ON THE CONVERGENCE OF ALGEBRAIC CONTINUED FRACTIONS WHOSE COEFFICIENTS HAVE LIMITING VALUES*

BY

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PADÉ † has shown that a power series $P(z)$ gives rise ordinarily to three types of regular continued fractions having the forms:

(I) $$\frac{U_0(z)}{V_0(z)} + \frac{a_2 z^{p_2}}{1} + \frac{a_3 z}{1} + \frac{a_4 z}{1} + \frac{a_5 z}{1} + \cdots,$$

(II) ‡ $$\frac{U_1(z)}{V_1(z)} + \frac{b_2 z^{p_2}}{c_2 z + d_2} + \frac{b_3 z^2}{c_3 z + d_3} + \frac{b_4 z^2}{c_4 z + d_4} + \frac{b_5 z^2}{c_5 z + d_5} + \cdots,$$

(III) $$\frac{U_2(z)}{V_2(z)} + \frac{e_2 z^{p_2}}{f_2 z + g_2} + \frac{e_3 z}{f_3 z + g_3} + \frac{e_4 z}{f_4 z + g_4} + \frac{e_5 z}{f_5 z + g_5} + \cdots,$$

in which $U_i(z)$, $V_i(z)$ denote certain polynomials with which we need not concern ourselves here. The object of the following note is to investigate the convergence of these three classes of continued fractions upon the hypothesis that the coefficients $a_n, b_n, \cdots, g_n$ have limiting values for $n = \infty$. The results obtained below for the first two types of continued fractions are in no wise dependent upon the value of p_2 nor upon the nature of the polynomials U_i, V_i, neither are they affected by the introduction of a finite number of irregularities into the continued fraction—that is to say, by the presence of a finite number of partial numerators or denominators of degree higher than the normal. This is not true of the third type of continued fractions.

§ 1. *Preliminary discussion for type* I.

Consider first a continued fraction of this type which is regular from the beginning. The result which will be proved is as follows:

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† *Thesis*, Annales de l' Ecole Normale, ser. 3, vol. 9, supplement (1892).

‡ If $1/x$ be substituted for z, type II appears in the familiar form:

$$\frac{U(x)}{V(x)} + \frac{b_2}{d_2 x + c_2} + \frac{b_3}{d_3 x + c_3} + \cdots.$$

THEOREM. *If in the continued fraction*

$$A(z) = \frac{1}{1} + \frac{a_2 z}{1} + \frac{a_3 z}{1} + \cdots, \tag{1}$$

lim $a_n = k$ *for* $n = \infty$, *the continued fraction will converge over the entire plane of* z *except* (1) *along the whole or a part of a rectilinear cut drawn from* $z = -1/4k$ *to* $z = \infty$ *with an argument equal to that of the vector from the origin to* $z = -1/4k$, *and except possibly*, (2) *at certain isolated points* $p_1, p_2, p_3, \cdots$. *Within the plane so cut the limit of the continued fraction is holomorphic except at the points* $p_1, p_2, p_3, \cdots$, *which are poles.*

This result I have previously established under certain restrictions in the Annals of Mathematics,* and application was there made to the continued fractions of GAUSS, HEINE and BESSEL. To remove these restrictions a new and simpler method of proof is adopted here.

The proof is founded upon the familiar formula:

$$A(z) = \frac{N_n}{D_n} + (-)^n a_2 a_3 \cdots a_n z^{n-1} \left(\frac{a_{n+1} z}{D_n D_{n+1}} - \frac{a_{n+1} a_{n+2} z^2}{D_{n+1} D_{n+2}} + \frac{a_{n+1} a_{n+2} a_{n+3} z^3}{D_{n+2} D_{n+3}} - \cdots \right), \tag{2}$$

in which N_n and D_n denote the numerator and denominator of the nth convergent of (1). Between three consecutive denominators their exists the relation

$$D_{n+1} = D_n + a_{n+1} z D_{n-1}, \tag{3}$$

which by hypothesis has the limiting form

$$D_{n+1} - D_n - kzD_{n-1} = 0.$$

Now POINCARÉ† has shown that when a limiting form exists for the recurrent relation and the roots of the auxiliary equation

$$x^2 - x - kz = 0 \tag{4}$$

are of unequal modulus, the quotient D_{n+1}/D_n will converge to a limit which is the one or the other of the roots of (4). Since these roots are

$$x = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} + kz},$$

their moduli will be equal only when $kz + \frac{1}{4}$ is a negative number, that is, when $|z| > 1/4|k|$ and z has the same argument as $-1/k$.

* Ser. 2, vol. 3 (1901), p. 13.

† American Journal of Mathematics, vol. 7 (1885), p. 213. POINCARÉ's theorem is cited for the special case of a recurrent relation of the 2d order.

We shall exclude such values of z by drawing the cut already described in the theorem. Then at any point not on the cut we have

$$\lim_{n=\infty} \frac{D_{n+1}}{D_n} = \beta, \tag{5}$$

in which β is a root of (4), and the ratio of the rth term of (2) to the preceding has the limit $-kz/\beta^2$. Since $-kz$ is the product of the roots of (4), the absolute value of this ratio will be less than 1 if β is the root which has the larger modulus, while it will be greater than 1 if β is the other root. Furthermore, it is clear from (5) that n can be taken so large that no one of the denominators on the right hand side of (2) will vanish at an assigned point z, not lying upon the cut. Hence the continued fraction will converge outside the cut at every point for which β is the root of (4) which has the greater modulus, while it increases indefinitely at the remaining points.*

§ 2. *Proof of Poincaré's Theorem.*

It will be necessary to show next that the last mentioned points can condense in infinite number only in the vicinity of a point upon the cut. With this ultimate object in view, it will be desirable to give here a new and simple proof of POINCARÉ's theorem. Let β', β'' denote the two roots of (4) or, more generally, of the auxiliary quadratic for *any* recurrent relation of the 2d order with a limiting form. If a sufficiently large value of n is taken, the recurrent relation can be expressed in the form

$$(D_{n+1} - \beta' D_n) - (\beta'' + \bar{\epsilon}_{n+1})(D_n - \beta' D_{n-1}) + \bar{\epsilon}'_{n+1} D_{n-1} = 0,$$

in which $|\bar{\epsilon}_{n+1}|$ and $|\bar{\epsilon}'_{n+1}|$ are smaller than a small positive quantity, arbitrarily prescribed. The last equation may be written

$$(D_{n+1} - \beta' D_n) - (\beta'' + \epsilon''_{n+1})(D_n - \beta' D_{n-1}) = 0, \tag{6}$$

where

$$\epsilon''_{n+1} = \bar{\epsilon}_{n+1} - \frac{\bar{\epsilon}'_{n+1} D_{n-1}}{D_n - \beta' D_{n-1}}.$$

Consequently, if there is any set of values of n, infinite it number, for which the point $\rho_n = D_n / D_{n-1}$ does not come at least once within less than an assigned distance of β', we may make $|\epsilon''_{n+1}|$ smaller than any given quantity ϵ by increasing n within the set. Similarly we have

$$D_{n+1} - \beta'' D_n = (\beta' + \epsilon'_{n+1})(D_n - \beta'' D_{n-1}) \qquad (|\epsilon'_{n+1}| < \epsilon), \tag{7}$$

* Thus far the work has been identical with that of PINCHERLE in his consideration of continued fractions of type II. Cf. Annales de l'Ecole Normale, ser. 3, vol. 6 (1889), p. 144; also Giornale di Matematiche, vol. 32 (1894), p. 234–6. PINCHERLE, however, goes no further and does not settle the character of the point set for which β is the smaller root of (4). In consequence, his result is an indefinite one, and he misses the theorems of the present paper.

for a sufficiently large n in any set of values for which ρ_n lies without some assigned distance of β''. The last equation may be written

$$(8) \qquad \rho_{n+1} - \beta'' = \frac{\beta' + \epsilon'_{n+1}}{\rho_n}(\rho_n - \beta''),$$

and its division by (6) gives

$$(9) \qquad \frac{\rho_{n+1} - \beta''}{\rho_{n+1} - \beta'} = \frac{\beta' + \epsilon'_{n+1}}{\beta'' + \epsilon''_{n+1}} \cdot \frac{\rho_n - \beta''}{\rho_n - \beta'} \quad (|\epsilon'_{n+1}| < \epsilon, \; |\epsilon''_{n+1}| < \epsilon).$$

Equation (9) holds for a sufficiently large value of n provided ρ_n lies without an arbitrarily small but prescribed distance of β' and β'', as I shall for a time assume. Denote this distance by d and consider the circle $|\rho - \beta''| = d$. By taking d sufficiently small, the ratio of the maximum to the minimum of

$$\left| \frac{\rho - \beta''}{\rho - \beta'} \right|$$

for points upon this circle can be made as nearly equal to 1 as we please. Let this ratio be $1 + \epsilon'$, in which ϵ' is taken small enough to fulfill the inequality which is given just below.

Suppose now that $|\beta'| > |\beta''|$. Then it is clear from (9) that upon increasing n by 1, the quotient $|\rho_n - \beta''| / |\rho_n - \beta'|$ is increased by a factor at least as great as the quantity

$$\frac{|\beta'| - \epsilon}{|\beta''| + \epsilon} > |1 + \epsilon'| > 1.$$

Hence as ρ_n, by hypothesis, lies without the circle, ρ_{n+1} does also. But when (9) holds for a series of successive values of n, we get a set of points ρ_n, ρ_{n+1}, $\rho_{n+2}, \cdots$, gravitating toward β' as a limit, and this continues until (9) breaks down. The last will not happen until ρ_{n+m} falls within circle $|\rho - \beta'| = d$. But then $\rho_{n+m+1} - \beta''$ and $\rho_{n+m} - \beta''$, by (8), are nearly identical vectors since ρ_{n+m} differs but little from β'. Hence the first of the points ρ_{n+m+1} $(i = 1, 2, \cdots)$ which falls without the circle $|\rho - \beta'| = d$ will differ infinitesimally from $\rho_{n+m+i-1}$ which lies within. As soon, however, as ρ_{n+m+i} falls without the circle, equation (9) operates immediately to draw it back again into the interior. If, finally, d be made smaller and smaller, it follows that the distance $\rho_n - \beta'$ becomes and remains eventually as small as we wish. Thus we conclude that unless ρ_n, from and after some fixed value of n, remains within an arbitrarily assigned distance of β'', it must approach β' as its limit. In other words, one of the two values β', β'' is its limit, as was to be proved.

§ 3. *Completion of discussion for type* I.

We return now to the continued fraction (1) and give to z a fixed value not on the cut. If a sufficient number of terms of (1) are omitted at the outset, a new continued fraction will be obtained,

$$A_m(z) = \frac{a_{m+1}z}{1} + \frac{a_{m+2}z}{1} + \frac{a_{m+3}z}{1} + \cdots, \tag{10}$$

in which all the partial numerators differ from their limit kz by as little as we please. For this continued fraction

$$D_0 = D_1 = 1, \qquad \rho_1 = \frac{D_1}{D_0} = 1.$$

Now the auxiliary equation has a root equal to 1 only if $k = 0$. Suppose first $k \neq 0$ and apply the reasoning of § 2 to (10), taking ϵ and d to be extremely small. Then

$$|\rho_1 - \beta''| > d, \qquad |\rho_1 - \beta'| > d.$$

Hence equation (9) takes effect at the very beginning of the continued fraction, and we obtain a sequence of points $\rho_1, \rho_2, \rho_3, \cdots$, approaching β' as a limit. For the particular case in which $k = 0$, we have $\beta' = 1$, and this root is *a fortiori* the limit of $\rho_n = D_n / D_{n-1}$. We conclude therefore in either case that the limit of ρ_n is that root of (4) which has the greater modulus, and consequently the continued fraction (10) will converge at the assigned point z, provided, of course, the value of m is taken sufficiently large.

The same argument holds simultaneously for values of z in the immediate vicinity of the chosen point. Take then a very small circle C enclosing the point. For this circle $|z|$ will have an upper limit U and $|\beta'|$ a lower limit L. Hence from and after some fixed value of n the series (2)—when constructed for the continued fraction (10)—will be comparable with a geometric progression in which the ratio of each term to the preceding is

$$\frac{kU(1+\epsilon)}{L^2(1-\epsilon)} < 1.$$

Hence the convergence of the series is uniform in this circle, and from this fact the analytic character of its limit immediately follows.

Let the nth convergent of (10) be denoted now by N'_n/D'_n. Then the $(n+m)$th convergent of (1) is

$$\frac{N_m + \dfrac{N'_n}{D'_n} N_{m-1}}{D_m + \dfrac{N'_n}{D'_n} D_{m-1}}.$$

Since N'_n/D'_n within the circle C has a limit $A_m(z)$, the continued fraction (1) must likewise converge to a limit

$$\frac{N_m + A_m(z) N_{m-1}}{D_m + A_m(z) D_{m-1}}, \tag{11}$$

except at the points of the circle C in which the denominator of (11) vanishes. If it should vanish identically, then $-D_m/D_{m-1}$ and $A_m(z)$ have the same expansion into a continued fraction. But this is impossible, since $A_m(z)$, by hypothesis, has an expansion (10) containing an infinite number of terms, whereas the development of a rational fraction D_m/D_{m-1} in a continued fraction of the form (10), as is well known, has only a limited number of terms. Accordingly the denominator of (11) is not 0 but an analytic function. Since the zeros of such a function are isolated, it follows that the continued fraction (1) is either holomorphic or meromorphic within the circle C. Now for the center of C any point not upon the cut can be chosen. Our conclusion therefore can be extended to the entire plane exterior to the cut. Thus the theorem stated at the beginning of § 1 is established.

The reasoning is not affected in any way if a finite number of partial numerators or denominators in (1) are replaced by polynomials.

§ 4. *The second class of continued fractions.*

The investigation of the second type of continued fractions when limiting values exist for the coefficients can be reduced quickly to the previous investigation for type I. To this end let (II) be written

$$(12)\quad \frac{U_1(z)}{V_1(z)} + \frac{\dfrac{b_2 z^{p_2}}{c_2 z + d_2}}{1} + \frac{\dfrac{b_3 z^2}{(c_2 z + d_2)(c_3 z + d_3)}}{1} + \frac{\dfrac{b_4 z^2}{(c_3 z + d_3)(c_4 z + d_4)}}{1} + \cdots.$$

Then make the substitution

$$(13)\qquad x = \frac{\sqrt{b}\, z}{cz + d}$$

in which b, c, d denote the limits of b_n, c_n, d_n $(n = \infty)$. Choose for x any fixed point of the finite plane or any value in its immediate vicinity. If enough partial fractions at the beginning of (12) are omitted, the remainder of the continued fraction may be written

$$(14)\qquad \frac{x^2(1 + \epsilon_{m+1})}{1} + \frac{x^2(1 + \epsilon_{m+2})}{1} + \cdots,$$

where $\epsilon_{m+1}, \epsilon_{m+2}, \cdots$ are rational functions of x, the absolute values of which will be less than a given positive number ϵ for a sufficiently small neighborhood about the fixed point.

The application of § 3 to the continued fraction (14) shows then that it converges uniformly in such a neighborhood, provided ϵ is small enough and the fixed point does not lie upon a cut along the imaginary axis of the x-plane exterior to the points $x^2 = -\frac{1}{4}$. To this cut there corresponds by (13) a cut along an arc of a circle in the z-plane, and to the fixed x-point there corresponds a z-point exterior to the cut. Returning now to the continued fraction (12) or (II), we conclude that it will converge uniformly in the vicinity of the z-point when a sufficient number of terms is omitted. Let $A_m(z)$ denote the limit after the omission in (II). Then for the $(n+m)$th convergent of (II) we have again such an expression as (11). But D_n/D_{n-1} when expanded into a continued fraction of type II, has only a finite number of terms,* whereas $A_m(z)$ has an infinite number. Consequently $D_m + D_{m-1}A_m(z)$ can not vanish identically. The following theorem therefore results:

If in the continued fraction (II)

$$\lim b_n = b, \qquad \lim c_n = c, \qquad \lim d_n = d,$$

it will converge over the entire plane of z except (1) *at isolated points and* (2) *upon the whole or a part of a cut along the arc of a circle into which the segments of the imaginary axis of x exterior to the points $x^2 = -\frac{1}{4}$ are converted by the transformation* (13). *In the plane thus cut the limit of the continued fraction is holomorphic except at these isolated points which are poles.*

If the alternate convergents of (I) are formed into two distinct sets, those of either set are by themselves the successive convergents of a continued fraction of type II, in which the partial fractions after the opening irregularity have the form

$$\frac{-a_{n-1}a_n z^2}{1 + a_{n+1}z + a_n z}.$$

It is easy to see that this expression has the same limiting form for odd and for even values of n if, and only if, a_{2n} and a_{2n+1} have limits. That then the analytic functions which are limits of the two sets of convergents are identical follows at once from a theorem which I have given in a previous paper.†

Thus when a_{2n} and a_{2n+1} in (I) *have separate limits, a theorem holds similar to that previously given except that the cut is in general the arc of a circle.*

§ 5. *Restrictions on the power-series.*

The relations between the coefficients of the continued fraction and of the corresponding power-series

$$P(x) = l_0 - l_1 z + l_2 z^2 - l_3 z^3 + \cdots$$

* This can be made clear to the reader who is familiar with PADÉ's thesis in the following manner: Let PADÉ's table be formed for the power series $P(x)$ which is the development of a rational fraction N'_p/D'_q with numerator and denominator of the p'th and q'th degrees respectively. His approximants U_p/V_q will be identical with N'_p/D'_q, for $p \geqq p'$, $q \geqq q'$. Now to a continued fraction of type II there corresponds in PADÉ's table a diagonal line of approximants, which from and after some fixed element of the line are identical with N'_p/D'_q. In other words, the continued fraction terminates.

† Transactions, vol. 2 (1901), p. 476.

have been given by FROBENIUS.* In case the continued fraction (1) is regular from the beginning we put†

$$A_n = \begin{vmatrix} l_0 & l_1 & \cdots & l_{n-1} \\ l_1 & l_2 & \cdots & l_n \\ \cdot & \cdot & \cdot & \cdot \\ l_{n-1} & l_n & \cdots & l_{2n-2} \end{vmatrix}, \qquad B_n = \begin{vmatrix} l_1 & l_2 & \cdots & l_n \\ l_2 & l_3 & \cdots & l_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ l_n & l_{n+1} & \cdots & l_{2n-1} \end{vmatrix},$$

also

$$m_{2n} = \frac{A_n^2}{B_{n-1} B_n}, \qquad m_{2n+1} = \frac{B_n^2}{A_n A_{n+1}}.$$

Then we have

$$a_n = \frac{1}{m_{n-1} m_n}$$

or

(15) $$a_{2n} = \frac{A_{n-1} B_n}{A_n B_{n-1}}, \qquad a_{2n+1} = \frac{A_{n+1} B_{n-1}}{A_n B_n}.$$

Suppose now that

(16) $$\lim a_{2n} = a', \qquad \lim a_{2n+1} = a''.$$

Then

$$\lim_{n=\infty} \sqrt[n]{a_1 a_2 a_3 \cdots a_{2n-1}} = \lim \sqrt[n]{\frac{A_n}{A_{n-1}}} = a' a'',$$

$$\lim_{n=\infty} \sqrt[n]{a_1 a_2 a_3 \cdots a_{2n}} = \lim \sqrt[n]{\frac{B_n}{B_{n-1}}} = a' a''.$$

Let us place

$$\frac{A_n}{A_{n-1}} = (a' a'')^n q_n', \qquad \frac{B_n}{B_{n-1}} = (a' a'')^n q_n'',$$

and substitute in (15), finally in (16). In this manner we find that *the series will generate a regular continued fraction* (1) *with limits for* a_{2n} *and* a_{2n+1}‡ *when, and only when, the following conditions are fulfilled*: §

(1) $A_n \neq 0$, $B_n \neq 0$ *for all values of* n. §

*Journal für Mathematik, vol. 90 (1881), p. 5.

†See also the memoir of STIELTJES, Annales de la Faculté des Sciences de Toulouse, vol. 8 (1894), J, pp. 26 and 3.

‡The conditions for the existence of other continued fractions of type I, which, though not regular throughout, possess a like property can be expressed similarly with the aid of the determinants $c_{\alpha\beta}$ of FROBENIUS. The continued fraction (1) which was selected above, corresponds to a step-like line of approximants starting from the corner of PADÉ's table, and is used much more than the remaining continued fractions of type I.

§If $A_n = 0$ and $B_n = 0$ for only a finite number of values of n, irregularities will occur for a time in the continued fraction, but limits for a_{2n} and a_{2n+1} will still exist, if the other two conditions are satisfied.

(2) *A common limit* a *exists for* $\sqrt[n]{A_n/A_{n-1}}$ *and* $\sqrt[n]{B_n/B_{n-1}}$.
(3) *Limits also exist for* q'_n/q''_{n-1} *and* q''_n/q'_n *in which*

$$q'_n = \frac{A_n}{a^n A_{n-1}}, \qquad q''_n = \frac{B_n}{a^n B_{n-1}}.$$

§ 6. *The third class of continued fractions.*

A parallel discussion of (III) leads to somewhat different results. Let it be expressed in the form

$$\frac{U_2}{V_2} + \frac{\frac{e_2 z^{p_2}}{f_2 z + g_2}}{1} + \frac{\frac{e_3 z}{(f_3 z + g_3)(f_2 z + g_2)}}{1} + \frac{\frac{e_4 z}{(f_4 z + g_4)(f_3 z + g_3)}}{1} \cdots,$$

and then put

(17) $$x' = x^2 = \frac{ez}{(fz + g)^2},$$

in which e, f, g denote the limits of e_n, f_n, g_n. We obtain such a continued fraction as (14).* A difference, however, manifests itself in passing thence to the equation of the form (11) which gives the $(n + m)$th convergent of (III). For it is not the case that the expansion of a rational fraction into a continued fraction of the third type will have necessarily only a finite number of terms. Consequently it is possible for the denominator of (11) to vanish identically, and the convergence or divergence of our continued fraction will depend upon its initial terms or upon any irregularities which may be introduced. Take, for example, the continued fraction

$$\frac{U_2(z)}{U_3(z)} - \frac{z}{1 + z} - \frac{z}{1 + z} - \frac{z}{1 + z} - \cdots.$$

If the first partial fraction be omitted, the nth convergent is

$$-z\frac{1 + z + \cdots + z^{n-1}}{1 + z + z^2 + \cdots + z^n},$$

which has the limit $-z$ if $|z| < 1$ and the limit -1 if $|z| > 1$. Hence the original continued fraction diverges for $|z| < 1$ if $U_3(z) = z$ and for $|z| > 1$ if $U_3(z) = 1$.

The form of cut to be made in limiting the region of convergence can be found by transforming the rectilinear cut for (14) by the substitution (17). This substitution, by a linear change in the x'- and z-coördinates, can be reduced to

$$x' = \tfrac{1}{2}\left(z' + \frac{1}{z'}\right),$$

* e_{m+n} and the convergents are still rational in z, though not in x.

which is discussed thoroughly in HOLZMÜLLERS's *Theorie der isogonalen Verwandschaften*, p. 143.

Attention should be called to one special case of frequent occurrence. Place first $g = 1$, which may be done without loss of generality. Then suppose $e = -f$. Equation (17) may be written

$$\tfrac{1}{2}\left(\frac{1}{x'} + 2\right) = \tfrac{1}{2}\left(ez + \frac{1}{ez}\right),$$

By this substitution the rectilinear cut along the real axis of the x-plane from $x' = -\frac{1}{4}$ to $x' = -\infty$ is converted into the entire circumference of a circle of the z-plane having the origin as its center and having a radius equal to $1/e$. The region of convergence is therefore in this case a circle.

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