THE COGREDIENT AND DIGREDIENT THEORIES OF MULTIPLE BINARY FORMS*

BY

EDWARD KASNER

The theory of invariants originally confined itself to forms involving a single set of homogeneous variables; but recent investigations, geometric as well as algebraic, have proved the importance of the study of forms in any number of sets of variables. In passing from the theory of the simple to the theory of the multiple forms, an entirely new feature presents itself: in the latter case the linear transformations which are fundamental in the definition of invariants may be the same for all the variables or they may be distinct, i. e., the sets of variables involved may be cogredient or digredient. Multiple forms thus have two distinct invariant theories, a cogredient and a digredient.

The object of this paper is to study the relations between these two theories in the case of forms involving any number of binary variables. Geometrically, such a form may be regarded as establishing a correspondence between the elements of two or more linear manifolds; in the digredient theory the latter are considered as distinct, thus undergoing independent projective transformations, while in the cogredient theory the linear manifolds are considered to be superposed, thus undergoing the same projective transformation. The first part of the paper, §§ 1–5, is devoted to the double forms. The extension of the results is made first, for convenience of presentation, to the triple forms in § 6, and then to the general case in § 7.

The case of the double binary forms is perhaps the most interesting geometrically. In addition to the general interpretation by means of an algebraic correspondence between two manifolds, such a form may be interpreted as an algebraic curve on a quadric surface, or as a plane algebraic curve from the view point of inversion geometry. In the former of these special interpretations the two binary variables are the (homogeneous) parameters of the two sets of generators on the quadric, while in the latter they are the parameters of the two sets of minimal lines in the plane. These interpretations suggest the

* Presented to the Society under other titles August 20 and December 27, 1901. Received or publication August 4, 1902.

contact with the theory of quaternary forms established by the author* in the first volume of the Transactions. This contact is made use of in § 3 of the present paper to connect the cogredient theory with that of quaternary forms.

The general results are applied in § 3 to the quadri-quadric, and in § 4 to bilinear forms.

§ 1. *The principle of adjunction.*

The transformations considered in the digredient theory are of the type:

$$(1) \qquad \begin{aligned} x_1 : x_2 &= l_{11}x_1' + l_{12}x_2' : l_{21}x_1' + l_{22}x_2', \\ y_1 : y_2 &= m_{11}y_1' + m_{12}y_2' : m_{21}y_1' + m_{22}y_2', \end{aligned}$$

involving six essential parameters

$$l_{11} : l_{12} : l_{21} : l_{22}; \qquad m_{11} : m_{12} : m_{21} : m_{22};$$

while the cogredient transformations are of the type:

$$(2) \qquad \begin{aligned} x_1 : x_2 &= n_{11}x_1' + n_{12}x_2' : n_{21}x_1' + n_{22}x_2', \\ y_1 : y_2 &= n_{11}y_1' + n_{12}y_2' : n_{21}y_1' + n_{22}y_2', \end{aligned}$$

involving three parameters

$$n_{11} : n_{12} : n_{21} : n_{22}.$$

The fundamental groups are thus a six-parametric G_6 for the digredient, and a three-parametric G_3 for the cogredient theory. The relation between the groups is obtained by observing that the transformations (2) leave invariant the equation

$$(3) \qquad T \equiv x_1y_2 - x_2y_1 = 0,$$

and that conversely any transformation (1) which leaves invariant this equation is of form (2). The relation may be stated as follows:

The cogredient group G_3 is a subgroup of the digredient group G_6, consisting of those transformations of the latter which leave invariant, except for a factor, the bilinear form $T \equiv x_1y_2 - x_2y_1$.

Consider now the general double binary form

$$(4) \qquad f = \sum_{k=1}^{k=n} \sum_{h=1}^{h=m} \binom{m}{h}\binom{n}{k} a_{hk} x_1^{m-h} x_2^h y_1^{n-k} y_2^k = a_x^m \alpha_y^n = b_x^m \beta_y^n = \cdots,$$

the relation between the symbolic and the real coefficients being

$$(5) \qquad a_{hk} = a_1^{m-h} a_2^h \alpha_1^{n-k} \alpha_2^k = b_1^{m-h} b_2^h \beta_1^{n-k} \beta_2^k.$$

* *The invariant theory of the inversion group: geometry upon a quadric surface,* Transactions of the American Mathematical Society, vol. 1 (1900), pp. 420–498.

A cogredient comitant* is any function possessing the invariant property with respect to the group G_3, and a digredient comitant one possessing it with respect to G_6. According to the Clebsch-Aronhold symbolism the cogredient comitants are expressible symbolically in terms of the types:

$$
\begin{array}{llll}
(ab), & (\alpha\beta), & (a\alpha), & \\
a_x, & \alpha_y, & a_y, & \alpha_x, \\
(xy). & & &
\end{array}
\tag{6}
$$

Of these types only the following four are also digredient:

$$
(ab), \qquad (\alpha\beta), \qquad a_x, \qquad \alpha_y,
\tag{7}
$$

and all digredient comitants are expressible in terms of these. Thus, as is obvious also from the relation of the groups, every digredient comitant is also a cogredient comitant, but the converse is not true.†

We consider now the digredient symbolic types belonging to the system

$$
f = a_x^m \alpha_y^n, \qquad T = t_x \tau_y = x_1 y_2 - x_2 y_1 = (xy).
\tag{8}
$$

These are as follows:

$$
\begin{array}{llll}
(ab), & (\alpha\beta), & (at)(\alpha\tau), & \\
a_x, & \alpha_y, & (ta)\tau_y, & (\tau\alpha)t_x, \\
t_x\tau_y, & (tt')(\tau\tau'). & &
\end{array}
\tag{9}
$$

The types not involving T are common to (9) and (6); the remaining types may be reduced by taking into account the special values of the coefficients of T,

$$
t_{11} = t_1\tau_1 = 0, \quad t_{12} = t_1\tau_2 = 1, \quad t_{21} = t_2\tau_1 = -1, \quad t_{22} = t_2\tau_2 = 0.
$$

Thus

$$
\begin{aligned}
(at)(\alpha\tau) &= (a_1 t_2 - a_2 t_1)(\alpha_1 \tau_2 - \alpha_2 \tau_1) \\
&= a_1 \alpha_1 t_2 \tau_2 - a_1 \alpha_2 t_2 \tau_1 - a_2 \alpha_1 t_1 \tau_2 + a_2 \alpha_2 t_1 \tau_1 \\
&= a_1 \alpha_2 - a_2 \alpha_1 = (a\alpha);
\end{aligned}
$$

and similarly we find the other relations in the following set:

$$
\begin{array}{ll}
(at)(\alpha\tau) = (a\alpha), & \\
(ta)\tau_y = a_y, & (\tau\alpha)t_x = \alpha_x, \\
(tt')(\tau\tau') = 2, & t_x\tau_y = (xy).
\end{array}
\tag{10}
$$

---

*The term *comitant* proposed by REUSCHLE (Verhandlungen, Züricher Kongress 1898, p. 123) seems destined to replace the unnecessarily cumbersome term *concomitant* due to SYLVESTER. Cf. Encyclopädie der mathematischen Wissenschaften, vol. 1, p. 325.

†CAPELLI, Giornale di Matematiche, vol. 17 (1879), p. 71; CLEBSCH-LINDEMANN, *Geometrie*, vol. 1, p. 951.

By means of these relations the sets (6) and (9) are observed to be intimately connected, and the connection between the corresponding invariants can now be obtained. Any digredient comitant of f and T may be expressed in terms of the types (9). If then we introduce for the coefficients of T the special values $0, 1, -1, 0$, or, what is equivalent, if we make use of the relations (10), we obtain an expansion in terms of the types (6), which is homogeneous in all the symbols involved and therefore represents a cogredient comitant of f. Thus every digredient comitant of f, T gives rise to a unique cogredient comitant of f.

The converse question is not so easily disposed of. A cogredient comitant of f is expressible in terms of the types (6), and by means of (10) thus gives rise to an expression in terms of the types (9). The result will not be a digredient comitant unless it is homogeneous not only in the coefficients of f but also in those of T; this homogeneity, however, can always be introduced by making use of the relation

$$(11) \qquad i \equiv \tfrac{1}{2}(tt')(\tau\tau') = 1.$$

Thus if the original comitant is $J_1 + J_2 + \cdots$, where the terms are products of factors of the types (6), all of the same degree in the symbolic coefficients of f, the transformed expression will be $J_1' + J_2' + \cdots$, where the terms are products of factors of the types (9) all of the same degree in the coefficients of f but of possibly different degrees in those of T. The differences between the degrees of the various terms, as proved by a simple enumeration, are necessarily even and therefore can be removed by introducing powers of i; so that the digredient comitant is of the form

$$i^{\kappa_1} J_1' + i^{\kappa_2} J_2' + \cdots.$$

It is thus seen that any cogredient comitant may be obtained by transformation from a digredient. It remains to show that the correspondence is essentially unique, i. e., the digredient comitants which give rise to the same cogredient comitant are identical except for factors of the form i^{κ}. The proof depends upon the following

LEMMA.—*If the cogredient comitant obtained by transformation from a digredient comitant vanishes identically, then the latter also vanishes identically.*

According to a theorem* of GORDAN, the cogredient comitant, since by assumption it vanishes identically, can be put into the form

$$J = \sum J_\kappa B_\kappa,$$

where B_κ is of the form

$$(qr)(st) + (rs)(qt) + (sq)(rt),$$

* GORDAN–KERSCHENSTEINER, *Vorlesungen über Invariantentheorie*, vol. II, p. 132.

q, r, s, t denoting any four of the symbols $a, \alpha, b, \beta, c, \gamma, \cdots$. Thus there are three possibilities:

1°. The four symbols involved may be all cogredient, as a, b, c, d; 2°, there may be three of one kind and the fourth of the opposite kind, as a, b, c, ϕ; 3°, there may be two of each kind, as a, b, ϕ, ψ. Consider now the symbolic expression B' obtained from B by means of the relations (10). In case 1°,

$$B' = (ab)(cd) + (bc)(ad) + (ca)(bd)$$

which vanishes by the fundamental identity in the theory of binary forms. In case 2°,

$$\begin{aligned} B' &= (ab)(ct)(\phi\tau) + (bc)(at)(\phi\tau) + (ca)(bt)(\phi\tau) \\ &= (\phi\tau)\{(ab)(ct) + (bc)(at) + (ca)(bt)\}, \end{aligned}$$

which also vanishes in virtue of the second factor. In case 3° it is necessary to introduce i in order to preserve homogeneity. Thus

$$B' = \tfrac{1}{2}(ab)(\phi\psi)(tt')(\tau\tau') + (bt)(\phi\tau)(at')(\psi\tau') - (at)(\phi\tau)(bt')(\psi\tau').$$

By permuting the equivalent symbols t, t' and τ, τ', and adding we have

$$\begin{aligned} 2B' &= (ab)(\phi\psi)(tt')(\tau\tau') \\ &\qquad + [(\phi\tau)(\psi\tau') + (\tau\psi)(\phi\tau')][(at')(bt) + (t'b)(at)] \\ &= (ab)(\phi\psi)(tt')(\tau\tau') + (\psi\phi)(\tau\tau')(ba)(t't) = 0. \end{aligned}$$

In all cases therefore B' vanishes identically. The digredient comitant corresponding to J is however expressible in the form

$$J' = \sum i^{\mu} J'_{\kappa} B'_{\kappa},$$

and therefore also vanishes identically, thus proving the lemma.

To apply the lemma, consider any two digredient comitants J'_1 and J'_2 which give rise to the same cogredient comitant J. Then J'_1 and J'_2 must be of the same degree in the coefficients of f, and the difference of their degrees in the coefficients of T must be even, say 2λ, so that we may build the comitant

$$J'_1 - i^{\lambda} J'_2.$$

By transformation this gives rise to the identically vanishing cogredient comitant $J - J$, and therefore by the lemma $J'_1 = i^{\lambda} J'_2$.

The fundamental results thus obtained may now be summed up in the

THEOREM.—*The cogredient theory of any system of double binary forms Σ is connected with the digredient theory of the enlarged system $\Sigma' = (\Sigma, T)$ obtained by adjoining the bilinear form $T = x_1 y_2 - x_2 y_1$. Every digredient comitant of the latter system gives rise to a cogredient comitant of the former* *system; conversely any cogredient comitant may be obtained in this way, the corresponding digredient comitant being unique except for a factor* i^λ.

As regards the three geometric interpretations of double binary forms mentioned in the introduction, the bilinear form T is represented by an homography, a conic, and a circle, respectively. We may therefore state the following results:

The study of an algebraic correspondence between two superposed linear manifolds is equivalent to the study of the same correspondence between distinct manifolds together with an homography T.

The digredient theory is illustrated by the projective geometry of algebraic curves on a quadric surface; the cogredient theory by that of the same curves together with the conic T.

The vanishing of a digredient invariant of a double binary form denotes a property of the corresponding plane curve which is unaltered by the transformations of the inversion group; the vanishing of a cogredient invariant denotes a relation between the curve and the circle T.

§ 2. *Relations between complete systems.*

Having considered the relation between the individual comitants we proceed now to the relation between the complete systems corresponding to

I: The cogredient theory of any set of double binary forms Σ; and

II: The digredient theory of the enlarged set Σ, T.

Let J_1', J_2', $\cdots$ be a complete system of comitants for II, so that any digredient comitant J' may be expressed in the form

$$J' = \sum c J_1'^k J_2'^l \cdots, \tag{1}$$

where the c's are numerical coefficients. Then if we denote by J_i the cogredient comitant obtained by transformation from J_i' we have, for any comitant of I,

$$J = \sum c J_1^k J_2^l \cdots. \tag{2}$$

That is,

*A complete system for II becomes on transformation a complete system for I.**

Passing to the converse question, let J_1, J_2, $\cdots$ denote any complete system for I, so that any cogredient comitant may be written in the form (2); and let the corresponding digredient comitants (unique except for factors i^λ) be J_1', $J_2' \cdots$. Consider the general digredient comitant J', and assume that the transformed comitant is J. Then passing from the right hand member of (2) to a corres-

* Moreover, in virtue of (11), those members of the first which when multiplied by a power of i are expressible in terms of the remaining members of the system may be omitted; the result is still a complete system for I. For an analogous result in a different domain compare STUDY, Leipziger Berichte, vol. 49 (1897), p. 458.

ponding digredient comitant (introducing if necessary powers of i to preserve homogeneity) we obtain

$$\sum c i^{\mu} J_1'^{k} J_2'^{l} \cdots \tag{3}$$

We thus have two digredient comitants, J' and (3), which on transformation give rise to identical cogredient comitants, and therefore can differ only by a factor i^{λ}. That is any comitant of II may be expressed in the form

$$i^{\lambda} J' = \sum c i^{\mu} J_1'^{k} J_2'^{l} \cdots.$$

Thus J_1', J_2', $\cdots$ do not necessarily constitute a complete system in the ordinary sense, since in the expression for J' the coefficients need not be numerical but may involve positive or negative powers of i. We may state the result however as follows:

To a complete system for I, there corresponds a system for II, such that every comitant of II multiplied by a proper power of i is rationally and integrally expressible in terms of the members of the system together with i; the new system is thus complete provided i is adjoined and i^{-1} is regarded as an integral invariant.

The digredient theory of any set of double binary forms

$$f = a_x^m \alpha_y^n, \qquad \cdots, \qquad T = t_x \tau_y,$$

including a bilinear form, is thus partially reduced to the cogredient theory of the same forms with the omission of T. The latter theory is simpler than the former, since by a theorem of CLEBSCH * it is always possible to obtain an equivalent set of simple binary forms. Thus, considering for simplicity only one double binary form f, the equivalent set of simple forms is

$$\phi_\kappa = (a\alpha)^\kappa a_x^{m-\kappa} \alpha_x^{n-\kappa} \qquad (\kappa = 0, 1, 2, \cdots, m),$$

where it is assumed that $n \geqq m$.

The problem of finding a system of digredient comitants for any set of double binary forms containing a bilinear form, possessing the property of a complete system when i is included in the domain of numerical coefficients, may be reduced to the problem of finding a complete system for a related set of simple binary forms.

When one pair of the variables is involved only linearly, say $f = a_x^n \alpha_y$, the related set contains only two members:

$$a_x^n \alpha_x, \qquad (a\alpha) a_x^n.$$

* CLEBSCH, *Ueber eine fundamentale Aufgabe der Invariantentheorie*, Göttinger Abhandlungen, vol. 17 (1872), p. 39.

Thus for the case $n = 1$, the set of digredient invariants of $a_x^2 \alpha_y$, $t_x \tau_y$ could be obtained from the invariants of the cubic form $C_x^3 = a_x^2 \alpha_x$ together with the linear form $M_x = (a\alpha) a_x$, which in turn are easily derived from the invariants and covariants of the cubic. The resulting three invariants together with the invariant i constitute the required system.

§ 3. *Relation to the theory of quaternary forms.*

In the writer's paper on *The Invariant Theory of the Inversion Group*, published in volume I of the Transactions, a connection was established between the digredient theory of double binary forms and the theory of a corresponding system of quaternary forms.* The comitants of a double form $f = a_x^n \alpha_y^n$ of the same degree in each of the binary variables (i. e., an "equi-form") may in fact be derived from the comitants of the system of quaternary forms $F = A_X^n$, $Q = p_X^2$.† This relation will now be combined with that just established between the cogredient theory of f and the digredient theory of f, T. The quaternary system corresponding to f, T contains, in addition to F and Q, a linear form $L = L_X$ corresponding to the bilinear form T.

The cogredient comitants of any system of equi-binary double forms may be obtained from the comitants of a related system of quaternary forms whose degrees are the same as those of the binary forms together with a quadric and a linear form. In particular the invariants of the binary forms correspond to the invariants and contravariants of the related quaternary forms and the quadric.

If we recall in addition the connection between the cogredient theory and the simple binary theory, we have in all four related systems. Assuming for simplicity that there is only one double binary form, these may be denoted as follows:

(A) Quaternary: $F = A_X^n$, $L = L_X$, $Q = p_X^2$;

(B) Digredient double binary: $f = a_x^n \alpha_y^n$, $T = t_x \tau_y$;

(C) Cogredient double binary: $f = a_x^n \alpha_y^n$;

(D) Simple binary: $\phi_\kappa = (a\alpha)^\kappa a_x^{n-\kappa} \alpha_x^{n-\kappa}$ $\qquad (\kappa = 0, 1, \cdots, n)$.

The comitants of any one of these four systems may be derived from the comitants of any other, provided, in the case of (B) and (C) only comitants of the same degree in x and y are considered. But only the theories (C) and (D) are *entirely* equivalent. Thus we have seen that a complete (B) system gives a complete (C) system, but that in passing from a (C) system to a (B) system

* Cf. chapter IV of the paper cited.

† Ibid., p. 468.

it is necessary to adjoin the invariant i and to consider its reciprocal as an integral invariant. A similar relation holds between the complete systems of (A) and (B): in passing from the latter to the former, $\Delta = \frac{1}{24}(pp'p''p''')^2$, the discriminant of Q, must be adjoined and its reciprocal counted integral.* Finally in passing from (C) or (D) to (A) it is necessary to adjoin the two invariants Δ and $i' = (pp'p''L)^2$ (the latter being the quaternary analogue of i), and to consider the reciprocals of both as integral. The relation between the first and last theories, (A) and (D), may be stated as follows:

The theory of any set of quaternary forms containing a quadric and a linear form is intimately connected with the theory of a related set of simple binary forms. A complete system of comitants for the former set gives a complete system for the latter, but the converse is only true provided the invariants Δ and i' are adjoined and their reciprocals counted integral.

As a concrete illustration consider the problem of finding the complete system of cogredient invariants of the quadri-quadric $a_x^2\alpha_y^2$. On the one hand we may pass to the related simple binary forms consisting of the biquadratic $a_x^2\alpha_x^2$, the quadratic $(a\alpha)a_x\alpha_x$, and the invariant $(a\alpha)^2$. The system of the biquadratic and the quadratic † contains six invariants, so that in all there are seven invariants in the required system. In order to obtain the actual expressions it is simpler, however, to pass to the related quaternary forms, which in this case are A_X^2, p_X^2, L_X. The required invariants are derived from the invariants and contravariants of the two quaternary quadrics, by means of the digredient invariants ‡ of $a_x^2\alpha_y^2$, $t_x\tau_y$. *The complete system of cogredient invariants of the quadri-quadric* $a_x^2\alpha_y^2$ is thus found to contain seven invariants:

$$\begin{aligned}
\phi' &= -(a\alpha)^2,\\
2I &= -(ab)^2(\alpha\beta)^2,\\
\psi' &= -(ab)(\alpha\beta)(a\alpha)(b\beta),\\
3J &= -(ab)(bc)(ca)(\alpha\beta)(\beta\gamma)(\gamma\alpha),\\
\chi' &= \{(ac)(\alpha\beta)(b\gamma)-(ab)(\alpha\gamma)(c\beta)\}^2,\\
24K &= \{(ac)(bd)(\alpha\beta)(\gamma\delta)-(ab)(cd)(\alpha\gamma)(\beta\delta)\}^2.\\
Z' &= \text{the transformed Jacobian of } \phi,\ \psi,\ \chi,\ \sigma'.
\end{aligned}$$

To illustrate the derivation consider the contravariant § $\psi = \frac{1}{2}(pABu)^2$ of the

* Transactions, vol. 1, p. 468.

† CLEBSCH, *Binäre Formen*, § 60. Cf. CAPELLI, *Sopra la corrispondenza* (2, 2), Giornale di Matematiche, vol. 17 (1879), p. 72.

‡ Transactions, vol. 1, pp. 477, 478.

§ ibid., p. 476.

quaternary quadrics A_X^2, p_X^2. This, on passing to the digredient theory, gives the invariant *

$$-(ab)(\alpha\beta)(at)(bt')(\alpha\tau)(\beta\tau')$$

of the double forms $a_x^2\alpha_y^2$, $t_x\tau_y$. Employing the relations (10) of § 1, namely

$$(at)(\alpha\tau)=(a\alpha), \qquad (bt')(\beta\tau')=(b\beta),$$

we derive the cogredient invariant

$$-(ab)(\alpha\beta)(a\alpha)(b\beta),$$

which is denoted by ψ' in the above list.

The non-symbolic values of I, J, K are given on p. 479 of the author's paper already cited; those of ϕ', ψ', χ' may be obtained by expanding the determinant:

$$\begin{vmatrix} a_{00} & a_{01}+a_{10} & a_{11}+\rho \\ a_{01}+a_{10} & 2a_{11}+a_{02}+a_{20}-2\rho & a_{12}+a_{21} \\ a_{11}+\rho & a_{12}+a_{21} & a_{22} \end{vmatrix} = 2\rho^3+\phi'\rho^2+\psi'\rho+\chi'.$$

§ 4. *The cogredient system for bilinear forms.*

The complete system of digredient comitants of any number of bilinear forms has been obtained by Peano.† Let the forms be

$$f_1 = a_{11}x_1y_1 + a_{12}x_1y_2 + a_{21}x_2y_1 + a_{22}x_2y_2 = a_x\alpha_y,$$
$$f_2 = b_{11}x_1y_1 + b_{12}x_1y_2 + b_{21}x_2y_1 + b_{22}x_2y_2 = b_x\beta_y, \text{ etc.}$$

Then the invariants are of the types:

$$I_{11} = (aa')(\alpha\alpha') = 2a_{11}a_{22} - 2a_{12}a_{21},$$
$$I_{12} = (ab)(\alpha\beta) = a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11},$$
$$K_{1234} = (ac)(bd)(\alpha\beta)(\gamma\delta) - (ab)(cd)(\alpha\gamma)(\beta\delta) = \begin{vmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \\ c_{11} & c_{12} & c_{21} & c_{22} \\ d_{11} & d_{12} & d_{21} & d_{22} \end{vmatrix},$$

* Ibid., p. 478.

† Giornale di Mathematiche, vol. 20 (1882), pp. 81–88. Cf. Transactions, vol. 1, pp. 444, 445.

and the covariants, in addition to $f_1, f_2, \cdots$, are of the types:

$$g_{123} = (ac)(\alpha\beta)b_x\gamma_y - (ab)(\alpha\gamma)c_x\beta_y = \begin{vmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \\ c_{11} & c_{12} & c_{21} & c_{22} \\ x_2y_2 & -x_2y_1 & -x_1y_2 & x_1y_1 \end{vmatrix},$$

$$L_{12} = (\alpha\beta)a_xb_x = (a_{11}b_{12} - a_{12}b_{11})x_1^2 + (a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11})x_1x_2 + (a_{21}b_{22} - a_{22}b_{21})x_2^2,$$

$$M_{12} = (ab)\alpha_y\beta_y = (a_{11}b_{21} - a_{21}b_{11})y_1^2 + (a_{11}b_{22} - a_{12}b_{21} - a_{21}b_{12} + a_{22}b_{11})y_1y_2 + (a_{12}b_{22} - a_{22}b_{12})y_2^2.$$

To find the cogredient system it is merely necessary to adjoin the fundamental bilinear form

$$T = t_x\tau_y = x_1y_2 - x_2y_1$$

(which it will be convenient to denote here by f_0) to the bilinear forms $f_1, f_2, \cdots$ above. The comitants of the enlarged system, in addition to the types written above, give certain new types, which may be obtained either symbolically or through the non-symbolic expressions. Thus the invariant of f_1, f_0 is $(at)(\alpha\tau)$, which, by the relations (10) § 1, reduces to $(a\alpha)$. The same invariant is obtained from the non-symbolic form

$$a_{11}t_{22} - a_{12}t_{21} - a_{21}t_{22} + a_{22}t_{11}$$

by introducing the special values of the coefficients of T

$$t_{11} = 0, \qquad t_{12} = 1, \qquad t_{21} = -1, \qquad t_{22} = 0,$$

namely,

$$a_{12} - a_{21}.$$

The two new types of invariants are thus found to be

$$I_{10} = (a\alpha) = a_{12} - a_{21},$$

$$K_{1230} = (ac)(b\gamma)(\alpha\beta) - (ab)(c\beta)(\alpha\gamma) = \begin{vmatrix} a_{11} & a_{12}+a_{21} & a_{22} \\ b_{11} & b_{12}+b_{21} & b_{22} \\ c_{11} & c_{12}+c_{21} & c_{22} \end{vmatrix};$$

and the new types of covariants,

$$g_{120} = -(ab)\alpha_x\beta_y - (\alpha\beta)a_yb_x = \begin{vmatrix} a_{11} & a_{12}+a_{21} & a_{22} \\ b_{11} & b_{12}+b_{21} & b_{22} \\ -x_2y_2 & x_1y_2+x_2y_1 & -x_1y_1 \end{vmatrix},$$

$$L_{10} = a_x\alpha_x = a_{11}x_1^2 + (a_{12}+a_{21})x_1x_2 + a_{22}x_2^2,$$

$$M_{01} = a_y\alpha_y = a_{11}y_1^2 + (a_{12}+a_{21})y_1y_2 + a_{22}y_2^2.$$

The complete system of cogredient comitants for any number of bilinear forms consists of the invariants of types

$$I_{10}, \quad I_{11}, \quad I_{12}, \quad K_{1230}, \quad K_{1234},$$

and the covariants of types

$$f_1, \quad L_{10}, \quad M_{01}, \quad L_{12}, \quad M_{12}, \quad g_{120}, \quad g_{123}.$$

Geometrically, a bilinear form represents a $(1, 1)$ correspondence or homography, and from this point of view such forms have been extensively studied.* The above system of invariants is thus fundamental in the theory of sets of homographies with superposed carriers. As regards the special interpretations mentioned in the introduction, a bilinear form represents a conic on the quadric surface or a circle in the inversion geometry of the plane. From the latter point of view the vanishing of the invariants above may be interpreted as relations between the set of circles $f_1, f_2, \cdots$ and the fundamental circle T.†

§ 5. *The group of a bilinear form.*

It has been seen in § 1 that the cogredient group may be regarded as a subgroup of the digredient group, namely, as that subgroup for which the special bilinear form $T = x_1 y_2 - x_2 y_1$ is invariant except for a factor. We proceed now to the subgroup defined by the general bilinear form

$$S = s_x \sigma_y = s_{11} x_1 y_1 + s_{12} x_1 y_2 + s_{21} x_2 y_1 + s_{22} x_2 y_2,$$

with non-vanishing determinant

$$I = \tfrac{1}{2}(ss')(\sigma\sigma') = s_{11} s_{22} - s_{12} s_{21},$$

and to the theory of invariants founded upon this subgroup.

The general digredient transformation involves six parameters, but since S is to be transformed except for a factor into itself, only three arbitrary constants remain; the group under consideration therefore involves three parameters and may be denoted by H_3.‡

---

* Most of the comitants, but not the complete system, are discussed from this point of view by : STEPHANOS, *Mémoire sur la représentation des homographies binaires*, etc., Mathematische Annalen, vol. 25 (1883), pp. 299–360 ;

BATTAGLINI, *Sulle forme binaire bilineari*, Giornale di Matematiche, vol. 25 (1887), pp. 281–297 ;

ANDOYER, *Théorie des formes*, vol. 1 (1900), p. 77.

† Cf. Transactions, vol. 1, p. 447.

‡ The theory of the transformations H_3, in invariant form, can be elaborated along lines similar to those employed by STUDY in his treatment of the automorphic transformations of a quadric, Leipziger Berichte, vol. 49 (1897), p. 454.

Since neither T nor S has a vanishing determinant, they are equivalent in the digredient group; there exists then a transformation Ω of the group G_6 which transforms the one into the other, or, in symbols,

$$T_\Omega = S.$$

It follows that the groups for which T and S respectively are automorphic, are also equivalent, and that H_3 is the transform of G_3 by Ω,

$$\Omega^{-1} G_3 \Omega = H_3.$$

From this equivalence of the groups results the equivalence of the corresponding invariant theories: the comitants with respect to the group H_3 may be obtained from the cogredient comitants by means of the transformation Ω, and conversely the latter comitants may be obtained from the former by means of the inverse transformation Ω^{-1}.

The fundamental symbolic types for the cogredient theory are given in (6), § 1; in the correspondence just indicated the analogue of $(a\alpha)$ is

$$[a\alpha] = (as)(\alpha\sigma).$$

Therefore

The fundamental symbolic types in the invariant theory of the group H_3 are

$$(ab), \qquad (\alpha\beta), \qquad [a\alpha] = (as)(\alpha\sigma),$$

with corrresponding types for the covariants.

The types are all seen to be digredient comitants after the adjunction of S to the forms considered.

The totality of comitants of any set of double binary forms with respect to the group H_3 may be obtained by adjoining S and treating the enlarged system digrediently.

§ 6. *Triple binary forms.*

We shall consider next in order forms involving three binary variables $x_1 : x_2$, $y_1 : y_2$, $z_1 : z_2$, represented symbolically * by

$$(1) \qquad f = a_x^l \alpha_y^m A_z^n = b_x^l \beta_y^m B_z^n = \cdots.$$

The independent transformations of the three variables now form a nine-parameter group G_9, while the cogredient group as before is a G_3 involving three

* Throughout the section the symbolic coefficients connected with the variables x, y, z are denoted by small italics, small Greek, and italic capitals respectively.

parameters. The cogredient transformations leave invariant the fundamental bilinear forms:

$$
\begin{aligned}
R &= x_1 y_2 - x_2 y_1 = r_x \rho_y, \\
S &= x_1 z_2 - x_2 z_1 = s_x S_z, \qquad (2)\\
T &= y_1 z_2 - y_2 z_1 = \tau_y T_z.
\end{aligned}
$$

In defining G_3 as a subgroup of G_9 it is, however, not necessary to include all three of these forms; any two are sufficient. Any one of the three is in fact a covariant of the other two. Thus the transvectant of R and S, with respect to x, the variable common to both,

$$(3) \qquad [R, S] = (rs)\rho_y S_z$$

reduces to $y_1 z_2 - y_2 z_1$, that is to T.

It may be proved, by a method entirely analogous to that employed for the double forms, that if the three bilinear forms are adjoined to f the digredient theory of the enlarged system

$$(4) \qquad f, R, S, T$$

is equivalent, with restrictions similar to those stated in the theorems of § 2, to the cogredient theory of the single form f. The relations between the symbolic types of the two theories are:

$$(5) \quad (a\alpha) = (ar)(\alpha\rho), \qquad (aA) = (as)(AS), \qquad (\alpha A) = (\alpha\tau)(AT),$$

with similar results for the covariant types. Since, however, T is a (digredient) comitant of R and S, it may be omitted from the system (4), i. e., the system (4) is digrediently equivalent to the reduced system

$$(6) \qquad f, R, S.$$

The cogredient theory of any set of triple binary forms is (roughly) equivalent to the digredient theory of the enlarged set formed by adjoining a pair of bilinear forms.

As to the symbolic relations, the first two of the set (5) remain unaltered, but in the third it is necessary to eliminate T. This can be effected by employing the result established above,

$$T = \tau_y T_z = (rs)\rho_y S_z.$$

From this it follows that

$$(\alpha\tau)(AT) = (rs)(\alpha\rho)(AS).$$

The fundamental relations for the invariant types are then,

(7) $(a\alpha) = (ar)(\alpha\rho), \quad (aA) = (as)(AS), \quad (\alpha A) = (rs)(\alpha\rho)(AS).$

Similarly, those for the covariant types are

$$a_y = (ra)\rho_y, \qquad \alpha_x = (\rho\alpha)r_x,$$
$$a_z = (sa)S_z, \qquad A_x = (SA)s_x,$$
$$\alpha_z = (rs)(\rho\alpha)S_z, \qquad A_y = (rs)(SA)\rho_y.$$

§ 7. *Multiple binary forms.*

In the discussion of the general case, where the forms involve any number of binary variables, it will be convenient to distinguish the several variables by accents, instead of using distinct letters as in the case of the double and the triple forms. Let the ν binary variables be

$$x_1' : x_2', \; x_1'' : x_2'', \; \cdots, \; x_1^{(\nu)} : x_2^{(\nu)},$$

and the corresponding symbolic coefficients,

$$A_1' : A_2', \; A_1'' : A_2'', \; \cdots, \; A_1^{(\nu)} : A_2^{(\nu)},$$

so that the general ν-tuple form is represented by

(1) $$f = A_{x'}'^{n_1} A_{x''}''^{n_2} \cdots A_{x^{(\nu)}}^{(\nu)n_\nu}.$$

The digredient group contains 3ν parameters, and the cogredient three; they may be denoted by $G_{3\nu}$ and G_3, respectively. The cogredient transformations leave invariant the following $\frac{1}{2}\nu(\nu - 1)$ forms:

(2) $$T_{ik} = (x^{(i)}x^{(k)}) = x_1^{(i)}x_2^{(k)} - x_2^{(i)}x_1^{(k)} = t_{x^{(i)}}^{(ik)}\tau_{x^{(k)}}^{(ik)}.$$

where (ik) denotes merely a double superscript. If the transvectant of any two of these forms which have one index in common is formed with respect to the common variable, the result is one of the forms of the same set. For the transvectant

$$[T_{ij}, T_{kj}] = (\tau^{(ij)}\tau^{(kj)})t_{x^{(i)}}^{(ij)}t_{x^{(k)}}^{(kj)},$$

on evaluation, reduces to $x_1^{(i)}x_2^{(k)} - x_2^{(i)}x_1^{(k)}$; therefore

(3) $$[T_{ij}, T_{kj}] = T_{ik}.$$

The set of $\frac{1}{2}\nu(\nu - 1)$ bilinear forms T_{ik} is thus a closed system, in the sense that the process of transvection, applied to forms having a variable in common, yields no additional forms.

The fundamental theorem connecting the two theories of multiple binary forms may now be stated as follows:

The cogredient comitants of any set of ν-tuple forms may be obtained from the digredient comitants of the enlarged set formed by adjoining the $\nu - 1$ bilinear forms $T_{12}, T_{13}, \cdots, T_{1\nu}$.

The proof follows the same lines as in the preceding section. In the first place the above relation holds when all the forms T_{ik} are adjoined. For then the fundamental symbolic cogredient comitants of the original forms f may be expressed as digredient comitants in the enlarged set f, T_{ik} as follows:

$$(A^{(i)}A^{(k)}) = (A^{(i)}t^{(ik)})(A^{(k)}\tau^{(ik)}),$$

$$A^{(i)}_{x^{(k)}} = (t^{(ik)}A^{(i)})\tau^{(ik)}_{x^{(k)}}.$$

In the second place, from (3), it follows that all the forms T_{ik} not belonging to the set

$$T_{12}, T_{13}, \cdots, T_{1\nu},$$

may be expressed as transvectants of this reduced set. Therefore the system f, T_{ik} may be reduced to f, $T_{12}, T_{13}, \cdots, T_{1\nu}$, the two being equivalent in the digredient theory. This completes the proof of the above theorem.

The fundamental symbolic relations are for the invariants,

$$(A'A^{(i)}) = (A't^{(1i)})(A^{(i)}\tau^{(1i)}),$$

$$(A^{(i)}A^{(k)}) = (t^{(1i)}t^{(1k)})(A^{(i)}\tau^{(1i)})(A^{(k)}\tau^{(1k)}),$$

and for the covariants,

$$A'_{x^{(i)}} = (t^{(1i)}A')\tau^{(1i)}_{x^{(i)}},$$

$$A^{(i)}_{x'} = (\tau^{(1i)}A^{(i)})t^{(1i)}_{x'},$$

$$A^{(i)}_{x^{(k)}} = (t^{(1i)}t^{(1k)})(\tau^{(1i)}A^{(i)})\tau^{(1k)}_{x^{(k)}},$$

where $i \neq k$, and both i and k are distinct from unity.

In the case of double forms there is only one bilinear form T, and by transformation the results obtained could be extended to the group for which an arbitrary bilinear form is automorphic (§ 5). If in the present general case we apply an arbitrary transformation of the group $G_{3\nu}$, i. e., independent linear transformations of the ν variables, it is possible to derive from $T_{12}, T_{13}, \cdots, T_{1\nu}$ an arbitrary set of $\nu - 1$ bilinear forms in the pairs of variables indicated by the subscripts, since in each form there is present a variable not occurring in any of the others. The forms resulting from the transformation of the remaining forms T_{ik} are then completely determined by the condition that they must be the transvectants of the set of $\nu - 1$ forms just considered. Applying the theorem above concerning the set T_{ik} we have the following result.

Consider any set of $\nu - 1$ *bilinear forms* $S_{12}, S_{13}, \cdots, S_{1\nu}$, *in which the subscripts indicate the two binary variables occurring in the form. If to these are added the transvectants of the forms taken two at a time, the result will be a closed system of* $\frac{1}{2}\nu(\nu - 1)$ *forms, in the sense that no new forms can be obtained by applying the process of transvection. Such a closed system* S_{ik} *can be transformed into itself by a triple infinity of digredient transformations forming a group* G'_3 *isomorphic to the cogredient group* G_3.

COLUMBIA UNIVERSITY, NEW YORK.