ORTHOCENTRIC PROPERTIES OF THE PLANE $n$-LINE

BY

F. MORLEY

In continuation of a memoir in these Transactions (vol. 1, p. 97) I consider the problem:

To find for $n$ lines of a plane natural metrical analogues of the elementary facts that the perpendiculars of 3 lines meet at a point (the orthocenter of the 3-line) and that the orthocenters of the 3-lines contained in a 4-line lie on a line.

I apply first to the special case of a 4-line the treatment sketched in § 7 of the memoir cited; this affords suggestions for the general case.

§ 1. The Deltoid.

To discuss with the minimum of trouble the metrical theory of a 4-line we should take, according to our purpose, the lines as tangents either of a parabola or of a hypocycloid of class three. We want here the latter curve. As it is to most geometers an incidental stationary thing and not a weapon, I will treat it ab initio. And as it is at least as good as other curves which have a given name I will call it a deltoid. It is hardly necessary to remark that it is the metrically normal form of the general rational plane curve of class three with isolated double line.

Denote by $t$ a turn or a complex number of absolute value 1; and think of $t$ as a point on the unit circle. We consider three points $t_1$, $t_2$, $t_3$ on the circle subject to the condition

$$s_3 = t_1 t_2 t_3 = 1.$$  

With this triad we associate a point $x$ by the equation

$$x = t_1 + t_2 + t_3 = s_1,$$  

which carries with it the conjugate equation

$$y = 1/t_1 + 1/t_2 + 1/t_3 = t_2 t_3 + t_3 t_1 + t_1 t_2 = s_2.$$  

We have from (1) and (2)

$$x = t_1 + t_2 + 1/t_1 t_2.$$  

*Presented to the Society at the Evanston meeting, September 2-3, 1902. Received for publication September 3, 1902.
Herein let \( t_1 \) be fixed, \( t_2 \) variable. Since \( x = t + 1/t \) is that simplest of all hypocycloids, the segment of a line, so also is (3) the segment of a line, of constant length 4, with center \( t_1 \) and ends where

\[ D_{t_1} x = 0, \]

or

\[ t_1 t_2^2 = 1, \]

or

\[ x = t_1 \pm 2/\sqrt{t_1}. \]

When \( t_1 \) varies, the deltoid is described by the motion of this variable segment; the ends move on the curve, and the segment touches the curve. And any point inside the curve is given by (3) as the intersection of 2 segments; the points of the curve itself are given when the segments are brought to coincidence, that is, are

\[ x = 2t + 1/t^2. \]

This equation expresses that while a point \( 2t \) is describing a circle of radius 2 another point \( x \) moves round it with an angular velocity opposite in sense and twice as great; thus the cycloidal nature of the curve is apparent.

A point of the curve and also a line of the curve is named by its parameter \( t \); thus the point \( t \) of the curve is given by (4), and the line \( t \) from (1) and (2) by

\[ xt^2 - yt = t^3 - 1. \]

\( x \) and \( y \) being always conjugate coördinates.

Now two lines \( t_1 \) and \( t_2 \) meet at

\[ x_1 = t_1 + t_2 + 1/t_1 t_2. \]

Symmetrize this equation for 3 lines of the curve by writing it

\[ x = s_1 - t_3 + t_3/s_3. \]

Omitting the suffix of \( t_3 \) we have the map-equation of the circumcircle. Hence: The circumcenter of the 3-line is \( s_1 \), and the circumradius is \( |1 - 1/s_3| \) or \( |1 - s_3| \).

The mean point of

\[ t_1 + t_2 + 1/t_1 t_2 \quad \text{and} \quad t_1 + t_3 + 1/t_1 t_3 \]

is

\[ m = \frac{1}{2} [s_1 + t_1 + (s_1 - t_1)/s_3]. \]

Hence the center of the circle which bisects the sides, the Feuerbach or nine-point circle, is

\[ c = \frac{1}{2} (s_1 + s_1/s_3). \]
The perpendicular from \( t_1 + t_2 + 1/t_1t_2 \) on the line \( t_3 \) is

\[
xt_3 + y = t_3(t_1 + t_2 + 1/t_1t_2) + 1/t_1 + 1/t_2 + t_1t_2 = s_2 + t_3s_1/s_3.
\]

Hence the orthocenter is

\[
p = s_1/s_3,
\]

as could of course be inferred from the fact that

\[
\text{orthocenter} + \text{circumcenter} = 2 \times \text{Feuerbach center}.
\]

Since then \( |p| = |s_1| \) we have:

**Theorem 1.** The center of a deltoid, of which three lines are given, is equidistant from the circumcenter and orthocenter of the 3-line.

The centre of the deltoid which touches 4 lines is thus determined as the intersection of 4 lines, one for each of 3 of the given 4.

The Feuerbach center for 3 of 4 lines is given by

\[
2c = (s_1 - t_4)(1 + t_4/s_4), \quad (s \text{ for } 4),
\]

\( s_1 \) and \( s_3 \) being replaced in (7) by symmetric functions of \( t_1, t_2, t_3, t_4 \). Thus the 4 such centers are included in

\[
2x = (s_1 - t)(1 + t/s_4),
\]

whose conjugate is

\[
2y = (s_3/s_4 - 1/t)(1 + s_4/t) = (s_3/t - s_4/t^2)(1 + t/s_4).
\]

But

\[
t^2 - s_1t + s_2 - s_3/t + s_4/t^2 = 0.
\]

Therefore

\[
2(tx + y) = s_2(1 + t/s_4),
\]

a line through \( s_2/s_4 \), perpendicular to the line \( t \) of the curve. Hence

**Theorem 2.** If from the Feuerbach center of any 3 of 4 lines a perpendicular be drawn to the remaining line, the 4 perpendiculars meet at a point, namely \( s_2/s_4 \).

§ 2. Extension to the n-line.

I employ now the notation of § 2 of the memoir cited, namely I write a line \( l_a \) in the form

\[
x t_a + y = x_a t_a = y_a.
\]

Denote the characteristic constants by \( a_a \),

\[
a_a = \sum \frac{x_1 t_1^{n-a}}{(t_1 - t_2) \cdots (t_1 - t_n)}, \quad (a = 1, 2, \ldots, n)
\]
and their conjugates by $b_a$, so that
\[ b_a = (-1)^{n-1} s_n a_{n+1-a} \]
where $s_n = t_1 t_2 \cdots t_n$, and in general $s_a = \sum t_i t_{i+1} \cdots t_n$ for $n$ $t$'s.

The circumcenter of a 3-line is $a_1$. The mean point of the joins of $l_1$, $l_2$ and $l_1$, $l_3$ is given by
\[ 2m = \frac{x_1 t_1 + x_2 t_2}{t_1 - t_2} + \frac{x_1 t_1 + x_3 t_3}{t_1 - t_3} \]
\[ = \frac{x_1 t_1 (2t_1 - s_1 + t_1)}{(t_1 - t_2)(t_1 - t_3)} + \frac{x_2 t_2 (2t_2 - s_1 + t_1)}{(t_2 - t_1)(t_2 - t_3)} + \frac{x_3 t_3 (2t_3 - s_1 + t_1)}{(t_3 - t_1)(t_3 - t_2)}, \]
and therefore the nine-point circle is
\[ (8) \quad 2m = 2a_1 - s_1 a_2 + a_2 t, \]
and its center is
\[ 2c = 2a_1 - s_1 a_2. \]

Hence the orthocenter is
\[ (9) \quad p = 2c - a_1 = a_1 - s_1 a_2. \]

The line about the points $a_1$ and $p$ (which is the locus of centers of inscribed deltoids) is, when $\tau$ is any turn,
\[ (10) \quad x = a_1 + s_1 a_2 = (x - a_1) \tau. \]

This for 3 of 4 lines is
\[ x - a_1 + t a_2 + (s_1 - t)(a_2 - t a_4) = (x - a_1 + t a_4) \tau, \]
or
\[ x - a_1 + s_1 a_2 - t(s_1 - t) a_3 = [x - a_1 + s_1 a_2 - (s_1 - t) a_2] \tau. \]

Now since for 4 lines $a_3/a_2$ is a turn we can equate $t a_3$ to $a_2 \tau$; that is, whatever turn $t$ may be, the line passes through the point
\[ (11) \quad p_1 = a_1 - s_1 a_2. \]

This is then the center of the inscribed deltoid of the 4-line.

For 4 of 5 lines this point is given by
\[ p = a_1 - t a_2 - (s_1 - t)(a_2 - t a_3) = a_1 - s_1 a_2 + t(s_1 - t) a_3, \]
or the conjugate equation
\[ q = b_1 - s_4 b_2 - \frac{1}{t} \left( \frac{s_4}{s_5} - \frac{1}{t} \right) b_3 = s_5 a_5 - s_4 a_4 + \left( \frac{s_4}{s_5} - \frac{s_4}{t^2} \right) a_3. \]

But for 5 things
\[ t^3 - t^2 s_1 + t s_2 - t s_3 + s_4 / t - s_5 / t^2 = 0. \]
Therefore

\[(12) \quad t(p - a_1 + s_1 a_2) - (q - s_6 a_5 + s_4 a_4) = (ts_2 - s_3) a_3,\]

that is, the line joining the \( p \) of 4 lines to the point \( a_1 - s_1 a_2 + s_2 a_3 \) is perpendicular to the remaining line; or

**Theorem 3.** If from the center of the inscribed deltoid of 4 of 5 lines a perpendicular be drawn to the line left out, the 5 perpendiculars meet at a point; namely the point

\[p_2 \equiv a_1 - s_1 a_2 + s_2 a_3.\]

Call this point the first orthocenter of the 5-line.

For 5 of 6 lines this point is

\[x = a_1 - ta_2 - (s_1 - t)(a_2 - ta_3) + (s_2 - ts_1 + t^2)(a_2 - ta_4)\]
\[= a_1 - s_1 a_2 + s_2 a_3 - t(s_2 - ts_1 + t^2)a_4\]
\[= p_2 - t(s_2 - ts_1 + t^2)a_4,\]

or, if the conjugates be written,

\[y = q_2 + (s_4/t - s_5/t^2 + s_6/t^3)a_5.\]

But for 6 things

\[t^3 - t^2 s_1 + ts_2 - s_3 + s_4/t - s_5/t^2 + s_6/t^3 = 0.\]

Hence

\[(13) \quad \frac{x - p_2}{a_4} - \frac{y - q_2}{a_3} = -s_3,\]

that is to say

**Theorem 4.** The first orthocenters of the 5-lines included in a 6-line lie on a line.

The argument is clearly general, so that if the point \( p_2 \equiv a_1 - s_1 a_2 + s_2 a_3 \) be constructed for a 6-line, the perpendiculars from such point for 6 of 7 lines on the line left out meet at a point, and for 7 of 8 lines these points lie on a line; and so on. Briefly, we have found an orthocenter for an odd number of lines, a directrix for an even number.

\[\S 3. \text{Construction of a series of points}.\]

The points to which attention is thus directed belong, for a given \( n \)-line, to the series

\[(14) \quad p_0 = a_1, \quad p_1 = a_1 - s_1 a_2, \quad p_2 = a_1 - s_1 a_2 + s_2 a_3, \text{ etc.}\]

Their construction is merely a matter of centroids, or centers of gravity. For we regard as known in an \( n \)-line:
F. Morley: Orthocentric Properties

January

\[ a_1, \text{ the center of the center-circle,} \]
\[ a_1 - t_1 a_2, \text{ the } n \text{ such points of the } (n-1)\text{-lines,} \]
\[ a_1 - (t_i + t_j) a_2 + t_i t_j a_3, \text{ the } \binom{n}{2} \text{ such points of the } (n-2)\text{-lines,} \]
\[ \ldots \]  

and taking the centroid of each set we have \( a_1, g_1, g_2 \ldots \) where

\[
\begin{align*}
ng_1 &= na_1 - s_1 a_2, \\
\left( \begin{array}{c} 2 \\ n \end{array} \right) g_2 &= \left( \begin{array}{c} 2 \\ n \end{array} \right) a_1 - (n-1) s_1 a_2 + s_2 a_3, \\
\left( \begin{array}{c} 3 \\ n \end{array} \right) g_3 &= \left( \begin{array}{c} 3 \\ n \end{array} \right) a_1 - \left( \begin{array}{c} 3 \\ 2 \end{array} \right) s_1 a_2 + (n-2) s_2 a_3 - s_3 a_4,
\end{align*}
\]

(15)

whence the \( p \)'s are easily constructed. It will be noticed that the last equations cease to be independent of the origin when \( a_n \) itself makes its appearance; thus \( g_{n-1} \) is the centroid of the points \( x_i \), the reflexions of the origin in the \( n \) lines. Hence also \( p_{n-1} \) is a point dependent on the origin, not a point of the \( n \)-line itself.

But a more vivid construction is indicated by the process by which \( p_1 \) for a 4-line was deduced (p. 4) from \( a_1 \) and \( p_1 \) for a 3-line. It will be clear on constructing \( p_1 \) for a 5-line.

We write as before (eq. 10), for a 4-line,

\[ x - p_1 = (x - a_1) z, \]

and extend this to a 5-line, observing that \( p_1 \) for 4 is \( p_1 + t (s_1 - t) a_2 \) for 5. Thus the extended equation is

\[ x - p_1 - t (s_1 - t) a_2 = (x - a_1 + t a_2) z = \{ x - p_1 - (s_1 - t) a_2 \} z. \]

If \( s a_2 = t a_3 \), we have \( x = p_1 \). Let then \( |z| = |a_3/a_2| \). That is, if we divide the known points \( p_1 \) and \( a_1 \) of 4 of 5 lines in the ratio \( |a_3/a_2| \), where the constants \( a_i \) refer to the 5-line, the 5 such circles meet at the point,

\[ p_1 = a_1 - s_1 a_2. \]

And \( p_1 \) being now known for a 5-line, we have a similar statement for a 6-line, whence \( p_1 \) is known in general. But again we know \( p_1 \) and \( p_2 \) for a 5-line. Write for \( n - 1 \) lines

\[ x - p_2 = (x - p_1) z, \]
and extend this. We have for an $n$-line,

$$x - p_2 + t(s_2 - ts_1 + t^2)a_4 = \{x - p_1 - t(s_1 - t)a_3\}z = \{x - p_2 + (s_2 - ts_1 + t^2)a_3\}z,$$

whence, as before, the point $p_2$ of the $n$-line has its distances from the points $p_2$ and $p_1$ of any included $n - 1$ lines in the fixed ratio $|a_4/a_3|$. And so in general:

**Theorem 5.** The point $p_n$ of an $n$-line has its distance from the points $p_n$ and $p_{n-1}$ of an included $(n - 1)$-line in the fixed ratio $|a_{n+1}/a_{n+1}|$.

Since $b_n = (-)^{n-1}s_{n+1-a}$, this fixed ratio is unity when

$$\alpha + 2 + \alpha + 1 = n + 1,$$

or

$$\alpha + 1 = \frac{1}{2}n,$$

that is, in a $2n$-line the point $p_{n-1}$ is equidistant from the points $p_{n-1}$ and $p_{n-2}$ of any included $(2n - 1)$-line.

Regarding the lengths $|a_n|$ as known, we have in Theorem 5 a construction for the points $p_n$ for $n$-lines, when the points $p_n$ for $(n - 1)$-lines are known.

§ 4. The curve $\Delta^{2n-1}$.

The peculiar appropriateness of the deltoid for the metrical theory of four lines makes it desirable to have an analogous curve for $2n$-lines. Such a curve is

$$(-)^n(xt^n - yt^{n-1}) = t^{2n-1} - 1 - (s_1t^{2n-2} - s_{2n-2}t) + \cdots + (-)^n(s_{n-2}t^{n+1} - s_{n+1}t^{n-2}),$$

where $c_a$ and $c_{2n-1-a}$ are conjugate. This is a curve $\Delta^{2n-1}$ of class $2n - 1$, order $2n$, with a line equation of the type

$$\xi^n\eta^n = \text{form in } \xi, \eta \text{ of order } 2n - 1.$$

For clearness I will take the case $n = 3$, next to the case $n = 2$ of § 1; the generalizations are immediate.

Any 6 lines are lines of a curve $\Delta^5$,

$$-(xt^6 - yt^5) = t^6 - 1 - (s_4t^7 - s_5t).$$

The map-equation of the curve is

$$- x = 3t^2 + 2t^3 - (2s_1t - s_4t^2).$$

Thus the curve is derived by addition from 2 concentric cycloids.
So $\Delta^{2n-1}$ is derived by addition from $n-1$ concentric cycloids; those points being added at which the tangents are parallel.

Let the common center of the cycloids be called the center of $\Delta^{2n-1}$.

The cusps of $\Delta^5$ are given from (19) by

$$6(t-t^{-4})-2(s_1-s_4t^{-3})=0,$$

or

$$3(t^5-1)=s_1t^4-s_4t.$$

Hence the cusp-tangents are such that

(20) \[ 3(xt^3-yt^2) = 2(s_1t^4-s_4t), \]

that is:

There are 5 cusp-tangents of $\Delta^5$; they touch a concentric $\Delta^3$. And so

**Theorem 6.** There are $2n-1$ cusp-tangents of $\Delta^{2n-1}$; they touch a concentric $\Delta^{2n-3}$.

Consider the common lines of $\Delta^5$ and any $\Delta^3$,

$$xt^2-yt=\alpha t^3+\beta t^2-\gamma t-\delta.$$

There are 5 common lines, and they are given by

(21) \[ t^5-1-(c_1t^4-c_4t)+t(\alpha t^3+\beta t^2-\gamma t-\delta)=0. \]

Hence the center of the $\Delta^3$ is

$$\beta = \sum t_1t_2 = s_2$$

where

$$s_5=1,$$

that is, the center of the deltoid touching 4 lines of $\Delta^5$ is

(22) \[ x = s_2 + s_1/s_4 \quad (s \text{ for } 4). \]

The perpendicular on a fifth line of $\Delta^5$ is

$$xt+y=t(s_2+s_1/s_4)+s_2/s_4+s_3=s_3+s_2t/s_5 \quad (s \text{ for } 5).$$

Hence the first orthocenter of 5 lines is $s_2/s_5$.

For 5 of 6 lines this point is

$$x = (t^2-ts_1+s_2)t/s_6$$

or

$$y = s_6/t^3-s_5/t^2+s_4/t,$$

whence the 6 first orthocenters lie on the line

(23) \[ s_6x + y = s_3. \]
The line of $\Delta^3$ perpendicular to a fifth line of $\Delta^5$ is
\[ xt^2 + yt + \alpha t^3 - \beta t^2 + \gamma t + \delta = 0, \]
where
\[ c_1 - \alpha = s_1, \quad \beta = s_2, \quad \gamma = s_3, \quad c_4 - \delta = s_4, \quad 1 = s_5; \]
or is
\[ xt^2 + yt + (c_1 - s_1 - 1/s_4) t^3 - (s_2 + s_1/s_4) t^2 - (s_3 + s_2/s_4) t \]
\[ + c_4 - s_4 - s_3/s_4 = 0 \quad (s \text{ for } 4), \]
or
\[ (24) \quad xt^2 + yt + c_1 t^3 - s_2 t^2 - s_4 + c_4 - s_3 t/s_5 - s_1 t^3/s_5 = 0 \quad (s \text{ for } 5). \]
That is:

**Theorem 7.** If of the deltoid touching any 4 of 5 lines we draw the line perpendicular to the omitted line, the 5 perpendiculars touch a deltoid.

The center of this deltoid is $s_2$. The first orthocenter was $s_2/s_5$. These are strokes of equal size. Hence:

**Theorem 8.** The locus of centers of curves $\Delta^5$ of which 5 lines are given is a line. And so for $\Delta^{2n-1}$.

The curve $\Delta^{2n-1}$ does then for 2n lines precisely what the deltoid $\Delta^3$ does for 4 lines; it replaces Clifford's $n$-fold parabola for metrical purposes. We have proved by its means the theorems of §2 over again with additions; in particular we have assigned a meaning to the point $p_1$ of a 5-line or $p_{n-2}$ of a $(2n - 1)$-line, for this is readily identified as the point $s_2$ of 5 lines of $\Delta^5$ or $s_{n-1}$ of $2n - 1$ lines of $\Delta^{2n-1}$. But at present I regard the use of this curve as more limited than the method of the $a$'s, to which I now return.

§5. The second circle of an n-line.

A curve of order $n$, whose highest terms in conjugate coordinates are
\[ tx^n + y^n, \]
has its asymptotes apolar to the absolute points $IJ$, that is, these asymptotes form an equiangular polygon. Such a curve depends on $\frac{1}{2} n(n + 1) + 1$ constants, and therefore a pencil can be drawn through $\frac{1}{2} n(n + 1)$ points in general, and the pencil determines $n^2 - \frac{1}{2} n(n + 1)$, or $\frac{1}{2} n(n - 1)$ other points. In the pencil are the imaginary curves
\[ x^n + a y^{n-1} + \ldots + a' y^{n-1} + \ldots = 0 \]
and
\[ y^n + b y^{n-1} + \ldots + b' y^{n-1} + \ldots = 0, \]
the pencil itself is
\[ (25) \quad tn^n + y^n + (ta + b') x^{n-1} + \ldots = 0, \]
the polar line of \( I \) is
\[ nt_x + ta + b' = 0, \]
and therefore the polar lines of \( I \) and \( J \) meet on a circle.*

Let now the \( \frac{1}{2} n(n + 1) \) points be the joins of \( n + 1 \) lines. We shall call the circle the second circle of the \( (n + 1) \)-line; the first circle being the center-circle (l. c., p. 99).

For a 3-line the second circle is the Feuerbach circle.

Now calculate the center and radius of the second circle of the \( n \)-line

\[ xt_a + y = x_a t_a = y_a \quad (a = 1, \ldots, n). \]

The pencil is
\[ \sum_a \frac{A_i}{nt_a + y - y_a} = 0. \]

The highest powers arise from
\[ \sum_a \frac{A_a}{xt_a + y}, \]
and are to be \( tw^{n-1} + y^{n-1} \), so that, if \( y/x = \lambda \),

\[ t + \lambda^{n-1} = \sum A_1(t_2 + \lambda)(t_3 + \lambda) \cdots (t_n + \lambda); \]

whence
\[ t = \sum A_1 t_2 \cdots t_n, \]
\[ t + (-t_1)^{n-1} = A_1(t_2 - t_1)(t_3 - t_1) \cdots (t_n - t_1). \]

Operating with \( D^{n-2}_{x_t} \) on
\[ \sum A_1(xt_2 + y - y_2) \cdots (xt_n + y - y_n), \]
we have for the polar line of \( I \)

\[ (n - 1)tx = \sum A_1 t_2 t_3 \cdots t_n(x_2 + x_3 + \cdots + x_n), \]
or if
\[ \sum x_a = ng_{n-1}, \]

\[ (n - 1)tx = ntg_{n-1} - \sum n_1 \frac{t + (-t_1)^{n-1}}{(t_2 - t_1)(t_3 - t_1) \cdots (t_n - t_1)}; \]
or since
\[ \sum x_1 \frac{t_2 \cdots t_n}{(t_2 - t_1) \cdots (t_n - t_1)} = a_1 - s_1 a_2 + \cdots + (-)^{n-1} s_{n-1} a_n = p_{n-1}, \]

the second circle is
\[ (n - 1)x = ng_{n-1} - p_{n-1} - s_{n-1} a_n / t. \]

Hence its radius is
\[ \frac{s_n a_2}{n-1}, \quad \text{i.e.,} \quad \frac{1}{n-1} |a_2|, \text{or:} \]

**Theorem 9.** The radius of the second circle of an \( n \)-line is \( 1/(n-1) \) of the radius of the first circle.

Also its center is given by
\[(n-1)c = ng_{n-1} - p_{n-1},\]
or explicitly by
\[(28) \quad (n-1)c = (n-1)a_1 - (n-2)s_1 a_2 + \cdots - (-)^n s_{n-2} a_{n-1}.\]

Omitting now the \( n \)th line we have for the second center of the rest
\[(n-2)c' = (n-1)g'_{n-2} - p'_{n-2},\]
whence
\[(n-1)c - (n-2)c' = x_n - (p_{n-1} - p'_{n-2}).\]

Here
\[p'_{n-2} = a_1 - ta_2 - (s_1 - t)(a_2 - ta_3) + \cdots + \{t^{n-2} - s_1 t^{n-3} + \cdots + (-)^n s_{n-2}\} (a_{n-1} - ta_n)\]
\[= p_{n-2} - a_n \{t^{n-1} - s_1 t^{n-2} + \cdots + (-)^n ts_{n-2}\}\]
\[= p_{n-1} - a_n \{t^{n-1} - s_1 t^{n-2} + \cdots - (-)^n s_{n-1}\}\]
\[= p_{n-1} + (-)^n a_n s_n/t\]
\[= p_{n-1} - b_1/t.\]

Therefore
\[(n-2)c' = (n-1)c - x_n + b_1/t_n.\]

But the reflexion \( r' \) of the first center \( a_1 \) in the omitted line is \( p_n - b_1/t_n \).

Hence
\[(29) \quad (n-2)c' = (n-1)c - r',\]
whence it follows at once that:

**Theorem 10.** If from the second center of each \( (n-1) \)-line of an \( n \)-line a perpendicular be drawn to the omitted line, these perpendiculars meet at a point; the point is the external center of similitude of the first and second circles of the \( n \)-line.

The point \( h \) so found is given by
\[(n-2)h = (n-1)c - a_1 = (n-2)(a_1 - s_1 a_2) + (n-3)s_2 a_3 - (n-4)s_3 a_4 + \cdots (-)^n s_{n-2} a_{n-1}.\]
Whereas the orthocenter of § 2 applied only to an odd number of lines the present one applies to any number. We have then for an odd \( n \)-line \textit{two} solutions of our problem, except when \( n = 3 \), in which case the points \( h \) and \( p \) coincide.

But when we have two orthocenters, we have a whole line of orthocenters, since evidently a perpendicular to one of the \( n \) lines, dividing the join of the two known points of the remaining \( n - 1 \)-lines in a fixed ratio, will divide the join of the two orthocenters in the same fixed ratio.

Thus we have found, \textit{for an even number of lines, one orthocenter; for an odd number of lines, a line of orthocenters; for an even number of lines, one directrix.}

\textsc{Knowlton, P. Q.,}
\textit{July, 1902.}