1. Denote by $G$ the group of the equation upon which depends the determination of the curves of order $n - 3$ having simple contact at $\frac{1}{2}n(n - 3)$ points with a given curve $C_n$ of order $n$ having no double points. The case in which $n$ is odd was discussed in the former paper (Transactions, January, 1902) and $G$ was shown to be a subgroup of the group defined by the invariants $\phi_3, \phi_4, \phi_5, \ldots$, the latter group being holoedrically isomorphic with the first hypoabelian group on $2p$ indices with coefficients taken modulo 2. For $n$ even, $G$ is contained in the group $H$ defined by the invariants $\phi_4, \phi_6, \ldots$, with even subscripts. JORDAN has shown (Traité, pp. 229–242) that $H$ is holoedrically isomorphic with the abelian linear group $A$ on $2p$ indices with coefficients taken modulo 2. The object of the present paper is to establish the latter theorem by a short, elementary proof, which makes no use of the abstract substitutions $[\alpha_i, \beta_i, \ldots, \alpha_p, \beta_p]$ of JORDAN, and which exhibits explicitly the correspondence† between the substitutions of the isomorphic groups.

2. We first define a non-homogeneous linear group $A_1$ on $2p$ indices which leaves the function $x_1y_1 + \cdots + x_py_p$ invariant modulo 2 and which is holoedrically isomorphic with the abelian group $A$. To the generators $M_i, L_i, N_{ij}$ of $A$ we make correspond the respective substitutions of $A_1$:

$$
\mu_i: \quad x'_i = y_i, \quad y'_i = x_i;
$$

$$
\lambda_i: \quad x'_i = x_i + y_i + 1;
$$

$$
\nu_{ij}: \quad x'_i = x_i + y_j, \quad x'_j = x_j + y_i.
$$

Then to the general substitution of $A$,

$$
S: \quad x'_i = \sum_{i=1}^p (\alpha_i x_j + \gamma_i y_j), \quad y'_i = \sum_{j=1}^p (\beta_j x_j + \delta_j y_j) \quad (i = 1, \ldots, p),
$$

will correspond the following substitution of $A_1$:

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† It is shown in § 6 that this correspondence is in accord with that given by JORDAN.
\[ x'_i = \sum_{j=1}^{p} (\alpha_j x_j + \gamma_j y_j) + \sum_{j=1}^{p} \alpha_j \gamma_j, \quad \sigma: \]
\[ y'_i = \sum_{j=1}^{p} (\beta_j x_j + \delta_j y_j) + \sum_{j=1}^{p} \beta_j \delta_j \quad (i = 1, \ldots, p). \]

In fact, the general correspondence \( S \sim \sigma \) includes the assumed correspondences

\[ M_i \sim \mu_i, \quad L_i \sim \lambda_i, \quad N_i \sim \nu_i \quad (i, j = 1, \ldots, p). \]

Moreover, if \( S_i \sim \sigma_i \), it is readily verified that

\[ M_i S_i \sim \mu_i \sigma_i, \quad L_i S_i \sim \lambda_i \sigma_i, \quad N_i S_i \sim \nu_i \sigma_i \quad (i, j = 1, \ldots, p). \]

Since the generators \( \mu_i, \lambda_i, \nu_i \) leave invariant the function \( x_1 y_1 + \cdots + x_p y_p \), the general substitution \( \sigma \) of the group \( A_1 \) will leave it invariant.

3. Theorem.*—The group \( A_1 \) may be represented as a doubly transitive substitution group on the \( R_p \equiv 2^{2p-1} - 2^{p-1} \) letters \((x_1, y_1, x_2, y_2, \ldots, x_p, y_p)\) in which \( x_1, y_1, \ldots, x_p, y_p \) assume every system of solutions, not all zero, of the congruence

\[ x_1 y_1 + x_2 y_2 + \cdots + x_p y_p \equiv 1 \pmod{2}. \]

That \( A_1 \) is transitive on the \( R_p \) letters may be shown by the usual methods of linear group theory, or directly by the following remark. Let \((\alpha_1, \gamma_1, \ldots, \alpha_p, \gamma_p)\) be an arbitrary one of the letters. Then \( \alpha_1 \gamma_1 + \cdots + \alpha_p \gamma_p \equiv 1 \pmod{2} \). One substitution which belongs to \( A_1 \) and which replaces \((11, 00, \ldots, 00)\) by \((\alpha_1, \gamma_1, \ldots, \alpha_p, \gamma_p)\) is the following:

\[
\begin{align*}
x'_i &= (\alpha_1 \gamma_1 + \alpha_i + \gamma_i) x_i + (\alpha_1 + 1) y_i + \sum_{i=2}^{p} ((\alpha_1 + 1) \gamma_i x_i + (\alpha_1 + 1) \alpha_i y_i) \\
&\quad + (\alpha_1 + 1)(\gamma_1 + 1), \\
y'_i &= (\gamma_1 + 1) x_i + (\alpha_1 \gamma_1 + \alpha_i + \gamma_i) y_i + \sum_{i=2}^{p} ((\gamma_1 + 1) \gamma_i x_i + (\gamma_1 + 1) \alpha_i y_i) \\
&\quad + (\alpha_1 + 1)(\gamma_1 + 1), \\
x'_j &= \alpha_j (\gamma_1 + 1) x_i + \alpha_j (\alpha_1 + 1) y_i + (\alpha_j \gamma_j + 1) x_j + \alpha_j y_j \\
&\quad + \sum \left( \alpha_j \gamma_i x_i + \alpha_j \alpha_i y_i \right) + \alpha_j (\alpha_1 + \gamma_1 + 1), \\
y'_j &= \gamma_j (\gamma_1 + 1) x_i + \gamma_j (\alpha_1 + 1) y_i + \gamma_j x_j + (\alpha_j \gamma_j + 1) y_j \\
&\quad + \sum \left( \gamma_j \gamma_i x_i + \gamma_j \alpha_i y_i \right) + \gamma_j (\alpha_1 + \gamma_1 + 1),
\end{align*}
\]

where \( \sum \) denotes the summation \( i = 2, \ldots, p; \ i \neq j \).

*For other applications one might employ the theorem that the group \( A_1 \) permutes transitively the \( 2^p \) functions \( a_1 x_1 + b_1 y_1 + \cdots + a_p x_p + b_p y_p + a_1 b_1 + \cdots + a_p b_p \).

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To prove that the group is doubly transitive, it now suffices to show that the subgroup leaving the letter \(11\ 00\ \ldots\ 00\) fixed is transitive on the remaining letters. The conditions that the general substitution \(\sigma\) of \(A_i\) shall leave fixed the letter \((11\ 00\ \ldots\ 00)\) are*

\[
\alpha_{il} + \gamma_{il} + \sum_{j=1}^{p} \alpha_j \gamma_{lj} \equiv \epsilon_{il}, \quad \beta_{il} + \delta_{il} + \sum_{j=1}^{p} \beta_j \delta_{lj} \equiv \epsilon_{il} \quad (i = 1, \ldots, p).
\]

With these conditions satisfied, \(S\) belongs \(\dagger\) to the second hypoabelian group (with \(x_i\) and \(y_i\) playing the special rôle). Employing these conditions, we may give to \(\sigma\) the form:

\[
x_i' = \sum_{j=1}^{p} (\alpha_j x_j + \gamma_j y_j) + \alpha_{il} + \gamma_{il} + \epsilon_{il},
\]

\[
y_i' = \sum_{j=1}^{p} (\beta_j x_j + \delta_j y_j) + \beta_{il} + \delta_{il} + \epsilon_{il}.
\]

It replaces \((11\ 10\ \ldots\ 00)\) by \((a_1 c_1\ a_2 c_2\ \ldots\ a_p c_p)\), where

\[
a_1 c_1 + \ldots + a_p c_p \equiv 1 \pmod{2},
\]

if

\[
\alpha_{il} + \epsilon_{il} = a_i, \quad \beta_{il} + \epsilon_{il} = c_i \quad (i = 1, \ldots, p).
\]

To show that the second hypoabelian group contains a substitution \(S\) whose coefficients satisfy the conditions (2), we note that the inverse \(S^{-1}\) is obtained by replacing \(\alpha_j, \beta_j, \gamma_j, \delta_j, \beta_{ji}, \gamma_{ji}, \alpha_{ki}\), respectively, so that the conditions (2) give the following conditions on \(S^{-1}\):

\[
\delta_{2i} \equiv a_i + \epsilon_{il}, \quad \beta_{2i} \equiv c_i + \epsilon_{il} \quad (i = 1, \ldots, p) \pmod{2}.
\]

Hence the coefficients of \(y_2'\) in \(S^{-1}\) are fully determined. Also

\[
\beta_{21} + \delta_{2i} + \sum_{i=1}^{p} \beta_{2i} \delta_{2i} \equiv a_1 + c_1 + (a_1 + 1)(c_1 + 1) + \sum_{i=2}^{p} a_i c_i
\]

\[
\equiv \sum_{i=1}^{p} a_i c_i + 1 \equiv 0 \pmod{2}.
\]

But \(\ddagger\) the second hypoabelian group contains such a substitution \(S^{-1}\).

4. **Theorem.—** The groups \(H\) and \(A_i\) are identical.

It is first shown that every substitution of \(A_i\) belongs to \(H\). By § 4 of the former paper, \(\mu\) and \(\nu\) (which have the same form as \(M_i\) and \(N_{\nu}\), respectively)

* Henceforth \(\epsilon_j\) denotes 1 if \(i = j\), but denotes 0 if \(i \neq j\).


leave the functions \( \phi_3, \phi_4, \phi_5, \ldots \) invariant. Next, \( \lambda_1 \) replaces the general term of \( \phi_4 \) by
\[
(x'_1 + y'_1 + 1 \ y'_1 \ldots)(x''_1 + y''_1 + 1 \ y''_1 \ldots)(x'''_1 + y'''_1 + 1 \ y'''_1 \ldots)
\]
\[
(x'_1 + x''_1 + x'''_1 + y'_1 + y''_1 + y'''_1 + 1 \ y'_1 + y''_1 + y'''_1 \ldots),
\]
which is seen to be a term of \( \phi_4 \). In like manner, it may be shown that \( \lambda_1 \)
leaves invariant \( \phi_6, \phi_8, \ldots \); but alters \( \phi_3, \phi_5, \ldots \).

It is next shown that every substitution of \( H \) belongs to \( A_1 \). Let \( L \) be an
arbitrary substitution of \( H \) and let it replace the letters
\[
\begin{align*}
\omega & = (001100 \ldots 00), \\
\omega_2 & = (101100 \ldots 00)
\end{align*}
\]
by certain letters \( l'_1, l'_2 \), respectively. By \S \, 3, \( A_1 \) contains a substitution \( L' \)
which replaces \( l_1 \) by \( l'_1 \) and \( l_2 \) by \( l'_2 \). Hence \( M = L'^{-1} L \) will belong to \( H \)
and will leave fixed the letters \( l'_1, l'_2 \). Since \( M \) does not alter \( \phi_4 \), it will leave
invariant the sum \( \psi \) of those terms of \( \phi_4 \) which contain the factor \( l_1 l_2 \). The
general term of \( \psi \) is therefore
\[
l_1 l_2 (x_1 y_1 x_2 y_2 x_3 y_3 \ldots) (x_1 + 1 \ y_1 x_2 y_2 x_3 y_3 \ldots).
\]
In view of (1), the last two expressions denote letters if, and only if,
\[
\sum_{i=1}^{p} x_i y_i \equiv 1, \quad y_1 \equiv 0 \pmod{2}.
\]
But the letters \( l_1 \) and \( l_2 \) satisfy these congruences. Hence \( \psi \) involves exactly
\( 2R_{p-1} \) letters. Hence \( M \) must permute amongst themselves the remaining
\( R_p - 2R_{p-1} = 2^{p-2} \) letters, the general one of which is
\[
(x_1 1 x_2 y_2 x_3 y_3 \ldots), \quad x_1 + \sum_{i=2}^{p} x_i y_i \equiv 1 \pmod{2}.
\]

The substitutions of \( A_1 \) which leave unaltered the letters \( l_1 \) and \( l_2 \) permute
transitively the \( 2^{p-2} \) letters (3).

Indeed, by \S \, 3, the substitutions of \( A_1 \) which leave \( l_1 \) fixed have the form
\[
x'_i = \sum_{j=1}^{p} (\alpha_{ij} x_j + \gamma_{ij} y_j) + \alpha_{i2} + \gamma_{i2} + \epsilon_{i2}, \quad (i=1, \ldots, p).
\]
\[
y'_i = \sum_{j=1}^{p} (\beta_{ij} x_j + \delta_{ij} y_j) + \beta_{i2} + \delta_{i2} + \epsilon_{i2}
\]
The latter leaves \( l_2 \) fixed if, and only if,
\[
\begin{align*}
\alpha_{11} &= 1, & \alpha_{21} &= 0, & \beta_{11} &= 0, & \beta_{21} &= 0, & \alpha_{i1} &= \beta_{i1} = 0 \quad (i=3, \ldots, p).
\end{align*}
\]
Let $\sigma_1$ denote the general substitution so defined and let $S_1$ denote the corresponding homogeneous substitution. Let $(c_1, c_2, d_2, c_3, d_3, \ldots)$ be an arbitrary letter of the form (3). The conditions that $\sigma_1$ shall replace $(01\ 11\ 00\ \ldots\ 00)$ by $(c_1, c_2, d_2, \ldots)$ are

\[(4) \quad c_1 = \gamma_{11}, \quad 1 = \delta_{11}, \quad c_2 = \gamma_{21} + 1, \quad d_2 = \delta_{21} + 1, \quad c_i = \gamma_{ii}, \quad d_i = \delta_{ii} \quad (i = 3, \ldots, p).\]

To prove that there exists a substitution $\sigma_1$ satisfying the conditions (4) we follow the method used at the end of § 3. We observe that $S_1^{-1}$ is the most general substitution of the second hypoabelian group (with $x_2, y_2$ playing the special rôle) which leaves the index $y_1$ unaltered. The conditions (4) give the following conditions modulo 2 on $S_1^{-1}$

\[\alpha_{11} = 1, \quad \gamma_{11} = c_1, \quad \alpha_{12} = d_2 + 1, \quad \gamma_{12} = c_2 + 1, \quad \alpha_{1i} = \gamma_{ii}, \quad \gamma_{1i} = c_i \quad (i = 3, \ldots, p).\]

Hence the coefficients of $x_1'$ in $S_1^{-1}$ are fully determined. Also, by (3),

\[\alpha_{12} + \gamma_{12} + \sum_{i=2}^{p} \alpha_{1i} \gamma_{1i} = 1 + c_1 + \sum_{i=2}^{p} c_i d_i \equiv 0 \pmod{2}.\]

But the second hypoabelian group contains a substitution of the form

\[y_1' = y_1, \quad x_1' = \sum_{i=1}^{p} (\alpha_{1i} x_i + \gamma_{1i} y_i), \ldots, \quad (\alpha_{12} + \gamma_{12} + \sum_{i=1}^{p} \alpha_{1i} \gamma_{1i} = 0, \quad \alpha_{11} = 1).\]

Next, let $M$ replace $l_3 = (01\ 11\ 00\ \ldots\ 00)$ by a letter $l_3'$ of the form (3). By the preceding result, $A_1$ contains a substitution $T$ which replaces $l_3$ by $l_3'$. Hence $M = T Q$, where $Q$ is a substitution of $H$ which leaves fixed the letters $l_1, l_2, l_3$. By § 9 of the former paper, $Q$ permutes amongst themselves the $R_{p-1}$ letters $(00\ x_2 y_2, x_3 y_3, \ldots)$. The theorem may now be established by induction from $p - 1$ to $p$. We proceed as in § 10 of the earlier paper, * deleting the functions $\phi_4$ and $\phi_4^{(p-1)}$. As a basis for the induction, we show that the theorem is true for $p = 2$, whence $R_2 = 6$. The six letters

\[(00\ 11), \quad (10\ 11), \quad (01\ 11), \quad (11\ 00), \quad (11\ 01), \quad (1110),\]

cannot be combined to give a term of $\phi_4$, so that the latter does not exist when $p = 2$. Evidently $\phi_4$ is the product of the six letters. Hence $H$ is the symmetric group on six letters. But the order of the quaternary abelian group modulo 2 is $2^4 - 1 \cdot 2^3 (2^2 - 1) \equiv 6$. Hence the groups $H$ and $A_1$ are identical when $p = 2$.

*One part of the proof by induction was there omitted, viz., the proof for the case $p = 2$, whence $R_2 = 6$. That $G_1$ and $G$ are identical follows from the equality of their orders (see § 11), or more simply since $Q$ is, for $p = 2$, either the identity or else is $M_2$, permuting $(11\ 01)$ with $(1110)$, and hence is hypoabelian.

† For a direct proof of the holoeidric isomorphism of the symmetric group on 6 letters and the quaternary abelian group modulo 2, see Linear Groups, p. 99.
5. It follows that the order $w_p$ of $H$ satisfies the recursion formula

$$w_p = R_p(R_p - 1)2^{2p-2} \frac{w_{p-1}}{R_{p-1}} \equiv (2^{2p} - 1)2^{2p-1}w_{p-1}.$$ 

Since $w_2 = (2^1 - 1)2^3(2^2 - 1)2$, we derive the result,

$$w_p = (2^{2p} - 1)2^{2p-1}(2^{2p-2} - 1)2^{2p-3} \cdots (2^2 - 1)2.$$ 

6. To show that the above correspondence of operators of the isomorphic groups $H$ and $A$ is in accord with that obtained by Jordan, we note that, in view of p. 241 of Traité des substitutions,

\[ [11 \ 00 \ \cdots \ 00] \sim M_1, \quad [10 \ 00 \ \cdots \ 00] \sim L_1, \quad [10 \ 10 \ 00 \ \cdots \ 00] \sim L_2 L_1 N_{12}. \]

Also (Traité, p. 230), \([11 \ 00 \ \cdots]\) leaves \((x_1 y_1 x_2 y_2 \cdots)\) fixed if \(x_1 + y_1 \equiv 0 \pmod{2}\), but replaces it by \((x_1 + 1 y_1 + 1 x_2 y_2 \cdots)\) if \(x_1 + y_1 \equiv 1\), and hence may be designated

$$\mu_1 : \quad x'_1 = y_1, \quad y'_1 = x_1.$$ 

Likewise, \([10 \ 00 \ \cdots]\) leaves \((x_1 y_1 x_2 y_2 \cdots)\) fixed if \(y_1 \equiv 1\), but replaces it by \((x_1 + 1 y_1 x_2 y_2 \cdots)\) if \(y_1 \equiv 0\), and hence may be designated

$$\lambda_1 : \quad x'_1 = x_1 + y_1 + 1.$$ 

Next, \([10 \ 10 \ 00 \ \cdots]\) leaves \((x_1 y_1 x_2 y_2 \cdots)\) fixed if \(y_1 + y_2 \equiv 1\), but replaces it by \((x_1 + 1 y_1 x_2 + 1 y_2 x_3 y_3 \cdots)\) if \(y_1 + y_2 \equiv 0\), and hence may be designated

$$\lambda_2 \lambda_1 \nu_{12} : \quad x'_1 = x_1 + y_1 + y_2 + 1, \quad x'_2 = x_2 + y_1 + y_2 + 1.$$ 

It follows that \(N_{12} \sim \nu_{12}\). In view of the symmetry, \(N_{ij} \sim \nu_{ij}\), etc.

The University of Chicago, January 10, 1902.