# ON REDUCIBLE GROUPS*

BY

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In the Transactions of the American Mathematical Society, January, 1903, pages 46–56 Professor LOEWY proves for the first time the theorem: " *Wie auch immer eine Gruppe G linearer homogener Substitutionen unter Hervorhebung ihrer irreduciblen Bestandteile in eine ähnliche Gruppe transformirt wird, so kann man die irreduciblen Bestandteilen, die sich bei irgend einer Darstellung ergeben, den irreduciblen Bestandteilen die sich bei irgend einer anderen Darstellung ergeben, eineindeutig so zuordnen, dass zwei zugeordnete irreducible Teilgruppen gleich viele Variablen haben und ähnliche Gruppen sind.*"

This says that if by transformation $G$ be decomposed into

$$(1)\qquad \begin{matrix} \alpha_{11} & 0 & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{\lambda 1} & \alpha_{\lambda 2} & \alpha_{\lambda 3} & \cdots & \alpha_{\lambda\lambda} \end{matrix}$$

and

$$(2)\qquad \begin{matrix} \beta_{11} & 0 & 0 & \cdots & 0 \\ \beta_{21} & \beta_{22} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta_{\mu 1} & \beta_{\mu 2} & \beta_{\mu 3} & \cdots & \beta_{\mu\mu} \end{matrix}$$

the irreducible groups

$$\begin{matrix} \alpha_{11} & \alpha_{22} & \cdots & \alpha_{\lambda\lambda} \\ \beta_{11} & \beta_{22} & \cdots & \beta_{\mu\mu} \end{matrix}$$

are similar to each other in *some* order, that the similar group $\alpha_{ii}$ and $\beta_{jj}$ have the same number of variables, and that $\lambda = \mu$.

The object of this note is to prove the same theorem more briefly. It is well to notice however that LOEWY's proof has the advantage of being direct, while mine, although much shorter, is indirect.

Suppose we have before us a reducible linear homogeneous group $G$ in $n$ variables, then there exists a system of linear differential equations which have

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$G$ for their group of rationality.* These equations are all of the same order and of the same *type*.† Let $P_n y = 0$ be some one of these linear differential equations whose group of rationality is $G$. Since by hypothesis $G$ is reducible, $P_n y = 0$ is also reducible.‡ We now decompose $P_n$ into irreducible factors; this may of course be accomplished in various ways:

$$P_n = Q_\lambda Q_{\lambda-1} \cdots Q_2 Q_1 = 0, \tag{3}$$

$$P_n = R_\mu R_{\mu-1} \cdots R_2 R_1 = 0. \tag{4}$$

To these decompositions of the equation there are corresponding decompositions of $G$, namely (1) and (2), where $\alpha_{ii}$ is the group of $Q_i = 0$ $(i = 1 \cdots \lambda)$ and $\beta_{jj}$ is the group of $R_j = 0$ $(j = 1 \cdots \mu)$.

Now LOEWY showed in his paper entitled *Ueber reduzible lineare homogene Differentialgleichungen* § for the decompositions (3) and (4) that $\lambda = \mu$, and that, for suitable values of $i$ and $j$, $Q_i$ and $R_j$ are of the same *order* and *type*.

Since $\lambda = \mu$ it follows that the number of irreducible groups $\alpha$ equals the number of irreducible groups $\beta$. Since $Q_i$ and $R_j$ are of the same order, the number of variables in $\alpha_{ii}$ is the same as the number of variables in $\beta_{jj}$ (the number of variables in the group being the same as the order of the differential equation). Finally, since $Q_i$ and $R_j$ are of the same type, $\alpha_{ii}$ and $\beta_{jj}$ must be similar.‖

The theorem may be stated in the following manner:** *Considering similar groups as equivalent, the irreducible constituents of a linear homogeneous group $G$ which are obtained by the various possible decompositions, are equivalent apart from the arrangement.*

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* JORDAN, Bulletin de la Société Mathematique de France, vol. 2, p. 104; KLEIN, *Höhere Geometrie,* II, pp. 296–7; LOEWY, Mathematische Annalen, January, 1903, p. 560. For this existence theorem, see SCHLESINGER, *Handbuch* II, 1, pp. 76–7, 93.

† The French *espèce* might be translated *species*, and the German *Art* as *kind.* But in LOEWY's paper in the Annalen of January, 1903, p. 560, it appears that the groups of these equations are all of the same type. For this reason I say that the *equations are of the same type.* In accordance with the customary practice, LOEWY calls such groups equivalent.

‡ SCHLESINGER, *Handbuch,* II, 1, p. 106.

§ Mathematische Annalen, January, 1903, p. 565.

‖ According to SCHLESINGER's *Handbuch,* II, 1, p. 121, $\alpha_{ii}$ and $\beta_{jj}$ are the same; but SCHLESINGER means by this that they are of the same type, i. e., can be transformed into each other. In this connection, cf. VESSIOT, Annales de l' École Normale Supérieure, 1892, p. 232.

** Cf. LOEWY, Transactions, January, 1903, p. 47.