ORTHOCENTRIC PROPERTIES OF THE PLANE n-LINE*

BY

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In continuation of a memoir in these Transactions (vol. 1, p. 97) I consider the problem :

To find for n lines of a plane natural metrical analogues of the elementary facts that the perpendiculars of 3 lines meet at a point (the orthocenter of the 3-line) and that the orthocenters of the 3-lines contained in a 4-line lie on a line.

I apply first to the special case of a 4-line the treatment sketched in § 7 of the memoir cited; this affords suggestions for the general case.

§ 1. *The Deltoid.*

To discuss with the minimum of trouble the metrical theory of a 4-line we should take, according to our purpose, the lines as tangents either of a parabola or of a hypocycloid of class three. We want here the latter curve. As it is to most geometers an incidental stationary thing and not a weapon, I will treat it *ab initio*. And as it is at least as good as other curves which have a given name I will call it a deltoid. It is hardly necessary to remark that it is the metrically normal form of the general rational plane curve of class three with isolated double line.

Denote by t a turn or a complex number of absolute value 1; and think of t as a point on the unit circle. We consider three points t_1, t_2, t_3 on the circle subject to the condition

(1) $$s_3 \equiv t_1 t_2 t_3 = 1.$$

With this triad we associate a point x by the equation

(2) $$x = t_1 + t_2 + t_3 \equiv s_1,$$

which carries with it the conjugate equation

$$y = 1/t_1 + 1/t_2 + 1/t_3 = t_2 t_3 + t_3 t_1 + t_1 t_2 \equiv s_2.$$

We have from (1) and (2)

(3) $$x = t_1 + t_2 + 1/t_1 t_2.$$

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Herein let t_1 be fixed, t_2 variable. Since $x = t + 1/t$ is that simplest of all hypocycloids, the segment of a line, so also is (3) the segment of a line, of constant length 4, with center t_1 and ends where

$$D_{t_2} x = 0,$$

or

$$t_1 t_2^2 = 1,$$

or

$$x = t_1 \pm 2/\sqrt{t_1}.$$

When t_1 varies, the deltoid is described by the motion of this variable segment; the ends move on the curve, and the segment touches the curve. And any point inside the curve is given by (3) as the intersection of 2 segments; the points of the curve itself are given when the segments are brought to coincidence, that is, are

(4) $$x = 2t + 1/t^2.$$

This equation expresses that while a point $2t$ is describing a circle of radius 2 another point x moves round it with an angular velocity opposite in sense and twice as great; thus the cycloidal nature of the curve is apparent.

A point of the curve and also a line of the curve is named by its parameter t; thus the point t *of the curve* is given by (4), and the line t from (1) and (2) by

(5) $$xt^2 - yt = t^3 - 1.$$

x and y being always conjugate coördinates.

Now two lines t_1 and t_2 meet at

$$x_1 = t_1 + t_2 + 1/t_1 t_2.$$

Symmetrize this equation for 3 lines of the curve by writing it

(6) $$x = s_1 - t_3 + t_3/s_3.$$

Omitting the suffix of t_3 we have the map-equation of the circumcircle. Hence: The circumcenter of the 3-line is s_1, and the circumradius is $|1 - 1/s_3|$ or $|1 - s_3|$.

The mean point of

$$t_1 + t_2 + 1/t_1 t_2 \qquad \text{and} \qquad t_1 + t_3 + 1/t_1 t_3$$

is

$$m = \tfrac{1}{2}[s_1 + t_1 + (s_1 - t_1)/s_3].$$

Hence the center of the circle which bisects the sides, the Feuerbach or nine-point circle, is

(7) $$c = \tfrac{1}{2}(s_1 + s_1/s_3).$$

The perpendicular from $t_1 + t_2 + 1/t_1 t_2$ on the line t_3 is

$$xt_3 + y = t_3(t_1 + t_2 + 1/t_1 t_2) + 1/t_1 + 1/t_2 + t_1 t_2 = s_2 + t_3 s_1/s_3.$$

Hence the orthocenter is

$$p = s_1/s_3,$$

as could of course be inferred from the fact that

$$\text{orthocenter} + \text{circumcenter} = 2 \times \text{Feuerbach center}.$$

Since then $|p| = |s_1|$ we have:

THEOREM 1. *The center of a deltoid, of which three lines are given, is equidistant from the circumcenter and orthocenter of the* 3*-line.*

The centre of the deltoid which touches 4 lines is thus determined as the intersection of 4 lines, one for each of 3 of the given 4.

The Feuerbach center for 3 of 4 lines is given by

$$2c = (s_1 - t_4)(1 + t_4/s_4), \qquad (s \text{ for } 4),$$

s_1 and s_3 being replaced in (7) by symmetric functions of t_1, t_2, t_3, t_4. Thus the 4 such centers are included in

$$2x = (s_1 - t)(1 + t/s_4),$$

whose conjugate is

$$2y = (s_3/s_4 - 1/t)(1 + s_4/t) = (s_3/t - s_4/t^2)(1 + t/s_4).$$

But

$$t^2 - s_1 t + s_2 - s_3/t + s_4/t^2 = 0.$$

Therefore

$$2(tx + y) = s_2(1 + t/s_4),$$

a line through $\frac{1}{2}s_2/s_4$, perpendicular to the line t of the curve. Hence

THEOREM 2. *If from the Feuerbach center of any* 3 *of* 4 *lines a perpendicular be drawn to the remaining line, the* 4 *perpendiculars meet at a point,* namely $\frac{1}{2}s_2/s_4$.

§ 2. *Extension to the n-line.*

I employ now the notation of § 2 of the memoir cited, namely I write a line l_a in the form

$$xt_a + y = x_a t_a = y_a.$$

Denote the characteristic constants by a_a,

$$a_a = \sum \frac{x_1 t_1^{n-a}}{(t_1 - t_2)\cdots(t_1 - t_n)}, \qquad (a = 1, 2, \cdots, n)$$

and their conjugates by b_a, so that

$$b_a = (-)^{n-1} s_n a_{n+1-a}$$

where $s_n = t_1 t_2 \cdots t_n$, and in general $s_a = \Sigma\, t_1 t_2 \cdots t_a$ for n t's.

The circumcenter of a 3-line is a_1. The mean point of the joins of l_1, l_2 and l_1, l_3 is given by

$$2m = \frac{x_1 t_1}{t_1 - t_2} + \frac{x_2 t_2}{t_2 - t_1} + \frac{x_1 t_1}{t_1 - t_3} + \frac{x_3 t_3}{t_3 - t_1}$$

$$= \frac{x_1 t_1 (2t_1 - s_1 + t_1)}{(t_1 - t_2)(t_1 - t_3)} + \frac{x_2 t_2 (2t_2 - s_1 + t_1)}{(t_2 - t_1)(t_2 - t_3)} + \frac{x_3 t_3 (2t_3 - s_1 + t_1)}{(t_3 - t_1)(t_3 - t_2)},$$

and therefore the nine-point circle is

(8) $$2m = 2a_1 - s_1 a_2 + a_2 t,$$

and its center is

$$2c = 2a_1 - s_1 a_2.$$

Hence the orthocenter is

(9) $$p = 2c - a_1 = a_1 - s_1 a_2.$$

The line about the points a_1 and p (which is the locus of centers of inscribed deltoids) is, when τ is any turn,

(10) $$x - a_1 + s_1 a_2 = (x - a_1)\,\tau.$$

This for 3 of 4 lines is

$$x - a_1 + t a_2 + (s_1 - t)(a_2 - t a_3) = (x - a_1 + t a_2)\tau,$$

or $$x - a_1 + s_1 a_2 - t(s_1 - t) a_3 = [x - a_1 + s_1 a_2 - (s_1 - t) a_2]\,\tau.$$

Now since for 4 lines a_3/a_2 is a turn we can equate $t a_3$ to $a_2 \tau$; that is, whatever turn t may be, the line passes through the point

(11) $$p_1 = a_1 - s_1 a_2.$$

This is then the center of the inscribed deltoid of the 4-line.

For 4 of 5 lines this point is given by

$$p = a_1 - t a_2 - (s_1 - t)(a_2 - t a_3) = a_1 - s_1 a_2 + t(s_1 - t) a_3,$$

or the conjugate equation

$$q = b_1 - \frac{s_4}{s_5} b_2 - \frac{1}{t}\left(\frac{s_4}{s_5} - \frac{1}{t}\right) b_3 = s_5 a_5 - s_4 a_4 + \left(\frac{s_4}{s_5} - \frac{s_5}{t^2}\right) a_3.$$

But for 5 things

$$t^3 - t^2 s_1 + t s_2 - t s_3 + s_4/t - s_5/t^2 = 0.$$

Therefore

(12) $$t(p - a_1 + s_1 a_2) - (q - s_5 a_5 + s_4 a_4) = (ts_2 - s_3) a_3,$$

that is, the line joining the p of 4 lines to the point $a_1 - s_1 a_2 + s_2 a_3$ is perpendicular to the remaining line; or

THEOREM 3. *If from the center of the inscribed deltoid of* 4 *of* 5 *lines a perpendicular be drawn to the line left out, the* 5 *perpendiculars meet at a point;* namely the point

$$p_2 \equiv a_1 - s_1 a_2 + s_2 a_3.$$

Call this point the *first orthocenter* of the 5-line.

For 5 of 6 lines this point is

$$\begin{aligned} x &= a_1 - ta_2 - (s_1 - t)(a_2 - ta_3) + (s_2 - ts_1 + t^2)(a_3 - ta_4) \\ &= a_1 - s_1 a_2 + s_2 a_3 - t(s_2 - ts_1 + t^2) a_4 \\ &= p_2 - t(s_2 - ts_1 + t^2) a_4, \end{aligned}$$

or, if the conjugates be written,

$$y = q_2 + (s_4/t - s_5/t^2 + s_6/t^3) a_3.$$

But for 6 things

$$t^3 - t^2 s_1 + ts_2 - s_3 + s_4/t - s_5/t^2 + s_6/t^3 = 0.$$

Hence

(13) $$\frac{x - p_2}{a_4} - \frac{y - q_2}{a_3} = - s_3,$$

that is to say

THEOREM 4. *The first orthocenters of the* 5-*lines included in a* 6-*line lie on a line.*

The argument is clearly general, so that if the point $p_2 \equiv a_1 - s_1 a_2 + s_2 a_3$ be constructed for a 6-line, the perpendiculars from such point for 6 of 7 lines on the line left out meet at a point, and for 7 of 8 lines these points lie on a line; and so on. Briefly, we have found an *orthocenter* for an odd number of lines, a *directrix* for an even number.

§ 3. *Construction of a series of points.*

The points to which attention is thus directed belong, for a given n-line, to the series

(14) $$p_0 = a_1, \qquad p_1 \equiv a_1 - s_1 a_2, \qquad p_2 \equiv a_1 - s_1 a_2 + s_2 a_3, \text{ etc.}$$

Their construction is merely a matter of centroids, or centers of gravity. For we regard as known in an n-line:

a_1, the center of the center-circle,

$a_1 - t_i a_2$, the n such points of the $(n-1)$-lines,

$a_1 - (t_i + t_k) a_2 + t_i t_k a_3$, the $\binom{n}{2}$ such points of the $(n-2)$-lines,

. . . .

and taking the centroid of each set we have $a_1, g_1, g_2 \ldots$ where

$$
(15) \qquad \begin{cases} n g_1 = n a_1 - s_1 a_2, \\ \binom{n}{2} g_2 = \binom{n}{2} a_1 - (n-1) s_1 a_2 + s_2 a_3, \\ \binom{n}{3} g_3 = \binom{n}{3} a_1 - \binom{n-1}{2} s_1 a_2 + (n-2) s_2 a_3 - s_3 a_4, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{cases}
$$

whence the p's are easily constructed. It will be noticed that the last equations cease to be independent of the origin when a_n itself makes its appearance; thus g_{n-1} is the centroid of the points x_i, the reflexions of the origin in the n lines. Hence also p_{n-1} is a point dependent on the origin, not a point of the n-line itself.

But a more vivid construction is indicated by the process by which p_1 for a 4-line was deduced (p. 4) from a_1 and p_1 for a 3-line. It will be clear on constructing p_1 for a 5-line.

We write as before (eq. 10), for a 4-line,

$$x - p_1 = (x - a_1) z,$$

and extend this to a 5-line, observing that p_1 for 4 is $p_1 + t(s_1 - t) a_3$ for 5. Thus the extended equation is

$$x - p_1 - t(s_1 - t) a_3 = (x - a_1 + t a_2) z = \{x - p_1 - (s_1 - t) a_2\} z.$$

If $z a_2 = t a_3$, we have $x = p_1$. Let then $|z| = |a_3 / a_2|$. That is, if we divide the known points p_1 and a_1 of 4 of 5 lines in the ratio $|a_3/a_2|$, where the constants a_i refer to the 5-line, the 5 such circles meet at the point,

$$p_1 \equiv a_1 - s_1 a_2.$$

And p_1 being now known for a 5-line, we have a similar statement for a 6-line, whence p_1 is known in general. But again we know p_1 and p_2 for a 5-line. Write for $n-1$ lines

$$x - p_2 = (x - p_1) z,$$

and extend this. We have for an n-line,

$$x - p_2 + t(s_2 - ts_1 + t^2)a_4 = \{x - p_1 - t(s_1 - t)a_3\}z$$
$$= \{x - p_2 + (s_2 - ts_1 + t^2)a_3\}z,$$

whence, as before, the point p_2 of the n-line has its distances from the points p_2 and p_1 of any included $n - 1$ lines in the fixed ratio $|a_4/a_3|$. And so in general:

THEOREM 5. *The point p_α of an n-line has its distance from the points p_α and $p_{\alpha-1}$ of an included $(n-1)$-line in the fixed ratio* $|a_{\alpha+2}/a_{\alpha+1}|$.

Since $b_\alpha = (-)^{n-1}s_{n+1-\alpha}$, this fixed ratio is unity when

$$\alpha + 2 + \alpha + 1 = n + 1,$$

or

$$\alpha + 1 = \tfrac{1}{2}n,$$

that is, in a $2n$-line the point p_{n-1} is equidistant from the points p_{n-1} and p_{n-2} of any included $(2n-1)$-line.

Regarding the lengths $|a_\alpha|$ as known, we have in Theorem 5 a construction for the points p_α for n-lines, when the points p_α for $(n-1)$-lines are known.

§ 4. *The curve* Δ^{2n-1}.

The peculiar appropriateness of the deltoid for the metrical theory of four lines makes it desirable to have an analogous curve for $2n$-lines. Such a curve is

$$(16)\quad \begin{aligned}(-)^n(xt^n - yt^{n-1}) = t^{2n-1} - 1 - (s_1t^{2n-2} - s_{2n-2}t)\\ + \cdots + (-)^n(s_{n-2}t^{n+1} - s_{n+1}t^{n-2}),\end{aligned}$$

where c_α and $c_{2n-1-\alpha}$ are conjugate. This is a curve Δ^{2n-1} of class $2n-1$, order $2n$, with a line equation of the type

$$(17)\quad \xi^n\eta^n = \text{form in } \xi,\ \eta \text{ of order } 2n-1.$$

For clearness I will take the case $n = 3$, next to the case $n = 2$ of § 1; the generalizations are immediate.

Any 6 lines are lines of a curve Δ^5,

$$(18)\quad -(xt^3 - yt^2) = t^5 - 1 - (s_1t^4 - s_4t).$$

The map-equation of the curve is

$$(19)\quad -x = 3t^2 + 2t^{-3} - (2s_1t - s_4t^{-2}).$$

Thus the curve is derived by addition from 2 concentric cycloids.

So Δ^{2n-1} is derived by addition from $n-1$ concentric cycloids; those points being added at which the tangents are parallel.

Let the common center of the cycloids be called the center of Δ^{2n-1}.

The cusps of Δ^5 are given from (19) by

$$6(t-t^{-4})-2(s_1-s_4t^{-3})=0,$$

or

$$3(t^5-1)=s_1t^4-s_4t.$$

Hence the cusp-tangents are such that

(20) $$3(xt^3-yt^2)=2(s_1t^4-s_4t),$$

that is:

There are 5 cusp-tangents of Δ^5; they touch a concentric Δ^3. And so

THEOREM 6. *There are* $2n-1$ *cusp-tangents of* Δ^{2n-1}; *they touch a concentric* Δ^{2n-3}.

Consider the common lines of Δ^5 and any Δ^3,

$$xt^2-yt=\alpha t^3+\beta t^2-\gamma t-\delta.$$

There are 5 common lines, and they are given by

(21) $$t^5-1-(c_1t^4-c_4t)+t(\alpha t^3+\beta t^2-\gamma t-\delta)=0.$$

Hence the center of the Δ^3 is

$$\beta=\sum t_1t_2=s_2$$

where

$$s_5=1,$$

that is, the center of the deltoid touching 4 lines of Δ^5 is

(22) $$x=s_2+s_1/s_4 \qquad (s \text{ for } 4).$$

The perpendicular on a fifth line of Δ^5 is

$$xt+y=t(s_2+s_1/s_4)+s_2/s_4+s_3=s_3+s_2t/s_5 \qquad (s \text{ for } 5).$$

Hence the first orthocenter of 5 lines is s_2/s_5.

For 5 of 6 lines this point is

$$x=(t^2-ts_1+s_2)t/s_6$$

or

$$y=s_6/t^3-s_5/t^2+s_4/t,$$

whence the 6 first orthocenters lie on the line

(23) $$s_6x+y=s_3.$$

The line of Δ^3 perpendicular to a fifth line of Δ^5 is

$$xt^2 + yt + \alpha t^3 - \beta t^2 - \gamma t + \delta = 0,$$

where

$$c_1 - \alpha = s_1, \qquad \beta = s_2, \qquad \gamma = s_3, \qquad c_4 - \delta = s_4, \qquad 1 = s_5;$$

or is

$$xt^2 + yt + (c_1 - s_1 - 1/s_4)t^3 - (s_2 + s_1/s_4)t^2 - (s_3 + s_2/s_4)t$$
$$+ c_4 - s_4 - s_3/s_4 = 0 \qquad (s \text{ for } 4),$$

or

(24) $$xt^2 + yt + c_1 t^3 - s_2 t^2 - s_4 + c_4 - s_3 t/s_5 - s_1 t^3/s_5 = 0 \qquad (s \text{ for } 5).$$

That is:

THEOREM 7. *If of the deltoid touching any* 4 *of* 5 *lines we draw the line perpendicular to the omitted line, the* 5 *perpendiculars touch a deltoid.*

The center of this deltoid is s_2. The first orthocenter was s_2/s_5. These are strokes of equal size. Hence:

THEOREM 8. *The locus of centers of curves* Δ^5 *of which* 5 *lines are given is a line.* And so for Δ^{2n-1}.

The curve Δ^{2n-1} does then for $2n$ lines precisely what the deltoid Δ^3 does for 4 lines; it replaces Clifford's n-fold parabola for metrical purposes. We have proved by its means the theorems of § 2 over again with additions; in particular we have assigned a meaning to the point p_1 of a 5-line or p_{n-2} of a $(2n-1)$-line, for this is readily identified as the point s_2 of 5 lines of Δ^5 or s_{n-1} of $2n-1$ lines of Δ^{2n-1}. But at present I regard the use of this curve as more limited than the method of the a's, to which I now return.

§ 5. *The second circle of an n-line.*

A curve of order n, whose highest terms in conjugate coördinates are

$$tx^n + y^n,$$

has its asymptotes apolar to the absolute points IJ, that is, these asymptotes form an equiangular polygon. Such a curve depends on $\frac{1}{2}n(n+1)+1$ constants, and therefore a pencil can be drawn through $\frac{1}{2}n(n+1)$ points in general, and the pencil determines $n^2 - \frac{1}{2}n(n+1)$, or $\frac{1}{2}n(n-1)$ other points. In the pencil are the imaginary curves

$$x^n + an^{n-1} + \cdots + a'y^{n-1} + \cdots = 0$$

and

$$y^n + by^{n-1} + \cdots + b'y^{n-1} \cdots = 0,$$

the pencil itself is

(25) $$tn^n + y^n + (ta + b')x^{n-1} + \cdots = 0,$$

the polar line of I is

$$ntx + ta + b' = 0,$$

and therefore the polar lines of I and J meet on a circle.*

Let now the $\frac{1}{2}n(n+1)$ points be the joins of $n+1$ lines. We shall call the circle the *second circle* of the $(n+1)$-line; the first circle being the center-circle (l. c., p. 99).

For a 3-line the second circle is the Feuerbach circle.

Now calculate the center and radius of the second circle of the n-line

$$xt_a + y = x_a t_a = y_a \qquad (\alpha = 1, \cdots, n).$$

The pencil is

$$\sum_a \frac{A_i}{nt_a + y - y_a} = 0.$$

The highest powers arise from

$$\sum_a \frac{A_a}{xt_a + y},$$

and are to be $tx^{n-1} + y^{n-1}$, so that, if $y/x = \lambda$,

(26) $$t + \lambda^{n-1} = \sum A_1(t_2 + \lambda)(t_3 + \lambda)\cdots(t_n + \lambda\ ;$$

whence

$$t = \sum A_1 t_2 \cdots t_n;$$

$$t + (-t_1)^{n-1} = A_1(t_2 - t_1)(t_3 - t_1)\cdots(t_n - t_1).$$

Operating with D_x^{n-2} on

$$\sum A_1(xt_2 + y - y_2)\cdots(xt_n + y - y_n),$$

we have for the polar line of I

$$(n-1)tx = \sum A_1 t_2 t_3 \cdots t_n(x_2 + x_3 + \cdots + x_n),$$

or if

$$\sum x_a = ng_{n-1},$$

$$(n-1)tx = ntg_{n-1} - \sum n_1 \frac{\{t + (-t_1)^{n-1}\}t_2\cdots t_n}{(t_2 - t_1)(t_3 - t_1)\cdots(t_n - t_1)};$$

or since

$$\sum x_1 \frac{t_2 \cdots t_n}{(t_2 - t_1)\cdots(t_n - t_1)} = a_1 - s_1 a_2 + \cdots + (-)^{n-1} s_{n-1} a_n = p_{n-1},$$

the second circle is

(27) $$(n-1)x = ng_{n-1} - p_{n-1} - s_n a_2/t.$$

* Cf. J. H. GRACE, Proceedings London Mathematical Society, vol. 33 (1900), p. 194.

Hence its radius is

$$\left|\frac{s_n a_2}{n-1}\right|, \qquad \text{i. e.,} \quad \frac{1}{n-1}\,|\,a_2\,|, \text{ or:}$$

THEOREM 9. *The radius of the second circle of an n-line is* $1/(n-1)$ *of the radius of the first circle.*

Also its center is given by

$$(n-1)c = ng_{n-1} - p_{n-1},$$

or explicitly by

(28) $$(n-1)c = (n-1)a_1 - (n-2)s_1a_2 + \cdots - (-)^n s_{n-2}a_{n-1}.$$

Omitting now the nth line we have for the second center of the rest

$$(n-2)c' = (n-1)g'_{n-2} - p'_{n-2},$$

whence $$(n-1)c - (n-2)c' = x_n - (p_{n-1} - p'_{n-2}).$$

Here

$$\begin{aligned} p'_{n-2} = a_1 - ta_2 - (s_1 - t)(a_2 - ta_3) + \cdots \\ + \{t^{n-2} - s_1t^{n-3} + \cdots + (-)^n s_{n-2}\}(a_{n-1} - ta_n) \\ = p_{n-2} - a_n\{t^{n-1} - s_1t^{n-2} + \cdots + (-)^n ts_{n-2}\} \\ = p_{n-1} - a_n\{t^{n-1} - s_1t^{n-2} + \cdots - (-)^n s_{n-1}\} \\ = p_{n-1} + (-)^n a_n s_n/t \\ = p_{n-1} - b_1/t. \end{aligned}$$

Therefore

$$(n-2)c' = (n-1)c - x_n + b_1/t_n.$$

But the reflexion r' of the first center a_1 in the omitted line is $x_n - b_1/t_n$. Hence

(29) $$(n-2)c' = (n-1)c - r',$$

whence it follows at once that:

THEOREM 10. *If from the second center of each* $(n-1)$*-line of an n-line a perpendicular be drawn to the omitted line, these perpendiculars meet at a point; the point is the external center of similitude of the first and second circles of the n-line.*

The point h so found is given by

(30) $$\begin{aligned} (n-2)h = (n-1)c - a_1 = (n-2)(a_1 - s_1a_2) + (n-3)s_2a_3 \\ - (n-4)s_3a_4 + \cdots (-)^n s_{n-2}a_{n-1}. \end{aligned}$$

Whereas the orthocenter of § 2 applied only to an odd number of lines the present one applies to any number. We have then for an odd n-line *two* solutions of our problem, except when $n = 3$, in which case the points h and p coincide.

But when we have two orthocenters, we have a whole line of orthocenters, since evidently a perpendicular to one of the n lines, dividing the join of the two known points of the remaining $n - 1$-lines in a fixed ratio, will divide the join of the two orthocenters in the same fixed ratio.

Thus we have found, *for an even number of lines, one orthocenter; for an odd number of lines, a line of orthocenters; for an even number of lines, one directrix.*

KNOWLTON, P. Q.,
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