QUADRIC SURFACES IN HYPERBOLIC SPACE*

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Introduction.

The object of this paper is to classify quadric surfaces in a three-dimensional space of constant negative curvature, and to exhibit some of their more striking metrical properties. The system of coördinates used is the usual projective one, and, as we shall always take a tetrahedron of reference self-conjugate with regard to the Absolute, the equation of the latter will appear in the form

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0.$$

The first broad line of demarcation is naturally between those surfaces which have a vanishing discriminant, and those which have a non-vanishing one. We shall occupy ourselves only with surfaces of the latter type.

The next great division is into ruled surfaces, non-ruled ones, and surfaces whose equations are definite forms. This classification depends upon the signs of the discriminant and its leading minors, according to certain familiar principles.

We then come to the metrical properties of quadrics. These arise from the various possible relations to the Absolute, so that the next problem is that of the classification of the mutual relations of two quadrics. This has been successfully solved by a number of writers, in particular, by HESSE † and CLEBSCH.‡ Yet from the strict point of view of a dweller in a hyperbolic space this last classification is not wholly satisfactory, for it is at once too inclusive and too ill-defined. It is too inclusive, for we are interested only in real surfaces, and must distinguish between real and imaginary curves of intersection. The Absolute has no real generators, so that we must exclude those cases where the two surfaces cut in one or more generators. It is not sharp enough, for if we search for the shape of our surface, we wish to know whether the curve which it cuts from the Absolute—the absolute curve, let us say—and the corresponding focal developable are real or imaginary; whether there be real vertices to the common self-conjugate tetrahedron; etc. Furthermore, we care not at all for those surfaces, analyt-

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† HESSE, *Vorlesungen über Geometrie des Raumes*, third ed., supplement 4, section 4.

‡ CLEBSCH, *Vorlesungen über Geometrie*, vol. 2, 2te Abtheilung, sections 10 and 12.

ically real, which lie entirely in the ultra-infinite or ideal portion of space. Nevertheless, it will be convenient to take CLEBSCH's classification as the basis of our work, and to use his numbering of the various cases for convenience of reference.

§ 1. *Central quadrics.*

Let us begin with case 1 (CLEBSCH) where the absolute curve is a twisted quartic. There are four cones, called the central cones, passing through this curve, whose vertices may all be real, or two may be conjugate imaginary. The important point is that they can not all be imaginary. Consider the question as follows. The vertices of the cones are also the vertices of the common self-conjugate tetrahedron of the surface and Absolute. If two vertices be conjugate imaginary points, the line joining them will be real, and the opposite edge, the polar of the first, will be real also. Now if we have two mutually polar lines with regard to a non-ruled quadric such as the Absolute, the one will fail to meet it in real points and, hence, bear an elliptic involution of conjugate points. This involution must have a real pair common with the involution of points conjugate with regard to the other surface * and these will be the two real vertices on that line. We shall at first restrict ourselves to the case where the four vertices are real. If O be such a vertex and a line through it meet the absolute in QQ_1 and the given surface in PP_1, we see, using VON STAUDT's symbol of projectivity, $QOQ_1P \barwedge Q_1OQP_1 \barwedge QP_1Q_1O$. Remembering CAYLEY's projective definition of distance, we see from this that $OP = P_1O$ or O is a centre to the surface. The surface has one actual, and three ideal centres. The opposite planes are planes of symmetry.

Let us now be more specific, and call those quadrics which cut the Absolute in a real curve, hyperboloids, and those which fail to meet it ellipsoids. We shall need to consider, not only the curve, but the focal developable. An ellipsoid is actual only when the developable is imaginary ; in the case of the hyperboloid, the nature of the developable determines the shape of the surface. A non-ruled hyperboloid is concave towards its centre when the developable is real, otherwise convex ; a ruled hyperboloid is two-sheeted in the former, and one-sheeted in the latter case. These indications will enable us to write the equations of the five central surfaces.

Ellipsoid,

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 - a_4x_4^2 = 0, \qquad a_1 > a_2 > a_3 > a_4 > 0.$$

Concave, non-ruled, hyperboloid,

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 - a_4x_4^2 = 0, \qquad a_1 > a_2 > a_4 > a_3 > 0.$$

* VON STAUDT, *Beiträge zur Geometrie der Lage*, p. 52.

Convex, non-ruled, hyperboloid,

$$a_1x_1^2 - a_2x_2^2 - a_3x_3^2 - a_4x_4^2 = 0, \qquad a_1 > a_4 > a_2 > a_3 > 0.$$

Two sheeted ruled hyperboloid,

$$a_1x_1^2 + a_2x_2^2 - a_3x_3^2 - a_4x_4^2 = 0, \qquad a_1 > a_4 > a_2 > 0.$$

One sheeted ruled hyperboloid,

$$a_1x_1^2 + a_2x_2^2 - a_3x_3^2 - a_4x_4^2 = 0, \qquad a_1 > a_2 > a_4 > 0.$$

What are the Cayleyan characteristics of the twisted quartic?* The osculating developable is of class 12 and order 8; there are 16 stationary planes, 38 lines in every plane which lie in two osculating planes, 2 lines meeting the curve twice pass through every point, 16 points in every plane lie on two tangents to the curve, and 8 planes through every point touch the curve twice. In interpreting these, let us remember that a conic having double contact with the Absolute comes under the general head of circle. If the chord of contact be ideal, the centre is actual, and we have a proper circle; if the chord be actual, the centre is ideal, and the curve is the locus of points at a fixed distance from the chord, an equi-distant curve, let us say. When the chord is tangent to the Absolute, the curve has four-point contact, and is called a horocycle; it is an orthogonal trajectory of a set of parallel lines. With these facts in view we see that:

Through every point will pass 12 planes cutting the surface in osculating parabolas;†

Through every line will pass 8 planes of parabolic section;

16 planes cut the surface in horocycles;

Every point is the centre of one section;

16 points in every plane are the centres of circular sections;

Through every point pass 8 planes of circular section.

Of these statistics, the most interesting are the ones that deal with cyclic and horocyclic sections. The circular sections are in planes tangent to the four central cones, the ray of tangency forming the chord of contact of circle and Absolute. Let us look more closely at the question of real and imaginary circles, for only the former possess any geometrical interest. To be precise we will start with the following hyperboloid:

$$a_1x_1^2 + a_2x_2^2 - a_3x_3^2 - a_4x_4^2 = 0, \qquad a_1 > a_2 > a_4 > 0.$$

*SALMON, *Geometry of Three Dimensions*, p. 312.

†STORY, American Journal of Mathematics, vol. 5 (1882), p. 358, calls this curve a "semi-circular parabola."

The equations of the four central cones are:

$$1) \qquad (a_1 - a_4)x_1^2 + (a_2 - a_4)x_2^2 - (a_3 + a_4)x_3^2 = 0,$$

vertex at $(0, 0, 0, 1)$, surrounds $(0, 0, 1, 0)$;

$$2) \qquad (a_1 + a_3)x_1^2 + (a_2 + a_3)x_2^2 - (a_3 + a_4)x_4^2 = 0,$$

vertex at $(0, 0, 1, 0)$, surrounds $(0, 0, 0, 1)$;

$$3) \qquad (a_1 - a_2)x_1^2 - (a_2 + a_3)x_3^2 - (a_4 - a_2)x_4^2 = 0,$$

vertex at $(0, 1, 0, 0)$, surrounds $(0, 0, 1, 0)$;

$$4) \qquad (a_2 - a_1)x_2^2 - (a_3 + a_1)x_3^2 - (a_4 - a_1)x_4^2 = 0,$$

vertex at $(1, 0, 0, 0)$, surrounds $(0, 0, 0, 1)$;

for each cone surrounds a vertex opposite to a face it cuts in imaginary lines. If two cones surround one another's vertices, every ray of each meets two rays of the other. This is the case with cones 1 and 2, hence every ray of 2 meets the Absolute in real points. A plane tangent to either of these cones will cut the surface in an equidistant curve. Cone 3 has real rays in $x_4 = 0$, hence some of its rays fail to meet the Absolute in real points. In the plane $x_3 = 0$ cone 4 has the lines

$$(x_2\sqrt{a_1 - a_2} + x_4\sqrt{a_1 - a_4})(x_2\sqrt{a_1 - a_2} - x_4\sqrt{a_1 - a_4}) = 0,$$

which are both ideal. Some rays of this cone also do not meet the Absolute.

Let us next consider the point $(0, 1, 0, 1)$. It lies without cones $1, 2, 3$, but within cone 4 and is on the Absolute. Then every point on the Absolute will be situated in that same way, or else be within the first three cones, and without the last one; for the question of externality is not altered as we move about the Absolute until we cross the curve, when the relation to each cone is reversed. There are also, clearly, points within the Absolute, actual points, which are without three cones, and within the fourth; but none which are without all four. For if there were such a point, we could pass thence to any other point outside all the cones, say a point in the close vicinity of $(1, 0, 0, 0)$ without ever being inside any cone, but this is impossible as at some time during our journey, we should have to pierce the Absolute. There are, however, actual points within all four cones. It is noticeable, by similar reasoning, that no actual point is without both cones 3 and 4. Summing up, we reach this theorem:

The maximum number of real planes of circular section through an actual point is six; in only two of these can we have proper circles.

It will be found that a like result holds in the case of the other hyperboloids. For the ellipsoid but two of the cones are real, one surrounding all actual points, the other surrounding none. There are thus two planes of circular section through each actual point, and these cut the surface in proper circles.

Let us now look at the sixteen horocyclic sections. The points of osculation with the absolute curve lie by fours in the faces of the tetrahedron.* Two faces will cut the curve in imaginary points and two in real ones, so that the hyperboloid has eight real horocyclic sections. The ellipsoid, naturally, has none.

A system of quadrics touching the same focal developable may properly be called a confocal system. The theorems connected therewith bear the closest analogy to the corresponding ones in Euclidean geometry. The quadrics touching the four planes of symmetry degenerate into conics, called the focal conics; each passes through two foci of the other. When the focal developable is imaginary, we may employ the usual proceeding and show that through each actual point will pass an ellipsoid, a ruled, and a non-ruled hyperboloid. When the developable is real we are driven to employ some other procedure, for instance the following. Through an actual point will pass two pairs of conjugate imaginary planes touching all of the confocal surfaces, hence there are three real mutually perpendicular planes which are conjugate with regard to the whole system. The generators at this point of the three quadrics meeting there are the pairs of intersections of the four planes first mentioned, only one pair being real. This shows that when the developable is real, through an actual point will pass one ruled and two non-ruled hyperboloids.

Our general class of central surfaces includes also those with two imaginary centres. These centres will lie on a line cutting the surface, the Absolute, and the two real central cones in pairs of an elliptic involution. The absolute curve and focal developable will both be real. The central cones will bear to one another the relation of cones 3 and 4 in the preceding discussion, so that through no actual point may we pass more than two real planes of circular section. Three confocals will pass through a point, and two focal conics will be real, though in the ideal region.

To form the equation, we may assume that $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ are the poles of $x_1 = 0$ and $x_2 = 0$, respectively; furthermore, the surface must cut $x_3 = 0$, $x_4 = 0$ in a pair of points separated by $(0, 0, 1, 1)$, $(0, 0, 1, -1)$. We thus get

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 - 2b x_3 x_4 + a_4 x_4^2 = 0, \qquad (2b)^2 > (a_3 + a_4)^2.$$

The surface is ruled if

$$a_1 a_2 > 0.$$

* SCHRÖTER, *Raumkurven 4ter Ordnung*, p. 85.

§ 2. *Paraboloids.*

Under this general heading, we shall include all those surfaces which touch the Absolute. The first are those that come under case 2 and have a nodal quartic as their absolute curve. There will be hyperbolic and elliptic sub-cases, according as the rest of the curve is real or imaginary. The properties of the latter may be quickly deduced from those of the ellipsoid. Two real planes of circular section will pass through an actual point, there are no horocyclic sections, and the developable is imaginary.

The hyperbolic paraboloids may be ruled or otherwise. Two centres coalesce at the node, two others lie in the tangent plane there and are, hence, real. The node may be an intersection of two branches of the curve, or an isolated point, but this difference is immaterial for our purposes.

There will be, as before, a distinction between those surfaces which have a real, and those which have an imaginary focal developable; the nodal tangent plane counting double in either case. When the developable is real, we shall have three real focal conics, otherwise but two. Three confocal surfaces will pass through a point. To find the simplest form of equation, we may assume once more that the product terms in x_1 and x_2 vanish, and that the surface touches $x_3 - x_4 = 0$ at $(0, 0, 1, 1)$.

We easily get

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + (a_4 - a_3) x_3 x_4 - a_4 x_4^2 = 0.$$

The criteria for the various sub-cases are obtained by regarding the sign of the discriminant and the question of reality for the cone from the node to the absolute curve and the focal conic in the special tangent plane. We thus get:

Elliptic paraboloid,

$$\left.\begin{matrix} 2a_1 \\ 2a_2 \end{matrix}\right\} a_3 + a_4, \qquad a_3 > a_4 > 0.$$

Tubular, hyperbolic paraboloid, non-ruled,

$$a_1 a_2 > 0, \qquad a_4 > a_3 > 0, \qquad 2a_1 > a_3 + a_4.$$

Cup-shaped, hyperbolic paraboloid, non-ruled,

$$\left.\begin{matrix} a_1 \\ a_2 \end{matrix}\right\} < 0, \qquad a_3 > a_4 > 0.$$

Open, ruled, hyperbolic paraboloid,

$$a_3 > a_4 > 0, \qquad a_2 < 0 < 2a_1 < a_3 + a_4.$$

Gathered, ruled, hyperbolic paraboloid,

$$2a_1 > a_3 + a_4 > 0, \qquad a_3 > a_4 > 0 > a_2.$$

Somewhat different results will be obtained under case 3 where the absolute curve has a cusp. Both curve and developable are necessarily real, there are two horocycles, and two real focal conics. The equation may be written as that of the most general quadric cutting the Absolute in the same curve as a cone whose vertex is at $(0, 0, 1, 1)$, and which touches $x_3 - x_4 = 0$ along the intersection with $x_1 = 0$, the polar of $(1, 0, 0, 0)$. We obtain:

$$a_1 x_1^2 + x_2^2 + 2a_2 x_2 (x_3 - x_4) + (a_3 + 1) x_3^2 - 2a_3 x_3 x_4 + (a_3 - 1) x_4^2 = 0 .$$

Writing the discriminant, we see that the surface is ruled if $a_1 < 0$.

In case 7 the absolute curve is two conics touching one another. Sub-cases arise through the option of real or imaginary conics. When the conics are real, there is one real central cone, so that two circular sections may pass through an actual point, the curve being, however, strictly speaking, an equidistant one. When the conics are imaginary, the cone is still real, but surrounds all actual points, as we see by a continuous change to the case where the two conics coalesce; there are no real circular sections. The number of horocycles is, in either case, singly infinite. The focal developable becomes imaginary with the conics; also, when the conics are real, if the surface be ruled. The developable is, in fact, two quadric cones.

To write the equations of the horocyclic paraboloids, we may take the vertex of the one central cone at $(1, 0, 0, 0)$, and have the planes of the conics harmonically separated by $x_1 = 0$ and $x_3 - x_4 = 0$. So we get

$$a_1 x_1^2 + x_2^2 + (x_3 - x_4) [(1 + a) x_3 + (1 - a) x_4] = 0 .$$

The hyperbolic case will arise if $(1 - a_1)/a > 0$. The surface is ruled if $a_1 < 0$.

Closely allied to this is case 10 where the absolute curve is a conic and two imaginary straight lines meeting on it. The conic must be real, since its plane is real and actual; the surface cannot be ruled. There is a singly infinite set of horocyclic sections, but no circular ones. The focal developable will be a cone and two imaginary generators. The equation takes the simple form:

$$x_1^2 + x_2^2 + x_3^2 + r x_1 (x_3 - x_4) - x_4^2 = 0 .$$

§ 3. *Surfaces of revolution.*

The next case, 6, is where the absolute curve is two intersecting conics. We shall restrict ourselves at first to the case where the line of intersection of the planes of the conics is ideal. The sections in planes through this line will be circles. Their centres will lie on the polar of the line, and their planes will be perpendicular thereunto, so that the surface is one of revolution. There will

be no horocyclic sections, and the developable will be a pair of cones. There will be four sub-cases:

1) The planes of the conics are both ideal.
2) The planes of the conics are both imaginary.
3) The planes of the conics are both actual.
4) One plane is actual and one ideal.

The first two suppositions give us ellipsoids. There is, however, an important distinction which arises from the following consideration. In an ellipse, the real common chords with the Absolute are perpendicular to the minor axis. The first sub-case will then be the oblate spheroid, while the second is the prolate. In neither case are there additional circular sections.

Under the third supposition we may get both ruled and non-ruled hyperboloids and among the latter there will be some which are convex to the centre, while others are concave. The first are obtained by rotating a convex hyperbola about the conjugate axis, the second by rotating it about the transverse axis, and the third by rotating a concave hyperbola.

In writing the equations we may take $x_3 = 0$, $x_4 = 0$ as our ideal line. The typical equation is

$$x_1^2 + x_2^2 + a_3 x_3^2 - a_4 x_4^2 = 0.$$

The criteria are:

Prolate spheroid, $a_3 > 1 > a_4 > 0.$

Oblate spheroid, $1 > a_3 > a_4 > 0.$

Concave hyperboloid of revolution, $a_3 > a_4 > 1.$

Convex hyperboloid of revolution, $a_3 < a_4 < 0.$

Ruled hyperboloid of revolution, $a_3 < 0 < a_4 < 1.$

Sub-case 4 is a very different configuration. The surface can not be ruled, and we easily see that it is obtained by rotating a semi-hyperbola about its line of symmetry. The focal developable consists of two cones, one being real. In finding the equation, we may take $x_3 = 0$ as the plane of the real conic: the other plane is to meet it in the same line as $x_4 = 0$. This gives:

$$x_1^2 + x_2^2 + a_3 x_3^2 - b x_3 x_4 - x_4^2 = 0, \qquad b^2 > (a_3 - 1)^2.$$

There are some more special surfaces of revolution which naturally come before us at this point. In case 8 we have a conic and two intersecting lines as our absolute curve. The plane of the conic may be actual or ideal. The focal developable consists of a cone, together with the two lines. The surface is a paraboloid of revolution, obtained by rotating a parabola about its line of symmetry.

There will be elliptic and hyperbolic paraboloids; the latter divided between tubular ones and cup-shaped ones, for which last the Absolute touches an ideal sheet of the surface. These, like the elliptic ones, yield an imaginary developable. In finding the equation, we may assume that the absolute curve lies in the planes $x_3 - x_4 = 0$, $x_4 = 0$ for the elliptic case, and $x_3 - x_4 = 0$, $x_3 = 0$ for the other. Hence

$$x_1^2 + x_2^2 + x_3^2 + (a_4 - 1)x_3x_4 - a_4x_4^2 = 0,$$

$$x_1^2 + x_2^2 + a_3x_3^2 + (1 - a_3)x_3x_4 - x_4^2 = 0.$$

For the elliptic paraboloid we must have $a_4^2 < 1$ to insure an actual surface. The hyperbolic paraboloid will be tubular if $1 > a_3 > 0$.

The last surface of revolution is the remarkable one coming under case 9. The absolute curve is two pairs of edges of a tetrahedron; a plane through one of the remaining real edges will cut the surface in a circle or equidistant curve. The surface may be obtained by rotating an equidistant curve about its line of symmetry, or moving a circle of constant radius along a line through its centre, perpendicular to its plane. It is the simplest type of canal surface, and is the exact analogue of Clifford's surface of parallel lines in elliptic space.* The equation is:

$$x_1^2 + x_2^2 + a(x_3^2 - x_4^2) = 0.$$

§ 4. *Surfaces of translation.*

The surface last considered serves as a natural bridge to our next class, where the planes of the two conics forming the absolute curve intersect in an actual line. Such a surface will be cut by every plane through this line in an equidistant curve, and is described by moving a conic of constant magnitude in such a way that its axes trace fixed planes whose intersection is ever perpendicular to the plane of the conic; the analogue, from one point of view, of a right cylinder in Euclidean space.

There will be elliptic and hyperbolic sub-cases. For the former we shall have two real central cones, whereof but one surrounds all actual points, so that through every actual point will pass two planes cutting the surface in proper circles. When the surface is hyperbolic there are two real cones situated like 2 and 4 in the discussion of § 1; no actual point lies without both, and they will yield, at best, two equidistant curves in planes through a chosen point.

In finding the equation, we may assume that the planes of the conics are harmonically separated by $x_1 = 0$, $x_2 = 0$, giving

$$a_1x_1^2 + a_2x_2^2 + x_3^2 - x_4^2 = 0.$$

*Cf. KLEIN, *Zur nicht-euklidische Geometrie*, Mathematische Annalen, vol. 37 (1890).

We shall have an ellipsoid if $a_1 > a_2 > 1$.

We shall have a hyperboloid if $a_1 > 1 > a_2$, ruled if $a_2 < 0$. The ruled surface will be obtained by transporting a convex hyperbola, otherwise we get the non-ruled surface.

§ 5. *Spheres.*

There remain to us only those surfaces which come under the general head of spheres. In case 12 the absolute curve is a conic counting twice. If the conic lie in an ideal plane, every section of the surface will be a circle, and we have a sphere whose centre is the pole of the plane. When the plane of the conic is actual, the perpendicular distance thence to the surface is constant, and we have an equidistant surface. The respective equations will be:

$$x_1^2 + x_2^2 + x_3^2 - a_4 x_4^2 = 0, \qquad 1 > a_4 > 0;$$

$$x_1^2 + x_2^2 + a_3 x_3^2 - x_4^2 = 0, \qquad a_3 > 1.$$

In case 13 the absolute curve is a pair of straight lines counted twice. Every plane through their intersection will cut the surface in a horocycle, other planes will cut it in circles. This is the horocyclic surface of zero curvature. Its equation will be:

$$x_1^2 + x_2^2 + (a+1)x_3^2 - 2ax_3x_4 + (a-1)x_4^2 = 0, \qquad a > 0.$$

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