\partial}{\partial u}\left(2\frac{\partial r}{\partial v}+\frac{\partial \log E}{\partial v}r\right)=0, \tag{42}
$$

for which the general integral is

$$
r=\frac{1}{\sqrt{E}}\left(\int\sqrt{E}\,V du+U\right), \tag{43}
$$

where U and V are functions of u and v respectively. From (9) we see that in the present case the coördinates of the mean surface of the congruence are given by

$$(43') \qquad \frac{\partial x}{\partial u} = -r\frac{\partial X}{\partial u} + \frac{\partial r}{\partial u}X, \qquad \frac{\partial x}{\partial v} = r\frac{\partial X}{\partial v} - \frac{1}{E}\frac{\partial (rE)}{\partial v}X.$$

Denoting by ρ the abscissa of one of the normal surfaces, we have

$$\frac{\partial \rho}{\partial u} = -\sum X\frac{\partial x}{\partial u}, \qquad \frac{\partial \rho}{\partial v} = -\sum X\frac{\partial x}{\partial v},$$

or, with the aid of equations (43′),

$$\frac{\partial \rho}{\partial u} = -\frac{\partial r}{\partial u}, \qquad \frac{\partial \rho}{\partial v} = \frac{\partial r}{\partial v} + \frac{\partial \log E}{\partial v}r.$$

By means of (43) we get from these equations

$$\rho = -r + 2\int V dv,$$

where V is the same function which appears in (43).

From (41′) and (42) it is seen that if C_1 is also to be a normal congruence we must have

$$\frac{\partial r}{\partial u} = 0, \qquad r\frac{\partial^2 \log E}{\partial u\,\partial v} = 0.$$

Excluding isotropic congruences, or, in other words, the case $r = 0$, we see that the necessary and sufficient conditions that C and C_1 be normal congruences is that r be a function of v alone and $\sqrt{E}$ be the product of a function of u and a function of v. If the latter condition is to be satisfied, either $U_1 = 0$, $U_2 = 0$, or $U_1 = \lambda U_2$, where λ is a constant; and conversely. Recalling our previous results, we see that for the first two cases the finding of the direction cosines is equivalent to the determination of a plane curve from its intrinsic equation and in the last case of a general helix from its intrinsic equations. Since both of these determinations require quadratures only, we have the

THEOREM.—*The complete determination of all normal congruences for which, when the ruled surfaces $u = const.$, $v = const.$ are the developables, the conjugate congruence also is normal, reduces to quadratures.*

§ 7. *When the system on the sphere is the image of the mean surfaces or of a family of developables.*

From (17) we see that the finding of all congruences for which one family of mean ruled surfaces is represented by a given family of great circles requires the determination of two functions r and ϕ satisfying the equation:

$$
(44)\quad \begin{aligned} &\frac{1}{\sqrt{E}}\frac{\partial^2 r}{\partial u^2} - \sqrt{E}\frac{\partial^2 r}{\partial v^2} + \frac{\partial}{\partial u}\frac{1}{\sqrt{E}}\frac{\partial r}{\partial u} - 3\frac{\partial}{\partial v}\sqrt{E} - 2\frac{\partial^2}{\partial v^2}\sqrt{E}\cdot r \\ &\qquad = \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial \phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\sqrt{E}\frac{\partial \phi}{\partial v}\right) + 2\sqrt{E}\phi. \end{aligned}
$$

From (31) we see that the necessary and sufficient condition that the ruled surfaces $u = \text{const.}$, $v = \text{const.}$ of C_1 also be its mean ruled surfaces is

$$\sum X\frac{\partial x}{\partial v} + \frac{\partial r}{\partial v} = 0,$$

or by (16)

$$\frac{\partial}{\partial u}(r - \phi) = 0,$$

from which we have

$$(45)\qquad r = \phi + V,$$

where V is a function of v alone. Substituting this expression for r in (44) we have

$$
(46)\quad \begin{aligned} &\frac{\partial^2 \phi}{\partial v^2} + \frac{\partial \log E}{\partial v}\frac{\partial \phi}{\partial v} + \left(\frac{1}{\sqrt{E}}\frac{\partial^2}{\partial v^2}\sqrt{E} + 1\right)\phi \\ &\qquad = -\tfrac{1}{2}\left[V'' + 3\frac{\partial \log \sqrt{E}}{\partial v}V' + 2\frac{1}{\sqrt{E}}\frac{\partial^2 \sqrt{E}}{\partial v^2}V\right], \end{aligned}
$$

where the accents denote differentiation with respect to v. From (45) and (46) we see that, when V is given an arbitrary form, the determination of r requires the solution of an ordinary differential equation of the second order. The expressions for the coördinates of one of the limit surfaces follow from (16).

When C is a normal congruence the above equation reduces to

$$V'' + 3\frac{\partial \log \sqrt{E}}{\partial v}V' + 2\frac{1}{\sqrt{E}}\frac{\partial^2 \sqrt{E}}{\partial v^2}V = 0.$$

Hence E must be a function of v alone, and consequently

$$\sqrt{E} = a\cos v + b\sin v,$$

where a and b are constants. But this is a subcase of that characterized by the equation

$$U_1 = \lambda U_2,$$

previously discussed; hence the direction-cosines X, Y, Z can be found by quadratures. When we put the above expression for $\sqrt{E}$ and $\phi = 0$ in (44), it reduces to a form from which r can be found by two quadratures. Thus, by (45),

$$r = V = \frac{k_1(a \sin v - b \cos v) + k_2}{(a \cos v + b \sin v)^2}.$$

where k_1 and k_2 are constants. Hence the

THEOREM.—*The complete determination of normal congruences whose mean ruled surfaces correspond to the mean ruled surfaces of the corresponding conjugate congruence reduces to quadratures.*

It is readily found that condition (32) is not satisfied, and therefore

The corresponding conjugate congruence is not a normal one.

Lastly, we consider the determination of congruences for which one system of developables has for its spherical representation the orthogonal trajectories of a given family of great circles. In this case equation (22) reduces to such a form that for any value of ϕ the finding of r requires two quadratures.

In order that the ruled surfaces $u = \text{const.}$ of the corresponding conjugate congruence C_1 be developable we must have* $f_1 = 0$ and hence by (31) $f = 0$. Consequently $f = f'$, so that C is a normal congruence, which is entirely in accord with the preceding results.

In order that the ruled surfaces $v = \text{const.}$ of C_1 be developables we must have by (31) and (21)

$$4\frac{\partial r}{\partial u} - \frac{\partial E}{\partial v}\phi = 0.$$

Eliminating r between this equation and the equation to which (22) reduces in this case, we find that ϕ satisfies a linear equation of the third order in the derivatives of ϕ with respect to u. Hence the

THEOREM.—*When the cosines X, Y, Z for a given system of great circles $u = \text{const.}$ and their orthogonal trajectories $v = \text{const.}$ are found, the determination of all congruences having these trajectories for the representation of one family of developables reduces to quadratures. Only when C is a normal congruence can the ruled surfaces $u = \text{const.}$ of the conjugate be developables. And the determination of all congruences for which the ruled surfaces $v = \text{const.}$ of the conjugate are developables requires the integration of an ordinary differential equation of the third order.*

PRINCETON, *October*, 1901.

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* BIANCHI, l. c., p. 252.