ON THE REDUCIBILITY OF LINEAR GROUPS*

BY

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The object of this note is a two-fold generalization of LOEWY's theorem proved in these Transactions, vol. 4, pp. 171–177. His theorem may be conveniently stated as follows: If R is the domain of all real numbers and C the domain of all complex numbers, any group of linear homogeneous transformations with coefficients in R which is irreducible in R, but reducible in C, can be transformed linearly into a decomposable group $\left(\begin{smallmatrix} G & 0 \\ 0 & \bar{G} \end{smallmatrix}\right)$, where G and $\bar{G}$ are two groups irreducible in C, with coefficients not all in R, such that the coefficients in every transformation of $\bar{G}$ are the conjugate imaginaries of the corresponding coefficients for G.

In seeking a generalization, we note that the domain C may be considered as derived from R by the adjunction of a root i of the quadratic equation $x^2 + 1 = 0$ belonging to and irreducible in R. For the generalization, R is replaced by a general domain F (or field not having a modulus) and $R(i)$ is replaced by the domain $F(\rho_0)$ given by the extension of F by the adjunction of a root ρ_0 of an equation $f(x) = 0$ of degree r belonging to and irreducible in F. The generalization will therefore be two-fold. Let the roots of $f(x) = 0$ be $\rho_0, \rho_1, \cdots, \rho_{r-1}$. If G_{11} is a group of transformations with coefficients $C_{ij}(\rho_0)$ in the domain $F(\rho_0)$, let $G_{11}^{(s)}$ denote the group of transformations with the coefficients $C_{ij}(\rho_s)$; in particular, $G_{11}^{(0)} = G_{11}$. The coefficients of $G_{11}, G'_{11}, \cdots, G_{11}^{(r-1)}$ are thus conjugate with respect to F. The generalized theorem is as follows:

Let G be a group of linear homogeneous transformations with coefficients in a domain F, such that G is irreducible in F but is reducible in the domain $F(\rho_0)$ given by the extension of F by the adjunction of a root ρ_0 of an equation belonging to and irreducible in F and having as its roots $\rho_0, \rho_1, \cdots, \rho_{r-1}$. Then G can be transformed linearly into a decomposable group †

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† When the irreducible equation is a normal equation, the groups $G_{11}^{(s)}$ $(s = 0, 1, \cdots, r-1)$ are all irreducible in the same (normal) domain. LOEWY's case furnishes an example.

$$\begin{matrix} G_{11} & 0 & 0 & \cdots & 0 \\ 0 & G'_{11} & 0 & \cdots & 0 \\ 0 & 0 & G''_{11} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & G^{(r-1)}_{11} \end{matrix}$$

where $G^{(s)}_{11}$ *is a group irreducible in* $F(\rho_s)$ *with coefficients not all in* F, *and* $G_{11}, G'_{11}, \cdots, G^{(r-1)}_{11}$ *are conjugate with respect to* F.

The proof starts as in Loewy, § 1. The first variation * occurs at the bottom of p. 173; we now take r-fold decomposable matrices

$$H = \begin{bmatrix} G & 0 & \cdots & 0 \\ 0 & G & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & G \end{bmatrix}, \qquad Q = \begin{bmatrix} P & 0 & \cdots & 0 \\ 0 & P' & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & P^{(r-1)} \end{bmatrix}.$$

Corresponding changes are to be made in the first two statements on p. 174. Thus, the diagonal groups in (6) are to be replaced by

$$G_{11},\ G_{22},\ G'_{11},\ G'_{22},\ G''_{11}, G''_{22},\ \cdots,\ G^{(r-1)}_{11},\ G^{(r-1)}_{22}.$$

In place of the transformation † (7), we have

$$(7') \qquad y_{jk} = \sum_{i=1}^{n} C^{(j)}_{ki} y^*_{ji} \qquad (k=1, \cdots, n;\ j=0, \cdots, r-1),$$

where $C^{(j)}_{ki}$ is a rational function of ρ_j with coefficients in F, and

$$(7_a) \qquad C^{(j)}_{ki} = 0 \qquad (k=1, \cdots, m;\ i=m+1, \cdots, n;\ j=0, \cdots, r-1).$$

Introduce two pairs each of rn new variables defined by

$$(8') \qquad y_{sk} = Y_{0k} + \rho_s Y_{1k} + \rho_s^2 Y_{2k} + \cdots + \rho_s^{r-1} Y_{r-1\,k}$$
$$(8'_1) \qquad y^*_{sk} = Y^*_{0k} + \rho_s Y^*_{1k} + \rho_s^2 Y^*_{2k} + \cdots + \rho_s^{r-1} Y^*_{r-1\,k} \qquad \begin{pmatrix} s=0, \cdots, r-1 \\ k=1, \cdots, n \end{pmatrix}.$$

This may be done since the determinant

$$\Delta \equiv \begin{vmatrix} 1 & \rho_0 & \rho_0^2 & \cdots & \rho_0^{n-1} \\ 1 & \rho_1 & \rho_1^2 & \cdots & \rho_1^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \rho_{r-1} & \rho_{r-1}^2 & \cdots & \rho_{r-1}^{n-1} \end{vmatrix} = \Pi(\rho_i - \rho_j) \neq 0.$$

* The statement on p. 173, lines 7–8, is apparently not used later; a proof follows readily from the main theorem under consideration.

† Loewy's notation is unwieldy even in his simple case. I write y_{0k}, y_{1k} for his y_k, z_k. The transformed variables are marked * instead of being primed.

Solving (8′) for fixed k, while $s=0, \cdots, r-1$, we get

$$(e) \qquad \Delta Y_{tk} = \sum_{s=0}^{r-1} (-1)^t D_{ts} y_{sk} \qquad (t=0, 1, \cdots, r-1),$$

where

$$D_{ts} \equiv \begin{vmatrix} 1 & \rho_0 & \cdots & \rho_0^{t-1} & \rho_0^{t+1} & \cdots & \rho_0^{r-1} \\ 1 & \rho_1 & \cdots & \rho_1^{t-1} & \rho_1^{t+1} & \cdots & \rho_1^{r-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \rho_{s-1} & \cdots & \rho_{s-1}^{t-1} & \rho_{s-1}^{t+1} & \cdots & \rho_{s-1}^{r-1} \\ 1 & \rho_{s+1} & \cdots & \rho_{s+1}^{t-1} & \rho_{s+1}^{t+1} & \cdots & \rho_{s+1}^{r-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \rho_{r-1} & \cdots & \rho_{r-1}^{t-1} & \rho_{r-1}^{t+1} & \cdots & \rho_{r-1}^{r-1} \end{vmatrix}.$$

Substituting for y_{sk} in (e) its value from (7′) and then eliminating y^*_{si} by $(8'_1)$, we obtain

$$(9) \qquad Y_{tk} = \sum_{\substack{i=1,\ldots,n \\ l=0,\ldots,r-1}} \alpha_{li}^{tk} Y^*_{li} \quad (k=1, \cdots n;\ t=0, \cdots r-1),$$

where

$$\alpha_{li}^{tk} \equiv \frac{(-1)^t}{\Delta} \sum_{s=0}^{r-1} (-1)^s D_{ts} C_{ki}^{(s)} \rho_s^l.$$

The coefficients of transformation (9) *belong to the domain F.* It suffices to show that each α_{li}^{tk} is unaltered by the interchange of ρ_0 with ρ_j (j being any one of the series $1, 2, \cdots, r-1$), since it is then a symmetric function of $\rho_0, \rho_1, \cdots, \rho_{r-1}$ with coefficients in F. To show that, for example, it is unaltered by the interchange of ρ_0 with ρ_1, we note that under this interchange, D_{t0} and D_{t1} are interchanged, $D_{ts}(s>1)$ is changed into $-D_{ts}$ while $C_{ki}^{(0)} \equiv C_{ki}(\rho_0)$ and $C_{ki}^{(1)} \equiv C_{ki}(\rho_1)$ are interchanged, and $C_{ki}^{(s)}(s>1)$ is unaltered. Hence the factor of α given by the sum is changed in sign; likewise the factor $1/\Delta$.

Moreover, from (7_a) follows at once

$$\alpha_{li}^{tk} = 0 \quad (i=m+1, \cdots, n;\ k=1, \cdots, m;\ t, l=0, 1, \cdots, r-1).$$

The group of transformations (9) is therefore of LOEWY's form (10), $\bar{\bar{H}}_{11}$ being always a matrix of rm rows and rm columns. The proof is then readily completed as in LOEWY's case (bottom of p. 175 and 176).

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