ON THE AUTOMORPHIC FUNCTIONS OF THE GROUP (0, 3; 2, 6, 6)*

BY

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In the recently published work of Fricke and Klein on automorphic functions,† attention is called to the desirability of studying groups, which, like the modular group, arise from the monodromy of the branch points of a given Riemann surface. On this suggestion I have examined a number of such cases, and present in the following pages some details relative to one of the simplest of these.

In § 1 the group arising from the monodromy of the branch-points of the given surface is determined and is found to be the group $(0, 3;\ 2, 6, 6)$.‡ In § 2 the "multiplicative" functions are considered, these being slightly more general than the Poincaré theta functions and including the latter as a subclass. Some general properties of such functions are discussed, and the simplest functions for the present group are determined. In § 3 definite analytic expressions are obtained for the multiplicative functions in terms of hyperelliptic theta functions with zero arguments. In § 4 certain groups which are related to the group $(0, 3;\ 2, 6, 6)$ are considered, and functions which are automorphic in these groups are expressed in terms of the functions already obtained.

§ 1. *The Riemann surface and the group associated with it.*

The Riemann surface considered, of genus $p = 3$, is defined by the equation

$$(1) \qquad y^6 = (x-a)^2(x-b)^2(x-c)^3(x-d)^5.$$

The period paths are chosen as indicated in figures 1, 2, 3.

The three integrals

$$u_1 = \int \frac{dx}{y}, \qquad u_2 = \int \frac{(x-a)(x-b)(x-c)^2(x-d)^4\,dx}{y^5},$$

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† *Automorphe Functionen*, Bd. 2, p. 136.

‡ See *Automorphe Functionen*, Bd. 1, p. 353, for this notation.

$$u_3 = \int \frac{(x-c)(x-d)\,dx}{y^2},$$

are of the first kind and linearly independent. Let A_i, B_i be the value of u_i

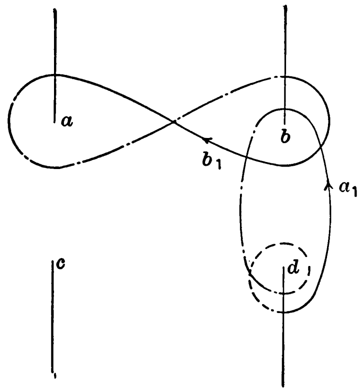


FIG. 1

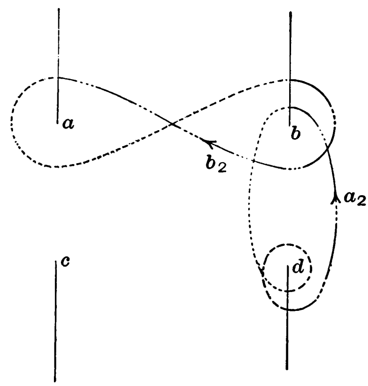


FIG. 2

when integrated around a_1 and b_1 respectively, and write ϵ for $e^{\pi i/3}$. Then the table of moduli of periodicity for these integrals is as follows:

	a_1	a_2	a_3	b_1	b_2	b_3
u_1	$-B_1$	B_1	$A_1-\epsilon^2 B_1$	A_1	$-A_1$	$\epsilon A_1 - B_1$
u_2	$-B_2$	B_2	$A_2+\epsilon B_2$	A_2	$-A_2$	$-\epsilon^2 A_2 - B_2$
u_3	$-B_3$	$-B_3$	$-A_3-\epsilon B_3$	A_3	A_3	$\epsilon^2 A_3 - B_3$

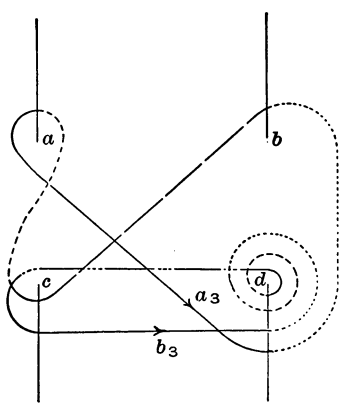
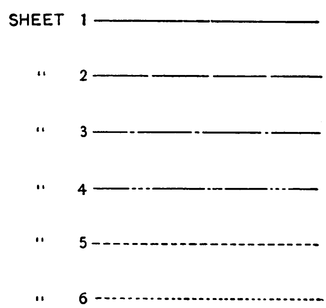


FIG. 3



Writing ω_i for the ratio $A_i : B_i$, the bilinear relations among the periods reduce to

$$\omega_2 = \frac{-\epsilon\omega_1 + 1}{\omega_1 - \epsilon^2}, \qquad \omega_3 = -\epsilon.$$

Instead of the system of moduli given in the preceding table, we introduce a simpler one by means of the canonical transformation

$$(2) \qquad \left|\begin{array}{ccc|ccc} 0 & 0 & 1 & & & \\ 0 & 1 & 0 & & 0 & \\ 1 & 1 & 0 & & & \\ \hline & & & 0 & 0 & 1 \\ & 0 & & -1 & 1 & 0 \\ & & & 1 & 0 & 0 \end{array}\right|.$$

Then, introducing the normal integrals v_1, v_2, v_3 in the usual way, the corresponding table of moduli for these will be:

$$(3) \qquad \begin{array}{c|ccc|ccc|} v_1 & \pi i & 0 & 0 & \pi i a & -2\pi i b & \pi i b \\ v_2 & 0 & \pi i & 0 & -2\pi i a & 2\pi i a & -\pi i a \\ v_3 & 0 & 0 & \pi i & \pi i b & -\pi i a & \pi i c \end{array},$$

in which

$$a = \frac{\epsilon^2\zeta^2 + \epsilon}{1 - \zeta^2}, \qquad b = \frac{1 + \dfrac{\epsilon + \epsilon^2}{\sqrt{2}}\zeta - \zeta^2}{1 - \zeta^2}, \qquad c = \frac{\epsilon - \frac{1}{2}\zeta^2}{1 - \zeta^2};$$

$$(4) \qquad \zeta = \frac{\sqrt{2}}{\omega_1 - \epsilon^2}.$$

We now proceed to determine the effect on the moduli of a monodromy of the branch points. First, let a and b be interchanged by a deformation of the Riemann surface which shall rotate positively the branch line joining these two points. The period paths a_k, b_k will be deformed into new paths a'_k, b'_k. Denote the value of u_i integrated along the old paths by $A_i^{(k)}$, $B_i^{(k)}$, respectively, and along the new paths by $A_i'^{(k)}$, $B_i'^{(k)}$. To express the new moduli in terms of the old, consider, for example, $A_i'^{(k)}$. Let T' denote the Riemann surface when cut along the paths a_k, b_k. Choose a closed path of integration for u_i in T' by starting from any point in a'_k and passing along this curve until it meets the boundary of T', then along the boundary to the point of a'_k on the opposite boundary, then along a'_k until it meets another boundary, and so on until the starting point is reached. The result of this integration is zero, whence, by transposing terms we obtain an expression for $A_i'^{(k)}$. The result is

$$A_i'^{(1)} = -B_i^{(1)} - B_i^{(3)}, \qquad B_i'^{(1)} = A_i^{(1)} + B_i^{(1)} + B_i^{(3)}.$$

From this follows

$$(5) \qquad \omega_1' = \frac{A_1'^{(1)}}{B_1'^{(1)}} = -\epsilon^2\omega_1 - 1.$$

Next, let a describe a closed path in the positive direction about c. This gives the transformation

$$A_i'^{(1)} = A_i^{(1)}, \qquad B_i'^{(1)} = A_i^{(1)} - A_i^{(2)} + B_i^{(1)} + B_i^{(3)},$$

whence

$$(6) \qquad \omega_1' = \frac{\omega_1}{\omega_1 + \epsilon}.$$

Corresponding to (5) and (6) we have for ζ the two transformations

$$(7) \qquad \zeta' = \epsilon\zeta,$$

$$(8) \qquad \zeta' = \frac{(\epsilon + 1)\zeta - \epsilon^2\sqrt{2}}{\zeta\sqrt{2} - (\epsilon^2 + 1)}.$$

By operating first with (7), then with (8) we get

$$(9) \qquad \zeta' = \frac{(\epsilon + \epsilon^2)\zeta - \epsilon^2\sqrt{2}}{\epsilon\zeta\sqrt{2} - (\epsilon + \epsilon^2)}.$$

Putting $\zeta = \eta/\sqrt{2}$ and $\zeta' = \eta'/\sqrt{2}$, (7) and (9) become

$$(10) \qquad \eta' = \epsilon\eta; \qquad \eta' = \frac{(\epsilon + \epsilon^2)\eta - 2\epsilon^2}{\epsilon\eta - (\epsilon + \epsilon^2)}.$$

The transformations of the η-group are evidently of the form

$$\begin{vmatrix} A + iB & 2(C + iD) \\ (C - iD) & A - iB \end{vmatrix}$$

when reduced to determinate unity. The coefficients are of two types.

Type 1:

$$A = \frac{a}{2}, \qquad B = \frac{b\sqrt{3}}{2}, \qquad C = \frac{c}{2}, \qquad D = \frac{d\sqrt{3}}{2},$$

in which a, b, c, d are integers.

Type 2:

$$A = \frac{a\sqrt{3}}{2}, \qquad B = \frac{b}{2}, \qquad C = \frac{c\sqrt{3}}{2}, \qquad D = \frac{d}{2},$$

in which a, b, c, d are integers.

The coefficients in (10) are evidently of these types. That any combination of these substitutions will have coefficients of the same types follows from the fact that the coefficients of the generating substitutions (and hence of any substitution produced by them) are of the form $a + b\epsilon$, a and b being integers, and that every such substitution may be reduced to one of determinant unity by multiplying all the coefficients by some power of $i\epsilon$, giving a substitution of the first or second type according as the power of $i\epsilon$ is even or odd.

Where η is replaced by $\zeta\sqrt{2}$ it is seen that the ζ-transformations are of the two types

$$\left|\begin{matrix} \dfrac{a+ib\sqrt{3}}{2} & \dfrac{c+id\sqrt{3}}{\sqrt{2}} \\[2ex] \dfrac{c-id\sqrt{3}}{\sqrt{2}} & \dfrac{a-ib\sqrt{3}}{2} \end{matrix}\right|, \qquad \left|\begin{matrix} \dfrac{a\sqrt{3}+ib}{2} & \dfrac{\sqrt{3}c+id}{\sqrt{2}} \\[2ex] \dfrac{c\sqrt{3}-id}{\sqrt{2}} & \dfrac{a\sqrt{3}-ib}{2} \end{matrix}\right|.$$

Introduce a new variable z by means of the equation

$$(11) \qquad \zeta = i\epsilon^2\frac{z-i}{z+i}.$$

After dividing all the coefficients by $-2i$ and substituting $a=\alpha$, $b=\gamma$, $c+d=-2\beta$, $3c-d=2\delta$, the transformation of the first type becomes

$$(\mathrm{I}) \qquad \left|\begin{matrix} \dfrac{\alpha+\beta\sqrt{6}}{2} & \dfrac{\gamma\sqrt{3}+\delta\sqrt{2}}{2} \\[2ex] \dfrac{-\gamma\sqrt{3}+\delta\sqrt{2}}{2} & \dfrac{\alpha-\beta\sqrt{6}}{2} \end{matrix}\right|,$$

in which

$$(12) \qquad \alpha^2 + 3\gamma^2 - 2(3\beta^2 + \delta^2) = 4.$$

Making the same substitution in type 2, dividing all the coefficients by $-2i$, and substituting $a=\alpha$, $b=\gamma$, $d+3c=-2\beta$, $d-c=2\delta$ we obtain

$$(\mathrm{II}) \qquad \left|\begin{matrix} \dfrac{\alpha\sqrt{3}+\beta\sqrt{2}}{2} & \dfrac{\gamma+\delta\sqrt{6}}{2} \\[2ex] \dfrac{-\gamma+\delta\sqrt{6}}{2} & \dfrac{\alpha\sqrt{3}-\beta\sqrt{2}}{2} \end{matrix}\right|$$

in which

$$(13) \qquad 3\alpha^2 + \gamma^2 - 2(\beta^2 + 3\delta^2) = 4.$$

Making the substitution (11) in (7) and (9) we have for the corresponding z-transformations

$$(14) \qquad z' = \frac{\frac{\sqrt{3}}{2}z + \frac{1}{2}}{-\frac{z}{2} + \frac{\sqrt{3}}{2}}, \qquad z' = \frac{\sqrt{2} - \sqrt{3}}{(\sqrt{2} + \sqrt{3})z}.$$

Denoting these by S and T respectively, we have now to show that *the group G generated by S and T consists of all the linear transformations contained in* (I) *and* (II).

It is evident that the fundamental region R of G may be taken to be the triangle bounded by the three circles (see Fig. 4)

$$(x \pm \sqrt{3})^2 + y^2 = 4,$$

$$x^2 + y^2 = (\sqrt{3} + \sqrt{2})^2 \qquad (x + iy = z).$$

It is further evident that G is contained within the group G' comprising all the substitutions (I) and (II), and hence the fundamental region R' of G' may be so chosen as to lie within R. It follows also from a general principle that R' may be so selected as to have only vertices that are fixed points for substitutions of the group. The mode of procedure would then be to write down the general expressions for the xy-coördinates of a fixed point, and subject them to the inequality conditions

$$(x \pm \sqrt{3})^2 + y^2 - 4 \leqq 0,$$

$$x^2 + y^2 - (\sqrt{3} + \sqrt{2})^2 \leqq 0,$$

these being the conditions that the point (x, y) shall be within or on the boundary of the triangle R. A somewhat lengthy analysis, but one that offers no particular difficulty, shows that the only fixed points of G' satisfying these conditions are the vertices of R itself [including the point $z = i(\sqrt{3} + \sqrt{2})$], and that all the substitutions of G' having these points as fixed points can be expressed as combinations of S and T.

Another form for the η-groups may be obtained by writing $a = 2\alpha + \beta$, $b = \beta$, $c = \gamma + \delta$, $d = \delta$ for substitutions of the first type, and $a = \beta$, $b = -2\alpha - \beta$, $c = \delta$, $d = -2\gamma - \delta$ for those of the second type whence it is seen that *the η-group consists of all the transformations of the two types*

$$\begin{vmatrix} \alpha + \beta\epsilon & 2(\gamma + \delta\epsilon) \\ \gamma - \delta\epsilon^2 & \alpha - \beta\epsilon^2 \end{vmatrix}, \qquad \begin{vmatrix} \alpha + \beta\epsilon & 2(\gamma + \delta\epsilon) \\ -\gamma + \delta\epsilon^2 & -\alpha + \beta\epsilon^2 \end{vmatrix}$$

in which α, β, γ, δ are integers subject to the condition

$$\alpha^2 + \alpha\beta + \beta^2 - 2(\gamma^2 + \gamma\delta + \delta^2) = 1.$$

§ 2. *The multiplicative functions.*

A function $\phi(\zeta)$ having the property

$$\phi\left(\frac{\alpha_k\zeta+\beta_k}{\gamma_k\zeta+\delta_k}\right)=\mu_k(\gamma_k\zeta+\delta_k)^d\phi(\zeta) \tag{15}$$

for all the substitutions of a given group, will be called a *multiplicative* function of degree d, with multipliers μ_k. We shall consider only functions without poles in the fundamental region.

To consider for the moment a more general case than the one under consideration let the fundamental region be a circle-arc polygon entirely within the fixed circle, whose vertices A_i are fixed points for elliptic substitutions of period l_i. Every function $\phi(\zeta)$ of given degree d will have a zero at A_i of an order not less than a certain number λ_i. There will also be a certain number N of simple zeros within (or in special cases on the boundary of) the region. This number is readily found to be*

$$N=\tfrac{1}{2}d(n-1)-\tfrac{1}{2}\sum_i\frac{d+2\lambda_i}{l_i},$$

in which n is the number of pairs of conjugate sides, and the summation relates to the different cycles of vertices.

To determine the value of λ_i, take one of the vertices $\zeta=a$ of the ith cycle. The elliptic substitution having a for fixed point may be written in the form

$$\frac{\zeta'-a}{\zeta'-a'}=e^{\frac{2\pi i}{l_i}}\frac{\zeta-a}{\zeta-a'},$$

in which a' is the other fixed point. Let μ_i be the multiplier corresponding to this substitution. Following the method of POINCARÉ† let a new function $\psi(\zeta)$ be defined by the relation

$$\psi(\zeta)=(\zeta-a')^d\phi(\zeta).$$

Then

$$\psi(\zeta')=\mu_i e^{-d\pi i/l_i}\psi(\zeta). \tag{16}$$

Assuming

$$\psi(\zeta)=\sum_{\nu=\lambda_i}^{\infty}A_\nu\left(\frac{\zeta-a}{\zeta-a'}\right)^\nu,$$

and substituting in (16) we find on equating the coefficients of the lowest powers in both members,

$$\mu_i=e^{\pi i(2\lambda_i+d)/l_i}. \tag{17}$$

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* Cf. FORSYTH, *Theory of functions*, § 307.

† Acta Mathematica, vol. 1 (1882), p. 193.

The degree d can be even, but cannot be odd for every group.

To show this, take the special case in which the fundamental region is a quadrilateral formed by two symmetrically situated triangles having like angles $\pi/l_i (i = 1, 2, 3)$, and let the triangles have in common the side adjacent to the angles π/l_1, π/l_3. Denote by S_1, S_2, S_3 the substitutions of periods l_1, l_2, l_3 having the vertices of one of the triangles of the quadrilateral for fixed points. Then $S_3 S_1 S_2 = 1$ and hence $\mu_1 \mu_2 \mu_3 = 1$, from which follows,

$$\sum_{i=1}^{3} \frac{2\lambda_i + d}{l_i} \equiv 0 \quad (\text{mod. } 2):$$

Assume $l_i = 2^{\rho_i} l'_i$ in which l'_i is odd and ρ_i is a positive integer or zero. Then the preceding relation becomes

$$\sum_i 2^{\rho_{i+1}+\rho_{i+2}} l'_{i+1} l'_{i+2} (2\lambda_i + d) = 2^{\rho_1+\rho_2+\rho_3} \cdot 2c,$$

in which c is an integer, and the subscripts are to be reduced (mod. 3) when they exceed 3. It is evident from this formula that d can be odd only when two of the exponents $\rho_{i+1} + \rho_{i+2}$ are equal and are less than the third. Suppose $\rho_i + \rho_j = \rho_i + \rho_k < \rho_j + \rho_k$. From this follows $\rho_j = \rho_k > \rho_i$. Hence, *the degree d can be odd only when two of the exponents $\rho_i (i = 1, 2, 3)$ are equal, and are greater than the third exponent.*

In the case of the ζ-group, $l_1 = l_2 = 6$, $l_3 = 2$, and accordingly $\rho_1 = \rho_2 = \rho_3 = 1$. Hence d cannot be odd. Let $d = 2m$. Then from (17)

$$\mu_i = e^{\pi i(\lambda_i + m)/3} \qquad (i = 1, 2),$$

$$\mu_3 = e^{\pi i(\lambda_3 + m)}.$$

The simplest cases are:

(a) $m = 1, \lambda_1 = 1, \lambda_2 = \lambda_3 = 0$; the function $\phi(\zeta)$ has a simple zero at e_1 (Fig. 4)* and at congruent points, and vanishes nowhere else.

(b) $m = 1, \lambda_2 = 1, \lambda_1 = \lambda_3 = 0$; $\phi(\zeta)$ has a simple zero at e_2 and at congruent points.

(c) $m = 3, \lambda_1 = \lambda_2 = 0, \lambda_3 = 0$; $\phi(\zeta)$ has a simple zero at e_3 and at congruent points.

The three different functions corresponding to these cases will be denoted by ϕ_1, ϕ_2, ϕ_3 respectively. It will be the object of the next section to prove that these functions actually exist.

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* This statement is used for brevity to mean that $\phi(\zeta)$ has a zero at the point in the ζ-plane which corresponds to the point e_1 by means of the transformation (11).

§ 3. *Transformation of the theta functions.*

The table of periods for u_1, u_2, u_3 (§ 1) can be reduced in an infinity of ways to the form

$$\begin{vmatrix} \omega_{11} & \omega_{12} & 0 & \omega'_{11} & \omega'_{12} & 0 \\ \omega_{21} & \omega_{22} & 0 & \omega'_{21} & \omega'_{22} & 0 \\ 0 & 0 & \omega_{33} & 0 & 0 & \omega'_{33} \end{vmatrix},$$

and hence the theta functions having the moduli of table (3) can be reduced to hyperelliptic theta functions in an infinity of different ways. A convenient reduction of this kind is obtained thus. Let the theta function with arguments zero be represented by the formula

$$(18) \qquad \vartheta\begin{bmatrix} g \\ h \end{bmatrix}(\zeta) = \sum_{m_1, m_2, m_3} e^{\sum\limits_{\kappa,\lambda}^{1,2,3} a_{\kappa\lambda}(m_\kappa+g_\kappa)(m_\lambda+g_\lambda)+\pi i \sum\limits_{\mu=1}^{3} h_\mu(m_\mu+g_\mu)},$$

in which

$$a_{11} = 2a_{22} = -a_{23} = \pi i a, \quad -2a_{12} = a_{13} = \pi i b, \quad a_{33} = \pi i c.$$

Introducing new letters of summation n_1, n_2, n_3 by means of the substitution

$$n_1 = m_1, \qquad n_2 = 2m_2 - m_3, \qquad n_3 = m_3,$$

the right member of formula (18) becomes

$$\vartheta\begin{bmatrix} g_1 & g_2 - g_3/2 \\ h_1 & h_2 \end{bmatrix}(\zeta)\,\vartheta\begin{bmatrix} g_3/2 \\ h_2 + 2h_3 \end{bmatrix}(2\epsilon)$$
$$+ \vartheta\begin{bmatrix} g_1 & (1 + 2g_2 - g_3)/2 \\ h_1 & h_2 \end{bmatrix}(\zeta)\,\vartheta\begin{bmatrix} (1 + g_3)/2 \\ h_2 + 2h_3 \end{bmatrix}(2\epsilon),$$

in which the hyperelliptic theta functions have the moduli a_{11}, a_{12}, a_{22} and the elliptic theta functions have the modulus 2ϵ, the arguments being zero in all cases. We proceed now to determine the effect of the transformations S and T, formulæ (7) and (9) on the functions $\vartheta\left[\begin{smallmatrix} g_1 & g_2 \\ h_1 & h_2 \end{smallmatrix}\right](\zeta)$.

To the transformation S on ζ corresponds the transformation

$$\begin{vmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & -2 & 0 & 0 \\ 0 & 4 & 2 & 1 \end{vmatrix}$$

on the theta moduli. This leads to the relation

$$(19)\quad \frac{1}{\sqrt{1-\zeta'^2}}\,\vartheta\begin{bmatrix} -\frac{1}{2}+g_1-2g_2+h_1 & g_2 \\ -g_1+2g_2 & -4g_2+2h_1+h_2 \end{bmatrix}(\zeta')$$

$$= \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\pi i[g_1^2-4g_1g_2+8g_2^2+2g_1h_1-4g_2h_1-\frac{1}{12}]}\,\vartheta\begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix}(\zeta),$$

in which $\zeta' = \epsilon\zeta$. The signs of the radicals are chosen so as to be alike when $\zeta = 0$.

To T corresponds the transformation

$$\left|\begin{array}{cc|cc} 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ \hline -2 & 0 & -1 & -1 \\ 0 & -4 & -2 & -1 \end{array}\right|$$

on the theta moduli and hence the relation

$$(20)\quad \frac{1}{\sqrt{1-\zeta'^2}}\,\vartheta\begin{bmatrix} g_1+2g_2-2h_1-h_2 & g_1+g_2-h_1-h_2 \\ 2g_1-h_1-h_2 & 4g_2-2h_1-h_2 \end{bmatrix}(\zeta')$$

$$= \frac{\sqrt{2}\epsilon\zeta-(\epsilon+\epsilon^2)}{\sqrt{1-\zeta^2}}\, e^{\pi i E}\,\vartheta\begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix}(\zeta),$$

in which

$$\zeta' = \frac{(\epsilon+\epsilon^2)\zeta-\epsilon^2\sqrt{2}}{\sqrt{2}\epsilon\zeta-(\epsilon+\epsilon^2)}$$

and

$$E = 2g_1^2+8g_1g_2+4g_2^2-8g_1h_1-4g_1h_2-8g_2h_1-8g_2h_2+4h_1^2+6h_1h_2+2h_2^2+\tfrac{1}{2}.$$

The signs of the radicals are to be alike when $\zeta' = \zeta$.

For brevity we write

$$\frac{1}{\sqrt{1-\zeta^2}}\,\vartheta\begin{bmatrix} g_1 g_2 \\ h_1 h_2 \end{bmatrix}(\zeta) = \Phi\begin{bmatrix} g_1 g_2 \\ h_1 h_2 \end{bmatrix}.$$

Then by means of formulæ (19) and (20) it is easy to verify that the functions ϕ_1, ϕ_2, ϕ_3 referred to at the end of the preceding section may be expressed in the form

$$\phi_2 = \Phi\begin{bmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{3} \end{bmatrix}\Phi\begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 \end{bmatrix},$$

$$\phi_1 = \frac{1}{\phi_2}\,\Phi\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\Phi\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\Phi\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\Phi\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

$$\phi_3 = \Phi\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\Phi\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}\Phi\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}\Phi\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\Phi\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\Phi\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}.$$

It is necessary to show that these functions do not vanish identically. For this purpose consider the value of the function $\Phi\begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 \end{bmatrix}$ when ζ is zero. It readily reduces to

$$e^{\frac{7\pi i}{72}} \sum_{n_1=-\infty}^{+\infty} e^{-\frac{\pi\sqrt{3}}{2}(n_1-\frac{1}{6})^2+\frac{\pi i}{2}(n_1^2-n_1)} \cdot \sum_{n_2=-\infty}^{+\infty} e^{-\pi\sqrt{3}(n_2+\frac{1}{6})^2}.$$

The last factor is evidently not zero. By writing for brevity $e^{-\pi\sqrt{3}/72} = t$, the first series may be put in the form

$$t(1 - t^{48} - t^{288}) + \sum_{n=1}^{\infty} [t^{(24n-1)^2} - t^{(24n+17)^2}]$$
$$+ \sum_{n=1}^{\infty} [t^{(24n+1)^2} - t^{(24n+7)^2}] + t^{25}(1 - t^{96} - t^{144})$$
$$+ \sum_{n=1}^{\infty} [t^{(24n+5)^2} - t^{(24n+11)^2}] + \sum_{n=1}^{\infty} [t^{(24n-5)^2} - t^{(24n+13)^2}].$$

Since $t^{48} = 0.02658 \cdots$, it follows that all the terms in this expression are positive, and hence the function does not vanish. Since the two factors Φ in ϕ_2 are interchanged by S, it follows that if one does not vanish identically, the other does not. Similarly it may be proved that ϕ_1 and ϕ_3 do not vanish identically.

Every multiplicative function (without poles) can be expressed as a product of the functions ϕ_1, ϕ_2, ϕ_3 and functions of the form $a\phi_1^6 + b\phi_2^6$, a, b being constants. Every automorphic function belonging to the group can be expressed in the form

$$\prod_{i=1}^{n} \frac{a_i\phi_1^6 + b_i\phi_2^6}{c_i\phi_1^6 + d_i\phi_2^6}.$$

§ 4. *The z-group and its relation to the ternary quadratic form.*

By comparison with the results of FRICKE-KLEIN (*Automorphe Functionen*, pp. 533–538) it is evident that the substitutions of the z-group (§ 1) correspond to linear transformations of the ternary form $2z_1^2 - z_1 z_2 - 3z_3^2$ into itself. Not all of the corresponding ternary substitutions have integer coefficients, but the z-group is *commensurable* * with the group whose corresponding ternary substitutions have integer coefficients. Denote the latter group by Γ'. According to the general theory † the substitutions of Γ' are comprised in eight different types, which, on account of the fact that p is even reduces to only four distinct types, viz.,

* Two groups are said to be *commensurable* when they have a common invariant subgroup of finite index under each.

† We use in what follows the notation and results of FRICKE-KLEIN, loc. cit.

I

$$\begin{vmatrix} \frac{a+b\sqrt{6}}{2} & \frac{c\sqrt{3}+d\sqrt{2}}{2} \\ \frac{-c\sqrt{3}+d\sqrt{2}}{2} & \frac{a-b\sqrt{6}}{2} \end{vmatrix},$$

II

$$\begin{vmatrix} \frac{a\sqrt{3}+b\sqrt{2}}{2} & \frac{c+d\sqrt{6}}{2} \\ \frac{-c+d\sqrt{6}}{2} & \frac{a\sqrt{3}-b\sqrt{2}}{2} \end{vmatrix},$$

III

$$\begin{vmatrix} \frac{a\sqrt{6}+b}{2} & \frac{c\sqrt{2}+d\sqrt{3}}{2} \\ \frac{-c\sqrt{2}+d\sqrt{3}}{2} & \frac{a\sqrt{6}-b}{2} \end{vmatrix},$$

IV

$$\begin{vmatrix} \frac{a\sqrt{2}+b\sqrt{3}}{2} & \frac{c\sqrt{6}+d}{2} \\ \frac{-c\sqrt{6}+d}{2} & \frac{a\sqrt{2}-b\sqrt{3}}{2} \end{vmatrix},$$

in which a, b, c, d are integers subject to the conditions,

(1) that each determinant is unity, and

(2) that a, c are even for substitutions of types I and II, and b, d are even for the other two types.

That the number of types is diminished by half when p is even is readily shown as follows. Suppose $p = 2\pi_1 p_2$, in which π_1, p_2 are relatively prime. Then, corresponding to the division of p into the two factors $2\pi_1$, p_2, we have the type

$$\begin{vmatrix} \frac{a\sqrt{2\pi_1 r_1}+b\sqrt{p_2 r_2}}{2}\sqrt{q_1} & \frac{c\sqrt{2\pi_1 r_2}+d\sqrt{p_2 r_1}}{2}\sqrt{q_2} \\ \frac{-c\sqrt{2\pi_1 r_2}+d\sqrt{p_2 r_1}}{2}\sqrt{q_2} & \frac{a\sqrt{2\pi_1 r_1}-b\sqrt{p_2 r_2}}{2}\sqrt{q_1} \end{vmatrix},$$

the coefficients of which must satisfy the conditions

$$2\pi_1 r_1 q_1 a^2 - p_2 r_2 q_1 b^2 + 2\pi_1 r_2 q c^2 - p_2 r_1 q_2 d^2 = 4,$$

$$a 2\pi_1 q_1 r_1 \equiv b p_2 q_1 r_2 \equiv c 2\pi_1 q_2 r_2 \equiv d p_2 q_2 r_1 \pmod{2}.$$

As p, q, r are relatively prime, p_2, q_1, r_2, q_2, r_1 are not divisible by 2 and hence b and d are even. Write $b = 2b'$, $d = 2d'$ and the above determinant becomes

$$\begin{vmatrix} \frac{a\sqrt{\pi_1 r_1}+b'\sqrt{2p_2 r_2}}{\sqrt{2}}\sqrt{q_1} & \frac{c\sqrt{\pi_1 r_2}+d'\sqrt{2p_2 r_1}}{\sqrt{2}}\sqrt{q_2} \\ \frac{-c\sqrt{\pi_1 r_2}+d'\sqrt{2p_2 r_1}}{\sqrt{2}}\sqrt{q_2} & \frac{a\sqrt{\pi_1 r_1}-b'\sqrt{2p_2 r_2}}{\sqrt{2}}q_1 \end{vmatrix}.$$

But these substitutions are the same as those of the type in which $s = 2$, and p

is separated into two factors π_1 and $2p_2$, while q, r are factored in the same way as before. Also, the determinant of the one type reduces to that of the other.

The same line of argument may be followed when r is even. Hence, *if p or r is even the number of distinct types is* $\leqq T_p T_q T_r$.

Suppose now that condition (2) is removed, and let the enlarged group be denoted by Γ. The z-group is then a subgroup of finite index under Γ consisting of all the substitutions of types I and II. We will denote this group by (I, II). Similarly we define the subgroups (I, III), (I, IV).

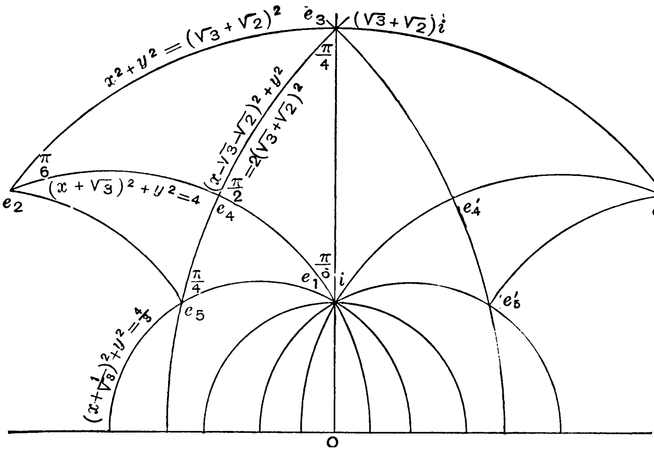

FIG. 4.

The fundamental regions for these groups are shown in Fig. 4, in which $e_1 e_4 e_3 e_4'$ is the fundamental region for Γ with $e_1 e_4$ conjugate to $e_1 e_4'$ and $e_3 e_4'$ conjugate to $e_3 e_4$; $e_1 e_2 e_2'$ is the fundamental region for (I, II) with $e_1 e_2$ and $e_3 e_2$ conjugate to $e_1 e_2'$ and $e_3 e_2'$ respectively; $e_1 e_5 e_3 e_5'$ is the fundamental region for (I, III) with $e_1 e_5$, $e_4' e_5'$, $e_4 e_5$ conjugate respectively to $e_1 e_5'$, $e_4' e_3$, $e_4 e_3$; and finally, $e_1 e_5 e_3 e_5'$ is the fundamental region for (I, IV) with $e_1 e_5$ and $e_3 e_5'$ conjugate to $e_1 e_5'$ and $e_3 e_5$.

In all these groups the substitutions of type I form an invariant subgroup of finite index which will be denoted by I. The fundamental region for I is $e_1 e_5 e_2 e_2' e_5'$, in which $e_3 e_2$, $e_1 e_5$, $e_5 e_2$ are conjugate to $e_3 e_2$, $e_1 e_5$, $e_5' e_2'$. The group I when represented by ζ-substitutions evidently consists of all operations formed by combinations of S and T in which S occurs an even number of times. Hence the group I may be generated by S^2, T, and STS^{-1}. Referring to section 2 it is easy to verify that, since I is of genus zero, every function which is automorphic for the group is expressible rationally in terms of $(\phi_1/\phi_2)^3$. Functions belonging to Γ are rational in $\tau = \phi_3^4/\phi_1^6\phi_2^6$, and those belonging to I, III are rational in $\sqrt{\tau} = \phi_3^2/\phi_1^3\phi_2^3$. Finally, functions belonging to (I, IV) are rational in $\sqrt{\tau + c}$, in which c is a non-vanishing constant. To prove the last statement, let $\lambda_1 = \phi_1^3 \phi_2^3$ and λ_2 be two particular multiplicative functions for the group (I, IV) such that every multiplicative function having a single zero within the fundamental region is expressible in the form $a_1 \lambda_1 + a_2 \lambda_2$. Suppose λ_2 is that particular function which is 0^4 at e_3 (Fig. 4). Then ϕ_3^4 being a multiplicative function for Γ is expressible in the form $\lambda_2 (a_1 \lambda_1 + a_2 \lambda_2)$, in which a_1, a_2 are chosen so that the second factor is 0^4 at e_5 and at points congruent to it in the group (I, IV). Solving the equation $\lambda_2 (a_1 \lambda_1 + a_2 \lambda_2) = \phi_3^4$ for the ratio $\lambda_1 : \lambda_2$ we find that it is expressible in the form $a + b\sqrt{\tau + c}$, and hence, every function belonging to (I, IV) is rational in terms of $\sqrt{\tau + c}$, since it is rational in $\lambda_2 : \lambda_1$. Conversely, every function which is rational in τ, $\sqrt{\tau}$, or $\sqrt{\tau + c}$ is automorphic in Γ, (I, III), or (I, IV) respectively.

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