ON THE POSSIBILITY OF DIFFERENTIATING TERM BY TERM THE DEVELOPMENTS FOR AN ARBITRARY FUNCTION OF ONE REAL VARIABLE IN TERMS OF BESSEL FUNCTIONS*

BY

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1. The developments for an arbitrary function $f(x)$ of the real variable x in terms of BESSEL's function $J_\nu(x)$ (ν real) may be classed into three general divisions as follows:

I.

$$\sum_1^\infty q_n J_\nu(\lambda_n x)$$

where

$$q_n = \frac{2}{J_\nu'^2(\lambda_n)} \int_0^1 x f(x) J_\nu(\lambda_n x)\, dx,$$

λ_n being one of the positive roots of the transcendental equation $J_\nu(x) = 0$.

II.

$$(2\nu + 2) \int_0^1 f(x) x^{\nu+1}\, dx + \sum_1^\infty q_n' J_\nu(\lambda_n' x)$$

where

$$q_n' = \frac{2}{J_\nu^2(\lambda_n')} \int_0^1 x f(x) J_\nu(\lambda_n' x)\, dx,$$

λ_n' being one of the positive roots of the transcendental equation

$$x J_\nu'(x) - \nu J_\nu(x) = 0.$$

III.

$$\sum_1^\infty q_n'' J_\nu(\lambda_n'' x)$$

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where

$$q''_n = \frac{2\lambda''^2_n}{\{h(2\nu + h) + \lambda''^2_n\} J^2_\nu(\lambda''_n)} \int_0^1 x f(x) J_\nu(\lambda''_n x)\, dx,$$

λ''_n being one of the positive roots of the transcendental equation

$$xJ'_\nu(x) - (h + \nu) J_\nu(x) = 0 \qquad (h \text{ real and } \neq 0).$$

With reference to these three developments it is our present purpose to determine a set of sufficient conditions for $f(x)$ under which the series obtained by differentiating series I or II term by term will converge to the limit $f'(x)$. The discussion naturally presupposes some facts concerning the convergence of the series in question, or of some related series, and thus we shall begin by stating the following established results: *

"If

$$(1) \qquad P_\nu(x) = \frac{J_\nu(x)}{x^\nu} = \frac{1}{2^\nu \Gamma(\nu + 1)} \left\{ 1 - \frac{x^2}{2(2\nu + 2)} + \frac{x^4}{2 \cdot 4 (2\nu + 2)(2\nu + 4)} - \cdots \right\}$$

and if $f(x)$ is an arbitrary function of the real variable x defined throughout the interval $0 \leqq x \leqq 1$ we shall have for any special value of x within an interval $(a', b')(0 < a' < b' < 1)$

$$(2) \qquad f(x) = \sum_1^\infty p_n P_\nu(\lambda_n x)$$

where

$$p_n = \frac{2}{P'^2_\nu(\lambda_n)} \int_0^1 f(x) x^{2\nu+1} P_\nu(\lambda_n x)\, dx,$$

λ_n being one of the positive roots of the transcendental equation $P_\nu(x) = 0$;

$$(3) \qquad f(x) = (2\nu + 2) \int_0^1 f(x) x^{2\nu+1} dx + \sum_1^\infty p'_n P_\nu(\lambda'_n x)$$

where

$$p'_n = \frac{2}{P^2_\nu(\lambda'_n)} \int_0^1 f(x) x^{2\nu+1} P_\nu(\lambda'_n x),$$

λ'_n being one of the positive roots of the transcendental equation $P'_\nu(x) = 0$;

$$(4) \qquad f(x) = \sum_1^\infty p''_n P_\nu(\lambda''_n x)$$

where

* These results in so far as they are independent of statements respecting *uniform* convergence may be found on pages 266, 267 of the *Serie di Fourier* of DINI, and I have reason to believe from a communication received from Professor DINI that the statements concerning uniform convergence have likewise been established by the same author, but remain as yet unpublished.

$$p''_n = \frac{2\lambda''^2_n}{\{h(2\nu+h)+\lambda''^2_n\}P^2_\nu(\lambda''_n)}\int_0^1 f(x)x^{2\nu+1}P_\nu(\lambda''_n x)\,dx,$$

λ''_n being one of the positive roots of the transcendental equation

$$xP'_\nu(x) - hP_\nu(x) = 0, \qquad (h \neq 0)$$

provided throughout that $\nu > -\frac{1}{2}$ and that $f(x)$ satisfies the following conditions.

"*Condition* (a): $f(x)$ when considered in the interval $0 \leqq x \leqq 1$ is finite and either continuous or made up of a finite number of continuous portions.

"*Condition* (b): $f(x)$ possesses finite first derivatives from the right and from the left at the point x.

"Also, the above statement is true when $-1 < \nu \leqq -\frac{1}{2}$ if in addition to these conditions we require that the function $|x^{2\nu}f(x)|$ be integrable in the neighborhood at the right of the point $x = 0$.

"Moreover, when $\nu > -\frac{1}{2}$ the series (2), (3) and (4) converge uniformly to the limit $f(x)$ when $a' < x < b'$ $(0 < a' < b' < 1)$ provided that the function $f(x)$ when considered in the interval $0 \leqq x \leqq 1$ satisfies condition (a), and when considered throughout the interval $a' \leqq x \leqq b'$ is continuous and possesses a finite first derivative from the right and from the left. And the same is true when $-1 < \nu \leqq -\frac{1}{2}$ provided that in addition to these requirements the function $|x^{2\nu}f(x)|$ is integrable in the neighborhood at the right of the point $x = 0$."

2. This premised, we shall now assume that we are dealing with a function $f(x)$ which satisfies condition (a), but instead of condition (b) it satisfies the following two conditions which place somewhat further restrictions upon it:

Condition (c): $f(x)$ when considered within the interval $0 < x < 1$ possesses a continuous derivative $f'(x)$ such that the function $|f'(x)|/x$ when considered in the neighborhood of the point $x = 0$ remains always less than a fixed constant c.

Condition (d): $f(x)$ possesses a finite second derivative from the right and from the left throughout the interval $a' \leqq x \leqq b'$.

Assuming then that $\nu > -1$ and that conditions (a), (c) and (d) are satisfied together with the condition when $-1 < \nu \leqq -\frac{1}{2}$ that the functions $|x^{2\nu}f(x)|$ and $|x^{2\nu-1}f'(x)|$ are integrable in the neighborhood at the right of the point $x = 0$, it is evident that for any special value of x such that $a' < x < b'$ condition (b) becomes satisfied so that in particular we shall have (2) for such a value of x. And, if we admit for the moment the possibility of differentiating the series term by term, we have for the same value of x

$$f'(x) = \sum_1^\infty p_n P'_\nu(\lambda_n x). \tag{5}$$

In order to justify (5) it suffices, as is well known,* to show that for the interval

* *Vid.* OSGOOD in American Journal of Mathematics, vol. 19, p. 155 *et seq.*

$a' < x < b'$ the series in (5) is uniformly convergent and we shall now show that this is the case when $f(x)$ satisfies the conditions which we have supposed, together with one other, viz., $f(1) = 0$. In passing, however, let us observe that from (1) we have

$$P_\nu'(\lambda_n x) = -\frac{\lambda_n^2 x}{2\nu + 2} P_{\nu+1}(\lambda_n x) \tag{6}$$

so that the series (5) may be written in the form

$$-\sum_1^\infty \frac{p_n \lambda_n^2 x}{2\nu + 2} P_{\nu+1}(\lambda_n x). \tag{7}$$

Now, utilizing the results stated at the beginning, we may write under the present hypothesis concerning $f'(x)$

$$\frac{f'(x)}{x} = \sum_1^\infty p_n'' P_{\nu+1}(\lambda_n'' x), \tag{8}$$

where

$$p_n'' = \frac{2\lambda_n''^2}{\{h(2\nu + 2 + h) + \lambda_n''^2\} P_{\nu+1}^2(\lambda_n'')} \int_0^1 f'(x) x^{2\nu+2} P_{\nu+1}(\lambda_n'' x)\, dx, \tag{9}$$

λ_n'' being one of the positive roots of the equation

$$x P_{\nu+1}'(x) - h P_{\nu+1}(x) = 0, \qquad (h \neq 0) \tag{10}$$

and from the results stated above we know that (8) holds uniformly when $a' < x < b'$. From (6) we have

$$P_{\nu+1}'(x) = -\frac{2\nu + 2}{x} P_\nu''(x) + \frac{2\nu + 2}{x^2} P_\nu'(x),$$

and hence (10) may be written

$$-P_\nu''(x) + \frac{1 + h}{x} P_\nu'(x) = 0, \tag{11}$$

so if we take $h = -2\nu - 2$ (which is consistent with $h \neq 0$ since $\nu > -1$) (10) reduces to

$$-P_\nu''(x) - \frac{2\nu + 1}{x} P_\nu'(x) = 0. \tag{12}$$

But from (1) we have

$$P_\nu''(x) + \frac{2\nu + 1}{x} P_\nu'(x) + P_\nu(x) = 0,$$

and hence (12) is equivalent to the equation $P_\nu(x) = 0$, so that having taken $h = -2\nu - 2$ we obtain a particular development of the form (4) in which

$\lambda''_n = \lambda_n$ and in which the coefficients p''_n as given by (9) reduce to the more simple form

$$p''_n = \frac{2}{P^2_{\nu+1}(\lambda_n)} \int_0^1 f'(x) x^{2\nu+2} P_{\nu+1}(\lambda_n x)\, dx,$$

or again, utilizing (6), to

$$p''_n = \frac{2\lambda_n^2}{(2\nu+2)^2 P'^2_\nu(\lambda_n)} \int_0^1 f'(x) x^{2\nu+2} P_{\nu+1}(\lambda_n x)\, dx. \tag{13}$$

In (13) let us now integrate once by parts, taking for this purpose

$$dv = f'(x)\, dx \qquad \text{and} \qquad u = x^{2\nu+2} P_{\nu+1}(\lambda_n x).$$

Then $v = f(x)$ and noting that

$$\frac{d}{dx}\left\{ x^{2\nu+2} P_{\nu+1}(x) \right\} = (2\nu+2) x^{2\nu+1} P_\nu(x)$$

we have $du = (2\nu+2) x^{2\nu+1} P_\nu(\lambda_n x)\, dx$, so that we may again write for p''_n

$$\begin{aligned} p''_n = {} & \frac{2\lambda_n^2}{(2\nu+2)^2 P'^2_\nu(\lambda_n)} \Big[x^{2\nu+1} f(x) P_{\nu+1}(\lambda_n x) \Big]_0^1 \\ & - \frac{2\lambda_n^2}{(2\nu+2) P'^2_\nu(\lambda_n)} \int_0^1 f(x) x^{2\nu+1} P_\nu(\lambda_n x)\, dx \\ = {} & \frac{2\lambda_n^2}{(2\nu+2)^2 P'^2_\nu(\lambda_n)} \Big[x^{2\nu+2} f(x) P_{\nu+1}(\lambda_n x) \Big]_0^1 - \frac{p_n \lambda_n^2}{2\nu+2}. \end{aligned}$$

Therefore, since $\nu > -1$, we have but to assume that $f(1) = 0$ in order to have the development (8) assume the form

$$\frac{f'(x)}{x} = -\sum_1^\infty \frac{p_n \lambda_n^2}{2\nu+2} P_{\nu+1}(\lambda_n x).$$

Thus the series (7) is a special form of the uniformly convergent series (8) and is therefore itself uniformly convergent ($a' < x < b'$).

Keeping the same hypotheses respecting ν and $f(x)$ we may show also that the series (3) when differentiated term by term will converge to the limit $f'(x)$ when $a' < x < b'$ ($0 < a' < b' < 1$).

We have, in fact, upon differentiating both members of (3)

$$f'(x) = \sum_1^\infty p'_n P'_\nu(\lambda'_n x) = -\sum_1^\infty \frac{p'_n \lambda'^2_n x}{2\nu+2} P_{\nu+1}(\lambda'_n x) \tag{14}$$

where the last series may be shown as follows to converge uniformly for

$$a' < x < b'.$$

From (6) the positive roots λ_n' which appear in (14) and which by hypothesis are roots of $P_\nu'(x) = 0$ are the same as the positive roots of the equation $P_{\nu+1}(x) = 0$, and hence by (2) and the results stated at the beginning, we have uniformly when $a' < x < b'$

$$\frac{f'(x)}{x} = \sum_1^\infty p_n P_{\nu+1}(\lambda_n' x), \tag{15}$$

where

$$p_n = \frac{2}{P_{\nu+1}'^2(\lambda_n')} \int_0^1 f'(x) x^{2\nu+2} P_{\nu+1}(\lambda_n' x)\,dx. \tag{16}$$

But

$$P_{\nu+1}(x) = (2\nu + 2)\frac{P_\nu'(x)}{x},$$

and hence

$$P_{\nu+1}'(x) = (2\nu + 2)\frac{P_\nu''(x)}{x} - (2\nu + 2)\frac{P_\nu'(x)}{x^2}.$$

Therefore

$$P_{\nu+1}'(\lambda_n') = (2\nu + 2)\frac{P_\nu''(\lambda_n')}{\lambda_n'};$$

or since in general

$$P_\nu''(x) + \frac{2\nu + 1}{x} P_\nu'(x) + P_\nu(x) = 0,$$

we may use the fact that $P_\nu''(\lambda_n') = -P_\nu(\lambda_n')$ and write

$$P_{\nu+1}'(\lambda_n') = -(2\nu + 2)\frac{P_\nu'(\lambda_n')}{\lambda_n'}.$$

Thus, formula (16) may be written

$$p_n = \frac{2\lambda_n'^2}{(2\nu + 2)^2 P_\nu^2(\lambda_n')} \int_0^1 f'(x) x^{2\nu+2} P_{\nu+1}(\lambda_n' x)\,dx,$$

and hence, with the present hypotheses concerning $f(x)$ we obtain, as in dealing with (13), the result that $p_n = -p_n'\lambda_n'^2/(2\nu + 2)$. Consequently the series (15) which we know is uniformly convergent for $a' < x < b'$ assumes the form

$$\frac{f'(x)}{x} = -\sum_1^\infty \frac{p_n'\lambda_n'^2}{2\nu + 2} P_{\nu+1}(\lambda_n' x)$$

from which the uniform convergence of the last series in (14) follows at once for the interval $a' < x < b'$.

Introducing into the developments (2) and (3) the function $J_\nu(x)$ instead of $P_\nu(x)$, recalling that $J_\nu(x) = x^\nu P_\nu(x)$, and applying our results to the function $x^{-\nu}f(x)$ instead of $f(x)$ we obtain the following

THEOREM: *Each of the series* I *and* II *converges, when* $a' < x < b'$ $(0 < a' < b' < 1)$, *to the limit* $f(x)$ *and each of the series obtained by differentiating these series term by term converges for the same values of* x *to the limit* $f'(x)$, *provided that* $\nu > -\frac{1}{2}$ *and that the function* $\phi(x) = x^{-\nu}f(x)$ *satisfies the following conditions:*

Condition A: $\phi(x)$ *when considered in the interval* $0 \leqq x \leqq 1$ *is finite and either continuous or made up of a finite number of continuous portions.*

Condition B: $\phi(x)$ *when considered in the interval* $0 < x < 1$ *possesses a continuous derivative* $\phi'(x)$ *such that the function* $|\phi'(x)|/x$ *when considered in the neighborhood of the point* $x = 0$ *is less than a fixed constant.*

Condition C: $\phi(x)$ *when considered in the interval* $a' \gtreqqless x \gtreqqless b'$ *possesses finite second derivatives from the right and from the left.*

Condition D: $\phi(1) = 0$.

Moreover, when $-1 > \nu \geqq -\frac{1}{2}$ the above theorem holds true if we require also that the functions $|x^\nu f(x)|$ and $|x^{\nu-1}f'(x)|$ be integrable in the neighborhood at the right of the point $x = 0$.

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