ON THE ENVELOPE OF THE AXES OF A SYSTEM OF CONICS PASSING THROUGH THREE FIXED POINTS*

BY

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In a recent number of the Annals of Mathematics† I have shown that the envelope of the asymptotes of a system of conics passing through three fixed points consists of two three-cusped hypocycloids, touching the three straight lines that join the three fixed points in pairs. I propose now to show that the envelope of the axes of the same system of conics consists of two three-cusped hypocycloids touching three concurrent straight lines.

The foci of a conic may be regarded as four of the vertices of a complete four-side circumscribing the conic, the other two vertices being the circular points at infinity; then the straight line at infinity is one diagonal line of this four-side, and the axes are the other two diagonal lines.

The coördinates of the circular points at infinity are $(1, -e^{Ci}, -e^{-Bi})$ and $(1, -e^{-Ci}, -e^{Bi})$; let us denote these points for the present by (x_1, y_1, z_1) and (x_2, y_2, z_2).

Let the equation of the conic be

$$U \equiv \lambda_1 yz + \lambda_2 zx + \lambda_3 xy = 0,$$

the three fixed points through which the conic is to pass being the vertices of the triangle of reference; and put

$$U_1 \equiv \lambda_1 y_1 z_1 + \lambda_2 z_1 x_1 + \lambda_3 x_1 y_1; \quad U_2 = \lambda_1 y_2 z_2 + \lambda_2 z_2 x_2 + \lambda_3 x_2 y_2;$$

$$U_1' \equiv x_1 \frac{\partial U}{\partial x} + y_1 \frac{\partial U}{\partial y} + z_1 \frac{\partial U}{\partial z}; \quad U_2' = x_2 \frac{\partial U}{\partial x} + y_2 \frac{\partial U}{\partial y} + z_2 \frac{\partial U}{\partial z}.$$

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† *On some curves connected with a system of similar conics*, Annals of Mathematics, 2d series, vol. 3 (1902), p. 154.

The equations of the tangents from the circular points at infinity are

$$U_1'^2 = 4UU_1$$

and

$$U_2'^2 = 4UU_2;$$

and the foci are the intersections of these two pairs of straight lines.

The equation $U_2 U_1'^2 = U_1 U_2'^2$ evidently represents a third pair of straight lines passing through the foci, and must therefore represent the axes.

Now the condition for similarity may be expressed in the form *

$$\sum(\lambda_1^2 \sin^2 A - 2\lambda_2\lambda_3 \sin B \sin C) = t^2(\lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C)^2,$$

where t is the tangent of the angle between the asymptotes; or,

$$\sum(\lambda_1^2 - 2\lambda_2\lambda_3 \cos A) = s^2(\lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C)^2,$$

that is,

$$U_1 U_2 = s^2 P^2,$$

where s is the secant of the angle between the asymptotes, and

$$P = \lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C.$$

Hence the equations of the axes may be expressed in the form

$$U_1 U_2' = sPU_1', \qquad \text{or} \qquad U_2 U_1' = sPU_2';$$

and

$$U_1 U_2' = -sPU_1', \qquad \text{or} \qquad U_2 U_1' = -sPU_2'.$$

Using the first of these equations, we may write the tangential coördinates of the corresponding axis in the form

$$u = U_1(\lambda_2 z_2 + \lambda_3 y_2) - sP(\lambda_2 z_1 + \lambda_3 y_1),$$
$$v = U_1(\lambda_3 x_2 + \lambda_1 z_2) - sP(\lambda_3 x_1 + \lambda_1 z_1),$$
$$w = U_1(\lambda_1 y_2 + \lambda_2 x_2) - sP(\lambda_1 y_1 + \lambda_2 x_1).$$

Noticing that

$$\lambda_1(y_1 z_2 + y_2 z_1) + \lambda_2(z_1 x_2 + z_2 x_1) + \lambda_3(x_1 y_2 + x_2 y_1) = -2P,$$

we have, on multiplying these equations first by x_1, y_1, z_1 and adding, and then by x_2, y_2, z_2, and adding,

$$V \equiv x_1 u + y_1 v + z_1 w = -2PU_1 - 2sPU_1,$$
$$W = x_2 u + y_2 v + z_2 w = 2U_1 U_2 + 2sP^2.$$

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*See the paper entitled, *On some curves, etc.*, referred to above.

Hence, taking account of the relation $U_1U_2 = s^2P^2$, we find,

$$P = \frac{\sqrt{W}}{\sqrt{2s(s+1)}}, \qquad U_1 = -\frac{V\sqrt{s}}{\sqrt{2(s+1)W}}.$$

Now writing the coördinates in the form

$$\begin{aligned} u &= (z_2U_1 - z_1sP)\lambda_2 + (y_2U_1 - y_1sP)\lambda_3, \\ v &= (z_2U_1 - z_1sP)\lambda_1 + (x_2U_1 - x_1sP)\lambda_3, \\ w &= (y_2U_1 - y_1sP)\lambda_1 + (x_2U_1 - x_1sP)\lambda_2, \end{aligned}$$

and using the equation

$$P = \lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C,$$

we have, on eliminating λ_1, λ_2, λ_3 the relation

$$\begin{vmatrix} u & 0 & z_2U_1 - z_1sP & y_2U_1 - y_1sP \\ v & z_2U_1 - z_1sP & 0 & x_2U_1 - x_1sP \\ w & y_2U_1 - y_1sP & x_2U_1 - x_1sP & 0 \\ P & \cos A & \cos B & \cos C \end{vmatrix} = 0.$$

On substituting the values of P and U_1 found above, and reducing by means of the relations

$$x_2V + x_1W = 2(u - v\cos C - w\cos B), \text{ etc.},$$

we finally obtain the equation of the envelope in the form

$$\begin{vmatrix} u & 0 & u\cos B + v\cos A - w & w\cos A + u\cos C - v \\ v & u\cos B + v\cos A - w & 0 & v\cos C + w\cos B - u \\ w & w\cos A + u\cos C - v & v\cos C + w\cos B - u & 0 \\ 1/(s+1) & \cos A & \cos B & \cos C \end{vmatrix} = 0,$$

or

$$\sum \left[u(v\cos C + w\cos B - u)\{u\cos(B - C) - v\cos B - w\cos C\} \right]$$

$$-\frac{2}{s+1}(v\cos C + w\cos B - u)(w\cos A + u\cos C - v)(u\cos B + v\cos A - w) = 0.$$

It may be shown that this curve has the straight line at infinity for a double tangent, the circular points at infinity being the points of contact.

It must therefore be of the fourth order and have three cusps; and hence for all values of s (except $s = -1$) it is a three-cusped hypocycloid.

It may easily be shown that it always touches the perpendicular bisectors of the sides of the triangle of reference; in the special case, $s = -1$, the curve degenerates into the points at infinity on these three lines.

The two axes envelope the same curve only in the case of the equilateral hyperbola, for which $s = \infty$.

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