ON THE CLASS NUMBER OF THE CYLCLOTOMIC NUMBER FIELD

\[ k(e^{2\pi i/p^n})^* \]

BY

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Introduction. — The object of the present paper is to investigate the relation between the class numbers of the cyclotomic number fields \( k(e^{2\pi i/p^n}) \) and \( k(e^{2\pi i/p^{n-1}}) \) when \( p \) is any odd prime and \( n \geq 2 \). The method is similar to the method used by Weber \( \dagger \) for the case \( p = 2 \).

Let \( m = p^n, m' = p^{n-1}, \mu = \phi(m) = p^{n-1}(p - 1), \mu' = \phi(m') = p^{n-2}(p - 1) \),
\( r = e^{2\pi i/p^n}, r' = e^{2\pi i/p^{n-1}} \). Denote by \( h \) and \( h' \) the class numbers of \( k(r) \) and \( k(r') \) respectively, and set
\[ h = h' H. \]

We also set \( h = kh_1, h' = k'h_1' \), \( k = k'A, h_1 = h_1'B \) and hence
\[ H = AB, \]
where \( h_1 \) and \( h_1' \) are the class numbers of the real fields \( k(r + r^{-1}) \) and \( k(r' + r'^{-1}) \) respectively. Also let
\[ E = DE', \]
\( E \) and \( E' \) being the regulators of \( k(r + r^{-1}) \) and \( k(r' + r'^{-1}) \) respectively.

I. Expressions for \( A \) and \( B \).

If we set \( \theta = e^{2\pi i/\mu} \) and \( t \equiv g^r, \mod. m, g \) being a primitive root of \( m \), we have the following expressions for \( k \) and \( h_1: \dagger\dagger \)

\[ k = \frac{p_{p-1}^{1+p_{p-1}^n(p-1)-1} + n_{p-1}^{p_{p-1}^{n-1}(p-1)}}{2_{p-1}^{2_{p-1}^{n-1}(p-1)}} \prod X_1^{(s)}, \]

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Here $t$ runs through a complete residue system with respect to the modulus $m$ except multiples of $p$. In (1), $s$ takes all odd and in (2), all even values less than $\mu$ except zero.

If we denote by $Y$ the function $X$ corresponding to the field $k(\tau')$, then we have for every $s < \mu'$

$$X^{(\mu')} = Y^{(s)}.$$

For

$$f_{\mu s}(\tau^x) = \sum \theta_{\tau^x \tau^{\ell x}} = \sum \theta_{\tau^x \tau^\ell x} \quad (\theta' = \theta \ell)$$

and

$$f_{\mu s}(\tau^x) = \sum \theta_{\tau^x \tau^{\ell x}} (1 + \rho^x + \ldots \rho^{(p-1)x}) \quad (\rho = e^{2\pi i p}),$$

where $t'$ runs through a complete residue system, mod. $m'$, except multiples of $p$.

Hence

$$f_{\mu s}(\tau^x) = 0, \quad \text{if} \quad \lambda \equiv 0, \quad \text{mod. } p$$

and

$$f_{\mu s}(\tau^x) = p \sum \theta_{\tau^x \tau^{\ell x}} \quad \text{if} \quad \lambda \equiv 0, \quad \text{mod. } p.$$

In the last expression $\lambda = p\lambda'$ and $\gamma' \equiv t' \text{ mod. } m'$. From this (3) follows directly.

To obtain expressions for $h'$ and $h'_1$, we replace, in (1) and (2), $n$ by $n - 1$ and $X$ by $Y$. Making use of (3) we then get after a few reductions the following expressions for the factors $A$ and $B$:

$$(4) \quad A = \frac{p}{2} \prod_{s=2}^{\frac{np-2(p-1)^2}{2}} \prod_{s=1}^{\frac{np-2(p-1)^2}{2} - 1} X_1^{(s)},$$

$$(5) \quad B = \frac{p}{2} \prod_{s=2}^{\frac{np-2(p-1)^2}{2}} \prod_{s=1}^{\frac{np-2(p-1)^2}{2} - 1} X_2^{(s)}.$$

In (4), $s$ takes all odd and in (5), all even values less than $\mu$ except multiples of $p$. 
II. The factor $A$.

1. Simplification of the expression for $A$. We will now show how the expression for $A$ given above may be simplified so as to make it more convenient for numerical computation and also prove that $A$ is an integer.

Consider the function $f_s(r^\lambda)$. Two cases present themselves: $\lambda \not\equiv 0$, mod. $p$, and $\lambda \equiv 0$, mod. $p$.

1°. $\lambda \not\equiv 0$, mod. $p$. In this case, observing that

$$\theta^{\gamma_1} = \theta^{\text{ind } \lambda},$$

we get, after replacing $\lambda t$ by $t$,

$$f_s(r^\lambda) = \theta^{-s \text{ ind } \lambda} \sum_{t} \theta^{s \text{ ind } \gamma} t = \theta^{-\gamma_1}(\theta^s, r),$$

where $\gamma_1 = \text{ind } \lambda$.

2°. $\lambda \equiv 0$, mod. $p$. In this case set $\lambda = p\lambda$, and we have

$$f_s(r^\lambda) = \sum_{t} \theta^{s \gamma \nu \lambda t} = \sum_{\gamma} \theta^{s \gamma + (p-1)\lambda_1 t}.$$  

Let $a$ be the greatest common divisor of $s$ and $p-1$, and set $p-1 = ab$. Then the exponents in (7) fall into $a$ groups which are congruent to each other mod. $\mu$, the elements of each group being incongruent mod. $\mu$. Hence

$$f_s(r^\lambda) = a \sum_{\gamma} \theta^{s \gamma + (p-1)\lambda_1 t},$$

where

$$\gamma = 1, 1 + a, 1 + 2a, \ldots, 1 + (p^n - 1)b - 1) a.$$  

But the $\mu/a$ terms under the summation sign are the roots of the equation

$$x^{p^n - 1} - 1 = 0,$$

and hence

$$f_s(r^\lambda) = 0.$$  

Making use of (6) and (8), which hold for both even and odd values of $s$, we get

$$X^{(s)} = \frac{\pi i}{n^2} (\theta^s, r) \phi(\theta^s);$$

if we set

$$\phi(\theta^s) = \sum_{\lambda} \lambda \theta^{-\gamma},$$

where $\gamma = \text{ind } \lambda$ and $\lambda = 1, 2, \ldots, m - 1$ except multiples of $p$.  

The function $\phi(\theta^r)$ may however be simplified. Since

$$(m - \lambda) \theta^{-r \text{ ind } (m-\lambda)} = -(m - \lambda) \theta^{-r \text{ ind } \lambda},$$

we get

$$(11) \quad \phi(\theta^r) = \sum_{\lambda} (2\lambda - m) \theta^{-r},$$

where $\lambda = 1, 2, \ldots, (m - 1)/2$ except multiples of $p$. For $A$ we then obtain, observing that

$$\theta^{\mu - s} = \theta^{-s}(\theta^r, r)(\theta^{-r}, r) = (-1)^r p^n, *$$

the following expression

$$(12) \quad A = \prod_{s} \phi(\theta^r) = \frac{\prod_{s} \phi(\theta^r)}{2^{p-2(p-1)s} p^{\frac{n-2(p-1)s}{2}}},$$

where $s$ takes all odd values less than $\mu$.

2. Proof that $A$ is an integer. It is evident that $\phi(\theta^r)$ is an algebraic integer in the field $k(\theta)$. Now we have

$$\phi(\theta^r) = \sum_{i}^{\mu} \lambda_i \theta^{-r} = \sum_{i}^{\mu/2} (\lambda_i - \lambda_{i/2 + i}) \theta^{-r} \quad (i = \text{ ind } \lambda).$$

But since

$$\lambda_{\mu/2 + i} = g^{\mu/2 + i} = -g^{i} \equiv -\lambda_i \quad \text{mod. } m,$$

we have

$$\lambda_{\mu/2 + i} = m - \lambda_i,$$

and hence

$$\phi(\theta^r) = 2 \sum_{i}^{\mu/2} \lambda_i \theta^{-r} - m \sum_{i}^{\mu/2} \theta^{-r} = 2 \sum_{i}^{\mu/2} \lambda_i \theta^{-r} + \frac{2m}{1 - \theta^r}$$

or

$$(1 - \theta^r) \phi(\theta^r) = 2 \left[ (1 - \theta^r) \sum_{i}^{\mu/2} \lambda_i \theta^{-r} + m \right].$$

But

$$\prod_{i} (1 - \theta^r) = \frac{\prod_{i} (1 - \theta^t)}{\prod_{i} (1 - \theta^{t'})},$$

where $\theta' = \theta^r$, and $t$ and $t'$ take all odd values less than $\mu$ and $\mu'$ respectively. Hence, the quantities $\theta^t$ and $\theta^{t'}$, being the roots of the equations,

$$x^{\mu/2} + 1 = 0 \quad \text{and} \quad x^{\mu'/2} + 1 = 0$$

respectively, it follows that
\[ \prod_i (1 - \theta^i) = 1. \]

Hence we see that \( \Pi \phi(\theta^r) \) is divisible by \( 2^{m-2(p-1)^2/2} \).

To prove that \( \Pi \phi(\theta^r) \) is divisible by \( p \) we have
\[
(g - \theta^r) \phi(\theta^r) = \sum_i (g \lambda_i - \lambda_{i+1}) \theta^{-ir}.
\]

But
\[ g \lambda_i \equiv \lambda_{i+1}, \mod. m; \]
hence
\[ (g - \theta^r) \phi(\theta^r) = m\phi(\theta^r), \]
where \( \phi(\theta^r) \) is an algebraic integer, and therefore
\[
\prod_i (g - \theta^r) \phi(\theta^r) = m^{p-1} \prod_i \phi(\theta^r).
\]

But
\[
\prod_i (g - \theta^r) = \frac{\prod_i (g - \theta^r)}{\prod_i (g - \theta'^r)},
\]
where \( t \) and \( t' \) take all odd values less than \( \mu \) and \( \mu' \) respectively. Hence, reasoning as above, we find
\[
\prod_i (g - \theta^r) = \frac{g^{\mu/2} + 1}{g^{\mu'/2} + 1}
= g^{\frac{\mu'}{2}(p-1)} - g^{\frac{\mu'}{2}(p-2)} + \cdots + 1,
\]
or, since \( g^{\mu'/2} \equiv -1, \mod. m' \),
\[
\prod_i (g - \theta^r) \equiv p, \mod. m.
\]

We thus see that \( \prod (g - \theta^r) \) is divisible by \( p \) and by no higher power of \( p \). Therefore \( \Pi \phi(\theta^r) \) is divisible by \( p^{p-1(p-1)^{m/2-1}} \) and hence \( A \) is an integer.

If we now denote by \( A_n \) the factor \( A \) corresponding to \( m = p^n \), we get the following expression for the first factor \( k \) of the class number of \( k(r) \):

\[
k = k_1 A_2 A_3 \cdots A_n,
\]
where \( k_1 \) is the first factor of the class number of \( k(e^{2\pi i/p}) \).
III. The factor $B$.

1. Simplification of the expression for $B$. Making use of (6) and (8), $X^{(s)}_2$ may be written

$$X^{(s)}_2 = -\frac{(\theta^s, r)}{m} \sum_{\lambda} \theta^{-\gamma} \log \sin \frac{\lambda \pi}{m},$$

where $\gamma = \text{ind } \lambda$ and $\lambda = 1, 2, \ldots, m - 1$ except multiples of $p$. But since

$$\theta^{-\text{ind } (m-\lambda)} \log \sin \frac{(m-\lambda) \pi}{m} = \theta^{-\text{ind } \lambda} \log \sin \frac{\lambda \pi}{m},$$

we obtain

$$X^{(s)}_2 = -\frac{2(\theta^s, r)}{m} \sum_{\lambda} \theta^{-\gamma} \log \sin \frac{\lambda \pi}{m},$$

for $\lambda = 1, 2, \ldots, (m - 1)/2$ except multiples of $p$. From this we get after a few reductions the following expression for $B$:

$$BD = \prod_s \psi(\theta^s),$$

where $s$ takes all even values, less than $\mu$, not divisible by $p$ and

$$\psi(\theta^s) = \sum_i \theta^{-s_i} \log \sin \frac{\lambda_i \pi}{m} \quad (i = \text{ind } \lambda_i).$$

We will now show how the product $\prod \psi(\theta^s)$ can be expressed in the form of a determinant. We have

$$\psi(\theta^s) = \sum_i \theta^{-s_i} \log \frac{\sin \frac{\lambda_i \pi}{m}}{\sin \frac{\pi}{m}} + \log \sin \frac{\pi}{m} \sum_i \theta^{-s_i}$$

$$= \sum_i \theta^{-s_i} \log \tau_i = \sum_i \theta^{-s_i} l_i,$$

where $l_i = \log \tau_i$ and

$$\tau_i = \frac{\sin \frac{\lambda_i \pi}{m}}{\sin \frac{\pi}{m}} = r^{1-\lambda_i} - \frac{1 - r^{\lambda_i}}{1 - r};$$

or, if we set $\theta^s = \theta_1$, then, since $s$ is even, $\theta_1^{-\mu/2} = 1$ and

$$\psi(\theta^s) = \sum_i \theta_1^{-s_i} l_i.$$
Now consider the system of equations:

\[ \psi(\theta_1) = l_0 + l_1 \theta_1^{-1} + \cdots + l_{\frac{\mu}{2} - 1} \theta_1^{-\left(\frac{\mu}{2} - 1\right)}, \]

(20)

\[ \theta_1^{-1} \psi(\theta_1) = l_{\frac{\mu}{2} - 1} + l_0 \theta_1^{-1} + \cdots + l_{\frac{\mu}{2} - 2} \theta_1^{-\left(\frac{\mu}{2} - 2\right)}, \]

\[ \theta_1^{-\left(\frac{\mu}{2} - 1\right)} \psi(\theta_1) = l_1 + l_2 \theta_1^{-1} + \cdots + l_0 \theta_1^{-\left(\frac{\mu}{2} - 1\right)}; \]

from which we get, by eliminating the powers of \( \theta_1 \),

\[
\begin{vmatrix}
  l_0 - \psi & l_1 & l_2 & \cdots & l_{\frac{\mu}{2} - 1} \\
  l_{\frac{\mu}{2} - 1} & l_0 - \psi & l_1 & \cdots & l_{\frac{\mu}{2} - 2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  l_1 & l_2 & l_3 & \cdots & l_0 - \psi
\end{vmatrix} = 0.
\]

(21)

This equation is of degree \( \mu/2 \) and its roots are the quantities \( \psi(\theta^s) \) for \( s = 0, 2, 4, \ldots \mu - 2 \). The product of these roots is then expressed by the following determinant whose absolute value we denote by \( T_1 \):

\[
\Pi_s \psi(\theta^s) = \pm \begin{vmatrix}
  l_0 & l_1 & \cdots & l_{\frac{\mu}{2} - 1} \\
  l_1 & l_2 & \cdots & l_0 \\
  \vdots & \vdots & \ddots & \vdots \\
  l_{\frac{\mu}{2} - 1} & l_0 & \cdots & l_{\frac{\mu}{2} - 2}
\end{vmatrix} = \pm T_1.
\]

(22)

In a similar manner, making use of the fact that in this case \( \theta_1^{-\mu/2} = 1 \), we obtain \( \Pi_s \psi(\theta^s) \) where \( s \) runs through the even multiples of \( \mu \), from the following system of equations:

\[ \psi(\theta_1) = L_0 + L_1 \theta_1^{-1} + \cdots + L_{\frac{\mu'}{2} - 1} \theta_1^{-\left(\frac{\mu'}{2} - 1\right)}, \]

(23)

\[ \theta_1^{-1} \psi(\theta_1) = L_{\frac{\mu'}{2} - 1} + L_0 \theta_1^{-1} + \cdots + L_{\frac{\mu'}{2} - 2} \theta_1^{-\left(\frac{\mu'}{2} - 2\right)}, \]

\[ \theta_1^{-\left(\frac{\mu'}{2} - 1\right)} \psi(\theta_1) = L_1 + L_2 \theta_1^{-1} + \cdots + L_0 \theta_1^{-\left(\frac{\mu'}{2} - 1\right)}, \]

where

\[
L_i = l_i + l_{i + \frac{\mu'}{2}} + \cdots + l_{i + (p-1)\frac{\mu'}{2}} \quad (i = 0, 1, \ldots, \frac{\mu'}{2} - 1).
\]

(24)

Then as above \( \Pi \psi(\theta^s) \) is expressed by the following determinant whose absolute value we denote by \( T_2 \):
(25) \[ \prod_{\nu} \psi(\theta^\nu) = \pm \begin{vmatrix} L_0 & L_1 & \cdots & L_{\mu'}_{2-1} \\ L_1 & L_2 & \cdots & L_{0} \\ \vdots & \vdots & \ddots & \vdots \\ L_{\mu'}_{2-1} & L_0 & \cdots & L_{\mu'}_{2-2} \end{vmatrix} = \pm T_2. \]

Hence from (22) and (25) we get

(26) \[ BD = \frac{T_1}{T_2}. \]

From (24) it is seen that \( T_1 \) may be written

\[ \pm T_1 = \begin{vmatrix} L_0 & \cdots & L_{\mu'}_{2-1} & l_{\mu'}_{2} & \cdots & l_{\mu'}_{2-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ L_{\mu'}_{2-1} & \cdots & L_{\mu'}_{2-2} & l_{0} & \cdots & l_{\mu'}_{2-1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ L_0 & \cdots & L_{\mu'}_{2-1} & l_{0} & \cdots & l_{\mu'}_{2-1} \\ L_{\mu'}_{2-1} & \cdots & L_{\mu'}_{2-2} & l_{0} & \cdots & l_{\mu'}_{2-1} \end{vmatrix} \]

Introducing the set of units

\[ \tau'_i = r^{\frac{\lambda_i - \lambda_{i+\mu'}/2}{2}} \frac{1 - r^{\lambda_{i+\mu'}/2}}{1 - r^{\lambda_i}} = \frac{\sin \frac{\lambda_{i+\mu'}/2 \cdot \pi}{m}}{\sin \frac{\lambda_i \pi}{m}}, \]

with \( l'_i = \log \tau'_i \), and making use of the fact that \( l'_i = l_{i+\mu'/2} - l_i \), we get

\[ T_1 = T_2 \cdot T_3, \]

where

\[ T_3 = \pm \begin{vmatrix} l'_0 & l'_1 & \cdots & l'_{(p-1)\mu'/2-1} \\ l'_1 & l'_2 & \cdots & l'_{(p-1)\mu'/2} \\ \vdots & \vdots & \ddots & \vdots \\ l'_{(p-1)\mu'/2-1} & l'_{(p-1)\mu'/2} & \cdots & l'_{(p-1)\mu'/2-2} \end{vmatrix}, \]

and therefore

\[ BD = T_3. \]
2. Normal Units. In order to investigate the character of $D$, we have to consider the normal* units of $k(r + r^{-1})$. By a normal unit in $k(r + r^{-1})$ we understand a unit $\epsilon(r)$, different from $\pm 1$, which satisfies

\[(29) \quad \epsilon(r)\epsilon(pr)\epsilon(p^2r)\cdots\epsilon(p^{n-1}r) = \pm 1,\]

where $p = e^{2\pi i/p} = r^{p-1}$. This means that the relative norm of $\epsilon(r)$ in $k(r + r^{-1})$, with respect to $k(r' + r'^{-1})$, is $\pm 1$.

It is evident that no unit in $k(r' + r'^{-1})$, which is also a unit in $k(r + r^{-1})$, can be a normal unit. The units $\tau_i'$, considered above, are normal units. For

\[\tau_{i+a.\mu'/2}'(r) = \pm \tau_i'(p^{(i+a)/2}A_a r),\]

where

\[g^{a\mu'/2} = (-1)^a + A_a m';\]

and, since $(-1)^a A_a$ runs through a complete residue system with respect to the modulus $p$ when $a = 0, 1, \ldots, p - 1$, it follows that

\[(30) \quad \tau_i'(r)\tau_i'(pr)\cdots\tau_i'(p^{n-1}r) = \pm \prod_{a=0}^{p-1} \tau_i'(a_m^2)(r) = \pm 1.\]

A system of $v = (p - 1)\mu'/2$ normal units $\epsilon_0(r), \epsilon_1(r), \ldots, \epsilon_{v-1}(r)$ is said to be an independent system of normal units if

\[
\begin{vmatrix}
\log |\epsilon_0(r)| & \cdots & \log |\epsilon_{v-1}(r)| \\
\log |\epsilon_0(r^q)| & \cdots & \log |\epsilon_{v-1}(r^q)| \\
\cdots & \cdots & \cdots \\
\log |\epsilon_0(r^{q^{n-1}})| & \cdots & \log |\epsilon_{v-1}(r^{q^{n-1}})|
\end{vmatrix}
\neq 0;
\]

and the absolute value of the determinant is called the regulator of the system $\epsilon_0, \epsilon_1, \ldots, \epsilon_{v-1}$. The units $\tau_0', \tau_1', \ldots, \tau_{v-1}'$, form such an independent system of normal units; for its regulator, being the determinant $T_3$, is evidently different from zero.

Now let $\epsilon_0, \epsilon_1, \ldots, \epsilon_{v-1}$ be an independent system of normal units and let $L_{i, \kappa} = \log |\epsilon_i(r^{q^\kappa})|$. Then, if $\epsilon'(r)$ be any normal unit and $L'_\kappa = \log |\epsilon'(r^{q^\kappa})|$, we can determine $\xi_0, \xi_1, \ldots, \xi_{v-1}$ from the system of equations

\[(31) \quad L'_\kappa = \xi_0 L_{0, \kappa} + \xi_1 L_{1, \kappa} + \cdots + \xi_{v-1} L_{v-1, \kappa} \quad (\kappa = 0, 1, \ldots, v - 1).\]

That this equation also holds for any value of $\kappa$ follows immediately from (29) and (30). By applying the same reasoning as for independent systems of units

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in any algebraic number field \* we can prove that \( \xi_0, \ldots, \xi_{\nu-1} \) are rational, and hence that there exists an independent system of normal units whose regulator is the least possible. Such a system we call a fundamental system of normal units, and any normal unit can be written in the form

\[ \pm \epsilon_0^{m_0} \epsilon_1^{m_1} \cdots \epsilon_{\nu-1}^{m_{\nu-1}}, \]

where \( \epsilon_0, \epsilon_1, \ldots, \epsilon_{\nu-1} \) are a fundamental system and \( m_0, m_1, \ldots m_{\nu-1} \) are integers. The regulator of a fundamental normal system is, therefore, a divisor of the regulator of any (independent) system of normal units.

3. Study of \( D \). Let \( \epsilon_1, \epsilon_2, \ldots, \epsilon_{\mu/2-1} \) be a fundamental system of units in \( k(r + r^{-1}) \), with the conjugate logarithms \( \lambda_{1,\kappa}, \lambda_{2,\kappa}, \ldots, \lambda_{\mu/2-1,\kappa} \) and regulator \( E \), and let \( \epsilon_1', \epsilon_2', \ldots, \epsilon_{\mu/2-1}' \) be a fundamental system of units in \( k(r' + r'^{-1}) \) with the conjugate logarithms \( \lambda_{1,\kappa}', \lambda_{2,\kappa}', \ldots, \lambda_{\mu/2-1,\kappa}' \) and regulator \( E' \). Also let \( \omega_0, \omega_1, \ldots, \omega_{\nu-1} \) be a fundamental system of normal units in \( k(r + r^{-1}) \) with the conjugate logarithms \( L_{0,\kappa}, L_{1,\kappa}, \ldots, L_{\nu-1,\kappa} \) and regulator \( T_0 \). Then the units

\[ \epsilon_1', \epsilon_2', \ldots, \epsilon_{\mu/2-1}', \omega_0, \omega_1, \ldots, \omega_{\nu-1} \]

form an independent system of units in \( k(r + r^{-1}) \). For since

\[ \lambda_{i,\kappa + \mu/2} = \lambda_{i,\kappa} \]

and

\[ L_{i,0} + L_{i,\mu/2} + \cdots + L_{i,(\mu-1)\mu/2} = 0, \]

we get for the regulator \( R \) of the system (32),

\[ R = p^{\mu/2-1} E' T_0, \]

which shows that \( R \neq 0 \) and hence that (32) form an independent system of units.

We can then determine rational numbers \( m_{i,\kappa} \) and \( M_{i,\kappa} \) such that

\[ p\lambda_{i,\kappa} = m_{i,\kappa} + \lambda_{i,\kappa} + \cdots + m_{\mu/2-1,\kappa} + M_{0,\kappa} L_{0,\kappa} + \cdots + M_{\nu-1,\kappa} L_{\nu-1,\kappa} \]

\[ \left( \kappa = 0, \ldots, \frac{\mu}{2} - 2; i = 1, \ldots, \frac{\mu}{2} - 1 \right). \]

We now wish to prove that \( m_{i,\kappa} \) and \( M_{i,\kappa} \) are integers. From (34) we get

\[ \lambda_{i,\kappa} + \lambda_{i,\kappa + \mu/2} + \cdots + \lambda_{i+(\mu-1)\mu/2} = m_{i,\kappa} + \lambda_{i,\kappa} + \cdots + m_{\mu/2-1,\kappa} \lambda_{\mu/2-1,\kappa}, \]

and, since

\[ \epsilon_i(r) \epsilon_i(r^{\mu/2}) \cdots \epsilon_i(r^{(\mu-1)\mu/2}) \]

is a unit in \( k(r' + r'^{-1}) \), it follows that \( m_{1, \epsilon}, m_{2, \epsilon}, \ldots, m_{\mu'/2-1, \epsilon} \) are integers.

We also obtain from (34)

\[
\lambda_{i, \kappa} + \ldots + \lambda_{i, \kappa + (p-1)\mu/2} - p\lambda_{i, \kappa} = -M_{0, \kappa} L_{0, \kappa} - \cdots - M_{v-1, \kappa} L_{v-1, \kappa},
\]

and since

\[
\epsilon_i(r) \epsilon_i(r^{p^m/\mu}) \cdots \epsilon_i(r^{p^{(p-1)\mu/2}}) [\epsilon_i(r)]^p
\]

is a normal unit, it follows that \( M_{0, \kappa}, \ldots, M_{v-1, \kappa} \) are integers.

From (34)

(35)

\[
E = p^{-\frac{\mu}{2} + \frac{1}{2}} RM,
\]

where \( M \) is the determinant of the coefficients \( m_{i, \kappa} \) and \( M_{i, \kappa} \) and hence an integer. Formulae (33) and (35) then give

(36)

\[
D = p^{(\kappa' - \kappa)/2} MT_0.
\]

We now propose to investigate the character of \( M \). To do this let

\[
\lambda'_{i, \kappa} = n_{1, \kappa} \lambda_{i, \kappa} + \ldots + n_{\frac{\mu}{2}-1, \kappa} \lambda_{\frac{\mu}{2}-1, \kappa} \quad (i=1, 2, \ldots, \frac{\mu'}{2}-1),
\]

\[
L_{i, \kappa} = N_{1, \kappa} \lambda_{i, \kappa} + \ldots + N_{\frac{\mu}{2}-1, \kappa} \lambda_{\frac{\mu}{2}-1, \kappa} \quad (i=0, 1, \ldots, v-1),
\]

where \( n_{i, \kappa} \) and \( N_{i, \kappa} \) are integers. Denoting by \( N \) the determinant of the coefficients \( n_{i, \kappa} \) and \( N_{i, \kappa} \), we get

\[
R = EN
\]

and hence

(37)

\[
MN = p^{\frac{\mu}{2} - 1},
\]

i.e., \( M \) and \( N \) are both powers of \( p \). To determine the power of \( p \) by which \( M \) is divisible, we determine a system of integers \( a_1, a_2, \ldots, a_{\mu/2-1} \) without common divisor satisfying the system of equations

(38)

\[
a_1 m_{1,1} + a_2 m_{1,2} + \cdots + a_{\mu/2-1} m_{1, \mu/2-1} = 0 \quad (i=1, 2, \ldots, \frac{\mu'}{2}-1).
\]

Let

\[
a_1 M_{1,1} + a_2 M_{1,2} + \cdots + a_{\mu/2-1} M_{1, \mu/2-1} = \xi_i \quad (i=0, 1, \ldots, v-1),
\]

and we have

\[
p \sum_{i=1}^{\frac{\mu}{2}-1} a_i \lambda_{i, \kappa} = \xi_0 L_{0, \kappa} + \xi_1 L_{1, \kappa} + \cdots + \xi_{v-1} L_{v-1, \kappa},
\]

from which

\[
\sum_{i=1}^{\frac{\mu}{2}-1} a_i \lambda_{i, \kappa} + \sum_{i=1}^{\frac{\mu}{2}-1} a_i \lambda_{i, \kappa + \mu'/2} + \cdots + \sum_{i=1}^{\frac{\mu}{2}-1} a_i \lambda_{i, \kappa + (p-1)\mu'/2} = 0.
\]
Hence we infer that \( e_1^0 e_2^0 \cdots e_{\mu_0}^{2-1} \) is a normal unit and that \( \xi_0, \xi_1, \ldots, \xi_{\nu-1} \) are integers divisible by \( p \). It is then very easy to show, by applying the same reasoning as in the case \( p = 2, * \) that \( M \) is divisible by \( p^{(p-1)/2} \). Hence if we set

\[
M = p^{(p-1)/2} + \sigma,
\]

we obtain from (36)

\[
D = p^\sigma T_0 \quad (\leq \sigma \leq \mu' - 1).
\]

From (28) we then have

\[
B = p^{-\sigma} \frac{T_3}{T_0}.
\]

where \( T_3/T_0 \) is an integer, \( T_0 \) being the regulator of a fundamental system of normal units.

If we now denote by \( B \) the factor \( B \) corresponding to \( m = p^n \), we get the following expression for the second factor of the class number of \( k(r) \):

\[
h_1 = h''_1 B_2 B_3 \cdots B_n,
\]

where \( h''_1 \) is the class number of \( k(e^{2\pi i/p} + e^{-2\pi i/p}) \).

Comparing our results with those obtained by Weber for \( p = 2 \), we notice that, for all values of \( p \), \( A \) is an integer and \( B = p^{-\sigma} T_3/T_0 \), where \( T_3/T_0 \) is an integer. For \( p = 2 \), Weber proves that \( \sigma = 0 \) and that both \( A \) and \( T_3/T_0 \) and hence \( B \) are odd numbers. When \( p \) is an odd prime, the question whether \( A \) and \( T_3/T_0 \) are divisible by \( p \) or not, and what the value of \( \sigma \) is, remains unsettled. The writer, however, hopes to be able to come back to this question in a following paper.

Purdue University,
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