ON SUPEROSCULATING QUADRIC SURFACES*

BY

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In trying to determine a surface of the second order $F(x, y, z) = 0$ having with a given surface Φ, whose coördinates are given as functions of u and v, a contact of the third order, ten equations are obtained, viz., F and all the first, second and third derivatives of F with respect to u and $v = 0$.

It seems as if after the elimination of the nine constants of F one equation for u, v would result, i. e., one curve on Φ would exist in every point of which a contact of the above description takes place. HERMITE, however, has shown that this is not so. † He found that in general *two* equations result which are free from the coefficients of F, determining a finite number of points P on Φ, and that for every such point P there exists not only one surface F but a whole pencil having a contact of the third order with Φ. ‡

Instead of actually deducing the resulting two conditions by elimination from the above named ten equations we can obtain them in the following much simpler way, which at the same time affords a better insight into the nature of the points in question.

Let us consider the two straight lines a_1 and a_2 of F through a contact point P. Each of them has with a normal section of Φ through P a contact of the third order. These lines a_1 and a_2 are the inflexional tangents of Φ at P; they have then *four* points with the surface Φ at P in common. § Thinking of the inflexional tangents as osculating circles of the two respective normal sections of Φ, they appear now as those particular osculating circles having with their normal sections a contact of the *third* order. Of normal sections of this kind there exist in general three through every point of a surface. They are defined by the following equation : ||

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† HERMITE, *Cours d'Analyse* (1873), pp. 148, 149.

‡ A general investigation of this interesting fact also for superosculating surfaces of orders higher than 2, has been given by HALPHEN : *Sur le contact des surfaces*, Bulletin de la Société Mathématique de France, III, pp. 28–37.

§ An algebraic discussion of these points on algebraic surfaces has been made by CLEBSCH, Journal für reine und angewandte Mathematik, vol. 63 and SCHUBERT, Mathematische Annalen, vol. 11, p. 347.

|| KNOBLAUCH, *Einleitung in die allgemeine Theorie der krummen Flächen*, p. 94 (8).

$$(1) \qquad Pdu^3 + 3Qdu^2dv + 3Rdu\,dv^2 + Sdv^3 = 0.$$

It now follows that the $dv : du$ belonging to the inflexional tangents must satisfy the above equation, i. e., since the inflexional tangents are defined by the equation

$$(2) \qquad Ldu^2 + 2Mdu\,dv + Ndv^2 = 0,$$

the left side of (2) must be a factor of the left side of (1). The condition that a quadratic form be a factor of a cubic form is given by the identical vanishing of a certain cubic covariant. * This yields for our case the following four equations:

$$(3) \qquad \begin{cases} (4M^2 - LN)P - 6LMQ + 3L^2R = 0, \\ L^2S - 3LNQ + 2MNP = 0, \\ N^2P - 3LNR + 2LMS = 0, \\ (4M^2 - LN)S - 6MNR + 3N^2Q = 0. \end{cases}$$

These equations are in every given case equivalent to *two* only, since two of them can be deduced from the two others (in order to cover however all possible cases all four equations are necessary). *These are then the required two relations* defining in general on Φ a certain finite number of points P.

The case that the two straight lines a_1 and a_2 of F through P coincide, from which immediately also the coincidence of the two inflexional tangents of Φ follows, does not require any modification of the equations (3). In this case we have however

$$(4) \qquad LN - M^2 = 0,$$

and by means of this relation the equations (3) can be transformed into

$$(5) \qquad \begin{cases} NP - 2MQ + LR = 0, \\ NQ - 2MR + LS = 0. \end{cases}$$

The left sides of these equations are equal to

$$(EG - F^2)\frac{\partial K}{\partial u} \qquad \text{and} \qquad (EG - F^2)\frac{\partial K}{\partial v}$$

respectively, † where K denotes the Gaussian curvature. We have then for our case (4):

$$(6) \qquad \frac{\partial K}{\partial u} = 0, \qquad \frac{\partial K}{\partial v} = 0.$$

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* CLEBSCH, *Theorie der binären algebraischen Formen* (1872), p. 94.

† KNOBLAUCH, *loc. cit.*, p. 97 (5).

One might ask what surfaces Φ are such that in every point a contact of the third order with a F_2 is possible. Then the equations (3) must hold for every point of the surface.

Let us, in order to answer this question, in the first place suppose the surface Φ to be a developable surface. Then (4) holds, i. e., $K = 0$. But since now $K = 0$ for every point of the surface we have

$$\frac{\partial K}{\partial u} = 0, \qquad \frac{\partial K}{\partial v} = 0,$$

i. e., (6) from which conversely the equations (3) follow.

In the second place, when Φ is not a developable surface we can take the asymptotic curves on it as parameter lines. We assume then

$$L = 0, \qquad N = 0, \qquad M \neq 0.$$

The equations (3) are then reduced to

$$P = 0, \qquad S = 0,$$

from which immediately

(7) $$J_2 = 0, \qquad J_1'' = 0$$

follows.* Now in general the geodesic curvature of the parameter-lines on a surface is given by the formulas †

$$g_u = \sqrt{\frac{EG - F^2}{G^3}} \cdot J_1'', \qquad g_v = \sqrt{\frac{EG - F^2}{E^3}} \cdot J_2.$$

We see then from (7) that the asymptotic curves on Φ have everywhere the geodesic curvature zero. But for asymptotic curves the geodesic curvature is identical with the ordinary (first) curvature. The asymptotic curves are then straight lines and the surfaces Φ are surfaces of the second order.

The developable surfaces and the surfaces of the second order are then the only ones for every point of which a surface of the second order exists having a contact of the third order.

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* KNOBLAUCH, *loc. cit.*, p. 96 (14).

† KNOBLAUCH, *loc. cit.*, p. 248 (15).