CONSTRUCTIVE THEORY OF THE UNICURSAL CUBIC BY SYNTHETIC METHODS*

BY

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1. SCHROETER's† classic work on the general cubic leaves little to be desired in point of symmetry and generality. It is nevertheless interesting to build up the theory of the unicursal cubic, the curve being defined as the locus of the intersection of corresponding rays of two projective pencils, one of the first and the other of the second order. This has in fact been done by DRASCH.‡ The following discussion, based likewise on this definition, is materially simplified by the use of the properties of the point designated in § 7 by Σ. Incidentally the investigation brings to light a remarkable one-to-one correspondence between the points of the plane and the line elements on the cubic.

2. The locus described above has at least one and at most three points in common with any line in the plane. We assume the truth of this theorem, a proof of which may be found in the eleventh chapter of REYE's *Geometrie der Lage.*

Notations.—Throughout this paper we shall use the following notations: The pencil of the first order will be denoted by s, its center by S, and its rays by a, b, c, etc. The pencil of the second order (and also the conic enveloped by it) will be denoted by κ, and the rays by α, β, γ, etc. The cubic itself will be denoted by C.

3. THEOREM.—*No point of the cubic C lies within the conic κ.*

4. THEOREM.—*The cubic C touches the conic κ in at least one point, and at most in three.*

To prove this take S', a point on κ, for the center of a pencil s' of the first order perspective to κ. This pencil generates with s a conic which cuts κ in at least one and at most three other points besides S'. These are easily seen to be points on C.

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† SCHROETER, *Die Theorie der ebenen Kurven dritter Ordnung*, Teubner, Leipzig, 1888. See also BINDER, *Theorie der unicursalen Plancurven vierter bis dritter Ordnung*, Teubner, Leipzig, 1896.

‡ DRASCH, *Beitrag zur synthetischen Theorie der ebenen Curven dritter Ordnung mit Doppelpunkt*, Wiener Berichte, vol. 85 (1882), p. 534.

5. Theorem. *If S is outside of κ it is a double point of C. The rays of s which correspond to the tangents of κ through S are the principal tangents to C at S.*

The rays of κ through S meet their corresponding rays in that point. Every ray of s in this case meets C in three points; two at S, and one where it meets its corresponding ray of κ. These three points tend to coincide as the ray of κ approaches as a limiting position either tangent from κ through S.

6. Theorem.—*If S is on κ it is a cusp of C.*

7. Problem.—*Given S and κ with three pairs of corresponding rays, to construct C.*

Let the three given pairs be a, α; b, β; and c, γ. The intersection of a and α will be a point A on C. Draw through A the other tangent v to κ. Take this line perspective to κ. Draw through A any other line u. Take this perspective to s. The lines u and v are seen to be in perspective position. Their center of perspectivity may be found by joining any two pairs of corresponding points of u and v such as (v, β) with (u, b) and (v, γ) with (u, c). This center of perspectivity we shall denote by Σ. The importance of this point in the theory of the cubic depends upon the following theorem:

8. Theorem.—*The locus of Σ as u revolves about A is a straight line a' passing through S. The point row Σ is projective to the pencil u.*

For as u revolves, (u, b) and (u, c) project to the fixed points (v, β) and (v, γ) in two projective pencils which generate the locus of Σ. These pencils are perspective since they have the corresponding ray v in common. The locus of Σ is thus a straight line. It passes through S as is seen by giving u the direction AS. The rest of the theorem is sufficiently evident.

9. Theorem.—*The intersection of a' with v is a point A' on C.*

This is seen by drawing the ray of κ corresponding to a' (§ 8). It follows from this theorem that the line v meets the cubic in three real points. To find the third point A'' we must determine the ray of s corresponding to V. Let T be the point of tangency of V, and let (T, Σ) cut u in U. Then (U, S) meets v in A''.

10. Theorem.—*The tangents from Σ to κ meet u in points on C.*

This important theorem is seen to be true by drawing the rays of s corresponding to these tangents.

11. Problem.—*To find a line u through A corresponding to a given point Σ on a'.*

The required line passes through the intersection of c with the line joining Σ to (v, γ). It also passes through the intersection of b with the line joining Σ to (v, β). The solution of this problem furnishes a very expeditious method of constructing C by points. Construct for each point on a' the corresponding line u and get its intersections with C by the previous theorem. According as Σ lies within or without κ the line u will meet C in one or three points.

12. PROBLEM.—*To draw all the tangents from A to C.*

Take for Σ the points where a' meets κ. The line u by § 10 meets C in two coincident points. The points on u might also coalesce if Σ should fall upon u. In such a case Σ would lie on C, and would be either at S or A'. When Σ is at S the u line is AS which is not a true tangent. When Σ is at A', u coincides with v. One of the tangents from Σ to κ coincides with u and gives no definite intersection. The u line in this case is not a tangent to C, since A and A' are not in general coincident.

13. THEOREM.—*Two tangents or none may be drawn from A to C according as a' does or does not intersect κ.*

The case where a' is tangent to κ will be discussed later (§ 19).

14. THEOREM.—*If S lies outside of κ, two tangents or none may be drawn from any point on the cubic to the cubic. If S is inside κ, two such tangents may always be drawn. If S is on κ, one and only one such tangent may be drawn.*

This theorem follows from the fact (§ 8) that a' passes through S. In the third case one of the points where a' meets κ is at S, and does not give a true tangent (§ 12).

15. PROBLEM.—*To draw the tangent to C at A.*

Take for Σ the point where a' meets α. For in this case one of the tangents from Σ to κ meets u in A, so that u meets C in two coincident points at A.

16. THEOREM.—*Let the line u through $(a_1\alpha_1)$ meet C again in the points $(a_2\alpha_2)$ and $(a_3\alpha_3)$. Then as u revolves about $(a_1\alpha_1)$ the pencils a_2 and a_3 are in involution, as are also the pencils a_2' and a_3'.*

By the theorem of § 10, a_1' passes through (α_2, α_3). Also a_2' passes through (α_3, α_1) and a_3' through (α_1, α_2). As u revolves, (α_2, α_3) describes the fixed line a_1'. Therefore α_2 and α_3 are pairs of tangents to κ in involution. The corresponding rays a_2 and a_3 are therefore in involution. Further α_2 and α_3 cut out on α_1 two point rows in involution which project to S in the rays a_2' and a_3', which are therefore in involution.

Clearly if u is drawn tangent to C, the two rays a_2 and a_3 fall together and are thus a double ray in the involution. Again if C has a double point at S the principal tangents at S are also a pair of corresponding rays in the involution Now any pair of corresponding rays in an involution are harmonic conjugates with respect to the pair of double rays. From this follows easily the well-known theorem:

17. THEOREM.—*In a cubic with a double point S, if two rays be drawn through S harmonically conjugate to the principal tangents at S, these rays will meet the cubic in two points such that the tangents at those points will meet again on the cubic.*

18. THEOREM.—*The rays a and a' are corresponding rays of a pencil in involution.*

Using the notation of § 16 we see that α_2 cuts out on α_1 a point row projective to a_2 and perspective to a'_3, and therefore projective to a'_2 by § 16. The pencil a_2 is therefore projective to the pencil a'_2. We have also on the line v, A, A'; (v, b), (v, γ); (v, c), (v, β), three pairs of points in involution, being the section by v of the complete four point, S, Σ, (u, b), (u, c). By projecting to S we have a and a' corresponding to each other doubly.

19. PROBLEM.—*Given a line a' through S to find the corresponding point A on C.*

By the preceding theorem we derive the line a from a' in the same way in which we derive a' from a (§ 7). In the case of a cubic with a double point we may ask what is the line a' corresponding to one of the principal tangents at S. It is not difficult to see that the corresponding line is one of the tangents from S to κ. Referring to § 13 we have

20. THEOREM.—*A cubic is divided by its double point into two portions such that from a point in one portion two tangents may be drawn to the cubic, while from a point in the other portion no such tangent may be drawn.*

21. THEOREM.—*To every point in the plane there corresponds a point on C, with a direction through that point and conversely.*

Let the point be Σ. Join it to S. Call this line a' and by § 19 get the corresponding point A. To the point Σ on a' corresponds a definite line u, through A. An apparent exception exists for two points Σ which lie one on one tangent from S to κ and one on the other. The corresponding point on C is S for either point. The point S for one point Σ is to be considered as belonging to one branch of the curve; and for the other point Σ as belonging to the other branch.

22. THEOREM.—*The locus of points Σ which give tangents to C from the correspondiug points on C, is the conic κ.*

This follows from § 12.

23. THEOREM.—*The locus of points Σ which give tangents to C at the corresponding points, is a point row C' of the third order, having the same base conic κ and the same center S for its pencil of the first order.*

For Σ thus defined is the locus of the intersection of a' with α by § 15: But by the discussion of § 18 it is seen that these are projective pencils, one of the first and one of the second order.

24. THEOREM.—*There are at most three points of inflection on C, and at least one.*

For the cubic C' of the preceding paragraph touches κ in at least one point and at most in three. These points correspond to inflection points on C, because one tangent *from* the point on C, coincides with the tangent *at* the point.

25. THEOREM.—*If S is outside of κ, the "tangential cubic" C', defined above, has for its principal tangents the tangents from S to κ.*

To find the principal tangents T_1 and T_2 to the cubic C, we found the rays of s corresponding to the rays τ_1 and τ_2 of κ which pass through S (§ 5). We must then find for the cubic C' the Σ lines corresponding to T_1 and T_2, which by § 19 are t_1, and t_2, the rays of s tangent to κ.

26. This last theorem, which was noted by one of my students, MR. B. O. LACEY, may be made the basis of a proof that the cubic with real double point has only one real inflection. Other theorems may be obtained without difficulty, but the power of the method has already been sufficiently indicated.

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