ON CERTAIN COMPLETE SYSTEMS OF QUATERNION EXPRESSIONS, AND ON THE REMOVAL OF METRIC LIMITATIONS FROM THE CALCULUS OF QUATERNIONS*

BY

JAMES MILLS PEIRCE

§ 1. Every student of quaternions is familiar with the use, in connection with a system of three non-complanar vectors, α, β, γ, of a supplementary system of derived vectors, α', β', γ', which I shall here determine by the equations:

$$(1) \qquad \alpha' = \frac{\mathrm{V}\beta\gamma}{\mathrm{S}\gamma\beta\alpha}, \qquad \beta' = \frac{\mathrm{V}\gamma\alpha}{\mathrm{S}\gamma\beta\alpha}, \qquad \gamma' = \frac{\mathrm{V}\alpha\beta}{\mathrm{S}\gamma\beta\alpha}.$$

These equations give:

$$(2) \qquad \left\{ \begin{array}{c} \mathrm{S}\alpha\alpha' = \mathrm{S}\beta\beta' = \mathrm{S}\gamma\gamma' = -1, \\ \mathrm{S}\alpha\beta' = \mathrm{S}\alpha\gamma' = \mathrm{S}\beta\gamma' = \mathrm{S}\beta\alpha' = \mathrm{S}\gamma\alpha' = \mathrm{S}\gamma\beta' = 0, \\ \mathrm{S}\alpha\beta\gamma\,\mathrm{S}\alpha'\beta'\gamma' = 1, \\ \alpha = \dfrac{\mathrm{V}\beta'\gamma'}{\mathrm{S}\gamma'\beta'\alpha'}, \qquad \beta = \dfrac{\mathrm{V}\gamma'\alpha'}{\mathrm{S}\gamma'\beta'\alpha'}, \qquad \gamma = \dfrac{\mathrm{V}\alpha'\beta'}{\mathrm{S}\gamma'\beta'\alpha'}. \end{array} \right.$$

Thus α, β, γ can be derived from α', β', γ', just as α', β', γ' are derived from α, β, γ; and all the relations between these two sets of vectors are reciprocal.

The above equations give, for any vector ρ, the two identical expressions:

$$(3) \qquad \rho = -(\alpha\mathrm{S}\alpha'\rho + \beta\mathrm{S}\beta'\rho + \gamma\mathrm{S}\gamma'\rho) = -(\alpha'\mathrm{S}\alpha\rho + \beta'\mathrm{S}\beta\rho + \gamma'\mathrm{S}\gamma\rho).$$

It is to be noted that α', β', γ' are the negatives of the reciprocals of the vector perpendiculars dropped from the origin O on planes which pass through the points A, B, C, parallel to the planes OBC, OCA, OAB, respectively, and, in like manner, α, β, γ are the negatives of the reciprocals of the vector perpendiculars dropped from O on planes passing through A$'$, B$'$, C$'$, parallel to OB$'$C$'$, OC$'$A$'$,

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OA′B′. Moreover, the volumes of the tetrahedra OABC and OA′B′C′ are reciprocal to each other.

§ 2. The six vectors α, β, γ, α', β', γ', may be said to constitute a *complete system* of vectors. I wish to point out the existence of other complete systems both of vectors and of quaternions, and also to propose certain modifications of the quaternion method, which are closely related to such systems, and which tend to enlarge the freedom of quaternion reasoning.

It is often assumed that the calculus of Quaternions is, in its relation to geometry, essentially a metric, as distinguished from a graphic, or projective, system. But this character seems to me to belong merely to the forms under which it is usually presented, and not to be inherent in the calculus itself. I shall show that it is easy to introduce into quaternions the principle of the *dualism of points and planes* familiar in modern analysis, and the principle of *homogeneousness*, which gives so great an advantage in projective geometry to tetrahedral over Cartesian coördinates, also to assume arbitrarily *any four linearly independent quaternions* as the fundamental geometric elements, and thus entirely to discard from our system all metric ideas.

§ 3. If $\lambda = $ OL, where O is an assumed origin, we are accustomed to call λ "the vector of L" and L "the point λ." I purpose to employ the same symbol λ to denote also the *plane* which is polar to L relatively to the unit-sphere ($\rho^2 = -1$), having its center at the origin. Thus, a plane is to be denoted by the negative of the reciprocal of the vector perpendicular dropped on the plane from the origin; and we shall call any vector λ the vector of the plane which it represents in this system, as well as of the pole of that plane, and shall call the plane "the plane λ." The same vector symbol will therefore admit two interpretations. In this system, α', β', γ' defined as in § 1, are the vectors of the points A′, B′, C′, and also of the planes drawn through A, B, C, respectively, parallel to the opposite faces of the tetrahedron OABC; while α, β, γ admit corresponding double interpretations. In this system, again, $\lambda = 0$ for the origin and for the plane infinity; $\lambda = \infty$ for any point at infinity and for any plane through the origin. It will be shown later that infinites and indeterminates can be avoided by a further modification of the system (see § 8).

The condition of the *collocation* of a point ρ and a plane ω, that is, the condition of the point lying in the plane, or of the plane passing through the point, is:

$$S\omega\rho = -1. \tag{4}$$

This equation, which represents *any* linear scalar equation in ρ or in ω, is the equation of the plane ω, when ω is constant; of the point ρ, when ρ is constant; and when neither ω nor ρ is constant, of *any collocated point and plane.*

The relation between ρ and ω is analogous to that between the coördinates of a collocated point and plane in the ordinary analysis. This is clearly shown by writing, in terms of the complete system of § 1,

$$(5)\qquad \begin{cases} \rho = x\alpha + y\beta + z\gamma, \\ \omega = u\alpha' + v\beta' + w\gamma'; \end{cases}$$

whence (4) gives at once

$$(6)\qquad ux + vy + wz = 1,$$

which represents the Cartesian equation of the collocation of a point and a plane; their coördinates being in fixed ratios to x, y, z, u, v, w.

§ 4. Hamilton has pointed out in the *Elements*, Book I, chap. iii, § 3, the advantage which may sometimes be gained by the use of *four* vectors, instead of three, for determining position; and has indicated the relation of such a system to tetrahedral (which he calls anharmonic) coördinates, and its adaptation to the expression of double ratios. But his treatment of the matter is inadequate; because he has not employed vectors dualistically, to represent both points and planes, and has failed to make his system of vectors complete by the introduction of four complementary vectors. The complete system may be formed as follows:

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be any four vectors, not termino-complanar, determining therefore the four vertices of a tetrahedron $\mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\mathrm{A}_4$. Let us adopt the notation

$$(7)\qquad \begin{cases} a_1 = S\alpha_4\alpha_3\alpha_2, \quad a_2 = S\alpha_3\alpha_4\alpha_1, \quad a_3 = S\alpha_2\alpha_1\alpha_4, \quad a_4 = S\alpha_1\alpha_2\alpha_3, \\ A = a_1 + a_2 + a_3 + a_4. \end{cases}$$

By a well-known identity, which is equivalent to (3),

$$(8)\qquad a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0,$$

and since $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are not termino-complanar,

$$(9)\qquad A \neq 0.$$

Moreover, if we make the assumption that the origin (which is otherwise unlimited in position) lies in neither face of the tetrahedron, none of the quantities a_1, a_2, a_3, a_4, vanishes.

Let us now assume the vectors of the four faces of the fundamental tetrahedron as our four complementary vectors, $\beta_1, \beta_2, \beta_3, \beta_4$. We have then

$$(10)\qquad \begin{cases} \beta_1 = \dfrac{V(\alpha_2\alpha_3 + \alpha_3\alpha_4 + \alpha_4\alpha_2)}{a_1}, \quad \beta_3 = \dfrac{V(\alpha_4\alpha_1 + \alpha_1\alpha_2 + \alpha_2\alpha_4)}{a_3}, \\ \beta_2 = \dfrac{V(\alpha_1\alpha_4 + \alpha_4\alpha_3 + \alpha_3\alpha_1)}{a_2}, \quad \beta_4 = \dfrac{V(\alpha_3\alpha_2 + \alpha_2\alpha_1 + \alpha_1\alpha_3)}{a_4}. \end{cases}$$

Computing hence the values of

$$(11)\quad \begin{cases} b_1 = S\beta_4\beta_3\beta_2, \quad b_2 = S\beta_3\beta_4\beta_1, \quad b_3 = S\beta_2\beta_1\beta_4, \quad b_4 = S\beta_1\beta_2\beta_3, \\ B = b_1 + b_2 + b_3 + b_4, \end{cases}$$

we readily find that

$$(12)\qquad \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \frac{b_4}{a_4} = \frac{B}{A} = \frac{A^2}{a_1 a_2 a_3 a_4} = \frac{b_1 b_2 b_3 b_4}{B^2},$$

whence

$$(13)\qquad A^2 B^2 = a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4;$$

and we obtain also the results

$$(14)\quad \begin{cases} S\alpha_1\beta_1 = \dfrac{A}{a_1} - 1 = \dfrac{B}{b_1} - 1, \qquad S\alpha_3\beta_3 = \dfrac{A}{a_3} - 1 = \dfrac{B}{b_3} - 1, \\ S\alpha_2\beta_2 = \dfrac{A}{a_2} - 1 = \dfrac{B}{b_2} - 1, \qquad S\alpha_4\beta_4 = \dfrac{A}{a_4} - 1 = \dfrac{B}{b_4} - 1, \\ S\alpha_i\beta_j = -1, \text{ whenever } i \neq j. \end{cases}$$

From the symmetry of these equations, or by actual computation, we see that the α's are related to the β's just as the β's are related to the α's; so that if the β's are taken as the vertices of a tetrahedron $\mathrm{B}_1\mathrm{B}_2\mathrm{B}_3\mathrm{B}_4$, the α's are the vectors of the faces of that tetrahedron, and may be expressed in terms of the β's by forms exactly analogous to (10).

§ 5. Since any five vectors satisfy one linear equation in which the sum of the coefficients is zero, and only one such equation unless all the vectors are termino-complanar, it follows that any vectors ρ and ω may be written in the forms

$$(15)\quad \begin{cases} \rho = \dfrac{m_1 x_1 \alpha_1 + m_2 x_2 \alpha_2 + m_3 x_3 \alpha_3 + m_4 x_4 \alpha_4}{m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4}, \\ \omega = \dfrac{n_1 u_1 \beta_1 + n_2 u_2 \beta_2 + n_3 u_3 \beta_3 + n_4 u_4 \beta_4}{n_1 u_1 + n_2 u_2 + n_3 u_3 + n_4 u_4}, \end{cases}$$

where the four m's and the four n's are scalar constants arbitrarily chosen, provided none is taken equal to zero, while the x's and u's are scalar variables, of which the ratios in each of the two expressions are singly determinate for fixed values of ρ and ω.

Writing the first of equations (15) in the form

$$m_1 x_1(\rho - \alpha_1) + m_2 x_2(\rho - \alpha_2) + m_3 x_3(\rho - \alpha_3) + m_4 x_4(\rho - \alpha_4) = 0,$$

and observing that this equation is unique, we see, by the principle represented by equation (8), that

$$
\begin{aligned}
m_1x_1 : m_2x_2 : m_3x_3 : m_4x_4 &= S(\rho-\alpha_4)(\rho-\alpha_3)(\rho-\alpha_2) : S(\rho-\alpha_3)(\rho-\alpha_4)(\rho-\alpha_1) \\
&\quad : S(\rho-\alpha_2)(\rho-\alpha_1)(\rho-\alpha_4) : S(\rho-\alpha_1)(\rho-\alpha_2)(\rho-\alpha_3) \qquad (16) \\
&= \text{tet } A_4A_3A_2R : \text{tet } A_3A_4A_1R : \text{tet } A_2A_1A_4R : \text{tet } A_1A_2A_3R,
\end{aligned}
$$

where R is the point ρ. Hence, the x's may be identified with the tetrahedral coördinates of R referred to $A_1A_2A_3A_4$.

Again, the perpendiculars dropped from A_1, A_2, A_3, A_4, on the plane ω are in the continued proportion

$$(17)\qquad S\omega\alpha_1 + 1 : S\omega\alpha_2 + 1 : S\omega\alpha_3 + 1 : S\omega\alpha_4 + 1 = \frac{n_1u_1}{a_1} : \frac{n_2u_2}{a_2} : \frac{n_3u_3}{a_3} : \frac{n_4u_4}{a_4};$$

so that the u's may be identified with the tetrahedral coördinates of the plane ω referred to $A_1A_2A_3A_4$.

Substituting from (15) in $S\omega\rho = -1$, the equation of collocation of the point ρ and the plane ω, we have, by easy reductions,

$$\frac{m_1n_1}{a_1}u_1x_1 + \frac{m_2n_2}{a_2}u_2x_2 + \frac{m_3n_3}{a_3}u_3x_3 + \frac{m_4n_4}{a_4}u_4x_4 = 0,$$

or, if the m's and n's are so taken that

$$(18)\qquad m_1n_1 : m_2n_2 : m_3n_3 : m_4n_4 = a_1 : a_2 : a_3 : a_4,$$

$$(19)\qquad u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0,$$

the usual form of the equation of collocation of a point and a plane in tetrahedral coördinates.

§ 6. The dualism between points and planes can be further exhibited by the consideration of some familiar relations between vectors.

The condition of the collinearity of two vectors is that there is an equation of the form

$$(20)\qquad x_1\lambda_1 + x_2\lambda_2 = 0, \qquad \text{or that} \qquad V\lambda_1\lambda_2 = 0;$$

that of the complanarity of three vectors is that there is an equation of the form

$$(21)\qquad x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3 = 0, \qquad \text{or that} \qquad S\lambda_1\lambda_2\lambda_3 = 0;$$

while any four vectors in space satisfy an equation of the form

$$(22)\qquad \begin{cases} x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3 + x_4\lambda_4 = 0, \qquad \text{in which} \\ x_1 : x_2 : x_3 : x_4 = S\lambda_2\lambda_3\lambda_4 : S\lambda_1\lambda_4\lambda_3 : S\lambda_4\lambda_1\lambda_2 : S\lambda_3\lambda_2\lambda_1. \end{cases}$$

If the λ's are vectors of points, (20) expresses the condition that the line joining two points passes through the origin; (21) the condition that the plane containing three points passes through the origin; while (22) is true for any four points and is, indeed, the same as (8). If the λ's are vectors of planes, (20) expresses the condition of the parallelism of two planes, that is, that their common line lies in the plane infinity; (21) expresses the condition that three planes are parallel to one line, that is, that their common point lies in the plane infinity; while (22) is true for any four planes.

If, now, (20) is accompanied by the further condition

(23) $$x_1 + x_2 = 0, \qquad \text{which gives} \qquad \lambda_1 = \lambda_2,$$

the two points or two planes are coincident.

If (21) is accompanied by the condition

(24) $$x_1 + x_2 + x_3 = 0, \qquad \text{which gives} \qquad \mathrm{V}(\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2) = 0,$$

we have the condition that three points lie in one line or that three planes intersect in one line; since we easily deduce from (21) and (24) that for any value of σ,

(25) $$x_1(\mathrm{S}\lambda_1\sigma + 1) + x_2(\mathrm{S}\lambda_2\sigma + 1) + x_3(\mathrm{S}\lambda_3\sigma + 1) \equiv 0;$$

whence it follows that all values of σ which satisfy two of the three equations,

(26) $$\mathrm{S}\lambda_1\sigma + 1 = 0, \qquad \mathrm{S}\lambda_2\sigma + 1 = 0, \qquad \mathrm{S}\lambda_3\sigma + 1 = 0,$$

also satisfy the third; and conversely, if this be the case there must be an identity (25), whence (21) and (24) are easily obtained. If then the λ's denote three points, and σ a plane, every plane which contains two of the three points contains also the third, so that the three points are collinear, and conversely; but if the λ's denote three planes, and σ a point, every point belonging to two of the three planes lies also in the third, so that the three planes intersect in one line, and conversely.

It is shown, in like manner, that if, in (22),

(27) $$x_1 + x_2 + x_3 + x_4 = 0, \quad \text{that is, if} \quad \mathrm{S}(\lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_4\lambda_3 + \lambda_4\lambda_1\lambda_2 + \lambda_3\lambda_2\lambda_1) = 0,$$

we have the condition of four points lying in one plane, or of four planes intersecting in one point.

The theorems of this section can also be obtained more briefly. For those which concern points follow immediately from the well-known propositions concerning linear equations between vectors; and those which concern planes can then be deduced from the others by the reciprocal relations between poles and polars.

§ 7. We pass next from complete systems of vectors to those of quaternions. Let p_1, p_2, p_3, p_4 be any four linearly independent quaternions; and let q_1, q_2, q_3, q_4 be four quaternions such that

(28) $$\mathrm{S}p_1q_1 = \mathrm{S}p_2q_2 = \mathrm{S}p_3q_3 = \mathrm{S}p_4q_4 = -1, \qquad \text{while} \qquad \mathrm{S}p_iq_j = 0, \text{ if } i \neq j.$$

The four q's exist, and constitute a unique system; for each is determined by four linearly independent scalar equations of the first degree. Indeed, we know, by the theory of determinants of quaternions, that, using the notation

$$(29)\qquad \begin{cases} \bar{p}_1 = \mathrm{K}p_1, \qquad \bar{p}_2 = \mathrm{K}p_2, \quad \text{etc.,} \\ R_1 = \frac{1}{6}\,|\,\bar{p}_4 p_3 \bar{p}_2\,|, \qquad R_2 = \frac{1}{6}\,|\,\bar{p}_3 p_4 \bar{p}_1\,|, \quad \text{etc.,} \\ R = \frac{1}{24}\,|\,p_1 \bar{p}_2 p_3 \bar{p}_4\,|; \end{cases}$$

$$(30)\qquad q_1 = \frac{R_1}{R}, \qquad q_2 = \frac{R_2}{R}, \qquad q_3 = \frac{R_3}{R}, \qquad q_4 = \frac{R_4}{R}.$$

Writing $p_1 = c_1 + \alpha_1$, $p_2 = c_2 + \alpha_2$, etc., we may express these forms as follows:

$$(31)\qquad \begin{cases} q_1 = \dfrac{\mathrm{S}\alpha_4\alpha_3\alpha_2 + (c_2\mathrm{V}\alpha_3\alpha_4 + c_3\mathrm{V}\alpha_4\alpha_2 + c_4\mathrm{V}\alpha_2\alpha_3)}{c_1\mathrm{S}\alpha_4\alpha_3\alpha_2 + c_2\mathrm{S}\alpha_1\alpha_4\alpha_3 + c_3\mathrm{S}\alpha_4\alpha_1\alpha_2 + c_4\mathrm{S}\alpha_3\alpha_2\alpha_1}, \\[2ex] q_2 = \dfrac{\mathrm{S}\alpha_3\alpha_4\alpha_1 + (c_1\mathrm{V}\alpha_4\alpha_3 + c_4\mathrm{V}\alpha_3\alpha_1 + c_3\mathrm{V}\alpha_1\alpha_4)}{c_1\mathrm{S}\alpha_2\alpha_3\alpha_4 + c_2\mathrm{S}\alpha_1\alpha_4\alpha_3 + c_3\mathrm{S}\alpha_4\alpha_1\alpha_2 + c_4\mathrm{S}\alpha_3\alpha_2\alpha_1}, \\[2ex] q_3 = \dfrac{\mathrm{S}\alpha_2\alpha_1\alpha_4 + (c_4\mathrm{V}\alpha_1\alpha_2 + c_1\mathrm{V}\alpha_2\alpha_4 + c_2\mathrm{V}\alpha_4\alpha_1)}{c_1\mathrm{S}\alpha_2\alpha_3\alpha_4 + c_2\mathrm{S}\alpha_1\alpha_4\alpha_3 + c_3\mathrm{S}\alpha_4\alpha_1\alpha_2 + c_4\mathrm{S}\alpha_3\alpha_2\alpha_1}, \\[2ex] q_4 = \dfrac{\mathrm{S}\alpha_1\alpha_2\alpha_3 + (c_3\mathrm{V}\alpha_2\alpha_1 + c_2\mathrm{V}\alpha_1\alpha_3 + c_1\mathrm{V}\alpha_3\alpha_2)}{c_1\mathrm{S}\alpha_2\alpha_3\alpha_4 + c_2\mathrm{S}\alpha_1\alpha_4\alpha_3 + c_3\mathrm{S}\alpha_4\alpha_1\alpha_2 + c_4\mathrm{S}\alpha_3\alpha_2\alpha_1}. \end{cases}$$

It is evident, moreover, from the symmetry of the equations (28) that the p's may be derived from the q's by the same rule by which the q's are derived from the p's. It is evident also that, since the p's are linearly independent, none of the above written determinants vanishes.

It is now easily seen that any quaternion r can be expressed identically in either of the following forms:

$$(32)\qquad \begin{cases} r = -\,(p_1\mathrm{S}q_1 r + p_2\mathrm{S}q_2 r + p_3\mathrm{S}q_3 r + p_4\mathrm{S}q_4 r) \\ \;\;\, = -\,(q_1\mathrm{S}p_1 r + q_2\mathrm{S}p_2 r + q_3\mathrm{S}p_3 r + q_4\mathrm{S}p_4 r). \end{cases}$$

These identities correspond to the two identities (3) for any vector ρ; and they may also be written

$$(33)\qquad |\,p_1\bar{p}_2 p_3 \bar{r}\,| = |\,q_1 \bar{q}_2 q_3 \bar{r}\,| = 0;$$

just as (3) may be written

$$(34)\qquad |\,\alpha\beta\gamma\rho\,| = |\,\alpha'\beta'\gamma'\rho\,| = 0.$$

§ 8. I have shown that the principle of dualism can be introduced into quaternion analysis by using one and the same vector symbol to denote either a point or a plane. But in order to free our calculus entirely from the restrictions of metric geometry, we need also the principle of homogeneousness, whereby we shall be able to dispense with infinites and indeterminates. This is accomplished by representing a point or a plane by a *quaternion*, and not by a mere vector. We shall at first assume for this purpose a quaternion such that the ratio of its vector to its scalar shall be the vector of the point or plane in the system expounded in § 3. Thus, the quaternion of a point or a plane shall have an undetermined scalar factor m, so that p and mp represent the same point or plane, so long as m is scalar. In this system, when the quaternion of a point is any vector, the point is at infinity, in the direction of that vector; when it is any scalar, the point is at the origin. When the quaternion of a plane is any vector, the plane passes through the origin, perpendicular to the vector; when any scalar, it is the plane infinity. Every equation which expresses a true geometric relation is necessarily homogeneous, in this system. The equation of the unit-sphere having its center at the origin and the equation of collocation of a point and plane are respectively:

$$\mathrm{S}p^2 = 0, \qquad \text{and} \qquad \mathrm{S}pq = 0. \tag{35}$$

Let now the four vertices of a tetrahedron be determined in this system by four quaternions p_1, p_2, p_3, p_4, which are necessarily linearly independent, since otherwise the four vertices would lie in one plane; and let q_1, q_2, q_3, q_4 be derived from them by (28) or (31). Then any two quaternions, p and q, may be expressed in the forms: —

$$\begin{cases} p = x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4, \\ q = u_1 q_1 + u_2 q_2 + u_3 q_3 + u_4 q_4; \end{cases} \tag{36}$$

in which it is unnecessary to write arbitrary factors, m and n, as in (15), since such factors are implicitly contained in the p's and q's.

Substituting these expressions in the second equation of (35), we have at once, by (28), the ordinary equation of collocation for tetrahedral coördinates:

$$u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_4 = 0. \tag{37}$$

It is to be noted that the coördinates x and u, may be defined both here and in § 5, by certain anharmonic relations, and the forms p and q are therefore well adapted to the study of projective geometry. For if we assume a point p_0 and a plane q_0 such that

$$\begin{cases} p_0 = p_1 + p_2 + p_3 + p_4, \\ q_0 = q_1 + q_2 + q_3 + q_4, \end{cases} \tag{38}$$

where either the point may be assumed as any point not lying in a face of the fundamental tetrahedron, or the plane may be assumed as any plane not passing through a vertex, then for any point p,

$$x_h : x_i = (\mathbf{A}_j \mathbf{A}_k \cdot \mathbf{A}_h \mathbf{P}_0 \mathbf{A}_i \mathbf{P}); \tag{39}$$

where the expression in the second member denotes the double ratio of a pencil of four planes having $A_j A_k$ for its axis, and passing respectively through A_h, P_0, A_i, P, h and i denoting any two suffixes and j and k the other two.

In like manner, if any plane q and the fixed plane q_0 intersect $A_h A_i$ in points Q_{hi} and Q'_{hi},

$$u_h : u_i = (A_i Q'_{hi} A_h Q_{hi}). \tag{40}$$

§ 9. We ought, however, to take one further step, in order to give our system the utmost generality, namely, to eliminate from it the metric condition that there is a relation of perpendicularity involved in the dualistic representation of a point and a plane.

Let then *any four* linearly independent quaternions be arbitrarily chosen to denote the four vertices of our tetrahedron; and let four complementary quaternions be derived from them by (28) or (31). We may now assume arbitrarily any point p_0 or else any plane q_0, taking p_0 and q_0 as in (38); and then the point or plane represented by *any* value of p or q may be determined so as to satisfy (39) or (40). The point and plane which are denoted by the same symbol must be polar to each other relatively to the surface

$$Sp^2 = 0; \tag{41}$$

but this may be *any* non-singular quadric; it is not in general a sphere. Thus, if we assume arbitrarily the point p_0, we obtain by (41) the plane p_0, whence we obtain the plane q_0 by the aid of (31) and (40).

In this system, as in that of § 8, it follows from the anharmonic property that the condition of the *coincidence* of two points or of two planes is that their quaternions satisfy a linear equation,

$$ap + bq = 0, \qquad \text{i. e., that} \qquad |p\bar{q}| = 0; \tag{42}$$

that the condition of the *collinearity* of three points or of three planes, is

$$ap + bq + cr = 0, \qquad \text{i. e., that} \qquad |p\bar{q}r| = 0; \tag{43}$$

and that the condition of four points belonging to a *common plane*, or four planes to a *common point*, is

$$ap + bq + cr + ds = 0, \qquad \text{i. e., that} \qquad |p\bar{q}r\bar{s}| = 0; \tag{44}$$

moreover, that the *double ratio* of four collinear points or of four collinear planes may be written

$$(pqrs) = \frac{|p\bar{q}| \times |r\bar{s}|}{|q\bar{r}| \times |s\bar{p}|}. \tag{45}$$

This last equation may be used to determine the quaternion of any point or plane, when that of an assumed point or plane is known.

§ 10. The general equation of a surface of the second order or second class in the system of § 9 may be written

$$(46) \qquad \mathrm{S}p\mathrm{f}p = 0\,;$$

where f is a linear function, which may be reduced to its self-conjugate part, while p is the quaternion of a moving point or of a moving plane. The properties of the surface may then be deduced from those of the linear function of a quaternion.

The differentiating operator ∇ may be generalized by the aid of the complete system of § 7, so as to be applicable to any function of a variable quaternion. Defining ∇ as an operator which satisfies the identity

$$(47) \qquad \mathrm{d}_p = -\,\mathrm{S}\cdot \mathrm{d}p\nabla_p,$$

we may write

$$(48) \qquad \begin{cases} p = x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4, \\ \nabla_p = q_1 \mathrm{D}_{x_1} + q_2 \mathrm{D}_{x_2} + q_3 \mathrm{D}_{x_3} + q_4 \mathrm{D}_{x_4}. \end{cases}$$

Thus, if $p = w + xi + yj + zk$,

$$(49) \qquad \nabla_p = -\,\mathrm{D}_w + i\mathrm{D}_x + j\mathrm{D}_y + k\mathrm{D}_z.$$

It can easily be shown that the result of the application of the operator ∇_p to any function of p is independent of the arbitrary choice of the four fundamental quaternions, p_1, p_2, p_3, p_4.

It is, moreover, readily seen that the polar of any point or plane p, relatively to the quadric represented by (46), is the plane or point q, where (omitting the unnecessary factor $-\,2$)

$$(50) \qquad q = -\,\tfrac{1}{2}\nabla_p \mathrm{S}p\mathrm{f}p = \mathrm{f}p\,.$$

HARVARD UNIVERSITY,
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