ON HYPERCOMPLEX NUMBER SYSTEMS*

(FIRST PAPER)

BY

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Introduction.

The method invented by BENJAMIN PEIRCE† for treating the problem to determine all hypercomplex number systems (or algebras) in a given number of units depends chiefly, first, upon the classification of hypercomplex number systems into idempotent number systems, containing one or more idempotent numbers,‡ and non-idempotent number systems containing no idempotent number; and, second, upon the regularization of idempotent number systems, that is, the classification of each of the units of such a system with respect to one of the idempotent numbers of the system.

For the purpose of such classification and regularization the following theorems are required:

THEOREM I.§ *In any given hypercomplex number system there is an idempotent number (that is, a number $I \neq 0$ such that $I^2 = I$), or every number of the system is nilpotent.* ||

THEOREM II.¶ *In any hypercomplex number system containing an idempotent number I, the units $e_1, e_2, \cdots, e_n$ can be so selected that, with reference*

* Presented to the Society February 27, 1904. Received for publication February 27, 1904, and September 6, 1904.

† American Journal of Mathematics, vol. 4 (1881), p. 97. This work, entitled *Linear Associative Algebra*, was published in lithograph in 1870. For an estimate of PEIRCE's work and for its relation to the work of STUDY, SCHEFFERS, and others, see articles by H. E. HAWKES in the American Journal of Mathematics, vol. 24 (1902), p. 87, and these Transactions, vol. 3 (1902), p. 312. The latter paper is referred to below when reference is made to HAWKES' work.

‡ If the hypercomplex number $A \neq 0$ and $A^2 = A$, PEIRCE terms A "idempotent," loc. cit., p. 104. See p. 512 below.

§ PEIRCE, loc. cit., p. 113; HAWKES, loc. cit., p. 321.

|| If, for some positive integer m, $A^m = 0$, PEIRCE terms the hypercomplex number A "nilpotent," loc. cit., p. 104; see also p. 512 below.

¶ PEIRCE, loc. cit., p. 109; HAWKES, loc. cit., p. 314.

to I (termed the basis), they shall fall into one or other of four groups as follows:

$$1st\ group: \quad Ie_i = e_i = e_i I \qquad (i = 1, 2, \cdots, n'),$$

$$2d\ group: \quad Ie_i = e_i,\ e_i I = 0 \qquad (i = n' + 1, n' + 2, \cdots, n''),$$

$$3d\ group: \quad Ie_i = 0,\ e_i I = e_i \qquad (i = n'' + 1, n'' + 2, \cdots, n'''),$$

$$4th\ group: \quad Ie_i = 0 = e_i I \qquad (i = n''' + 1, n''' + 2, \cdots, n).$$

When so selected the units $e_1, e_2, \cdots, e_n$ of the number system are said to be *regular* with respect to I.* The units $e_1, e_2, \cdots, e_{n'}$ of the first group form a subsystem by themselves to which the idempotent number I belongs and of which I is the modulus; and I may, therefore, be chosen as one of the units of the first group.

THEOREM III.† *The basis or idempotent number I may be so chosen that there shall be no other idempotent number in the first group.*

THEOREM IV.‡ *In an idempotent number system § whose units are regular with respect to the basis or idempotent number I, chosen as one of the units, the remaining units of the first group can be taken to be nilpotent, ‖ and will then constitute a non-idempotent system by themselves, provided the first group contains no second idempotent number.*

Of these theorems requisite for the establishment of PEIRCE's method, his proofs of theorems I and IV are invalid. In a recent paper, cited above, by H. E. HAWKES, in these Transactions, he has fully established this chain of theorems, and thus, finally, placed PEIRCE's valuable method on a rigorous basis. But, whereas PEIRCE's methods are purely algebraic and involve only elementary considerations, HAWKES has recourse to the theory of transformation groups in establishing theorem IV, thus introducing conceptions unnecessary for the establishment of PEIRCE's theory and foreign to his methods. In the paper referred to, HAWKES has contributed a very valuable extension of PEIRCE's method in the conception of the regularization of all the units of a non-nilpotent number system with respect to each of certain idempotent numbers chosen as units.¶

The object of this paper at the outset was *to establish* PEIRCE's *method with-*

* HAWKES, loc. cit., p. 316.

† PEIRCE, loc. cit., p. 113; HAWKES, loc. cit., p. 316.

‡ PEIRCE, loc. cit., p. 117, 118; HAWKES, loc. cit., p. 322. This theorem may be expressed, otherwise, as follows. In a number system $e_1, e_2, \cdots, e_n$ with modulus e_n but no other idempotent number, the units $e_1, e_2, \cdots, e_{n-1}$ can be taken to be nilpotent and will then constitute by themselves a nilpotent system.

§ See p. 509; also p. 512.

‖ See note p. 509; also p. 512.

¶ HAWKES, loc. cit., p. 317; see also p. 512 below.

out recourse to the theory of groups. Employing methods purely algebraic, I give a proof of theorem I, which is, I think, somewhat simpler than that given by HAWKES; and I establish theorem IV by *the extension to number systems in general of the scalar function of quaternions*;—an extension suggested by my previous extension of the scalar function to matrices of any order,* and by C. S. PEIRCE's theorem that any number system can be represented by a matrix. †

The results of this paper in its original form were given at the February meeting of the American Mathematical Society of the current year; and the paper was to appear in the July number of these Transactions. But the employment of algebraic methods only, suggested to the editors an extension of the scope of the paper to the consideration of what may be designated as *number systems with respect to a domain R of rationality*, namely, the consideration of the totality of numbers

$$\sum_{i=1}^{n} a_i e_i$$

of a number system $e_1, e_2, \cdots, e_n$, where the a's are rational in the domain R of rationality of a given aggregate of real or ordinary complex numbers including the constants of multiplication of the system,—all transformations of the system being rational in R ‡. At their request I have revised the paper from this point of view; and I give generalizations of all of PEIRCE's theorems with reference to a domain of rationality. § These results constitute a step towards *the enumeration of the types of number systems to which all number systems in a given number of units are reducible by transformations rational in a given domain.* I find that the conceptions on which PEIRCE's method is based are applicable to this problem. In a subsequent paper I shall give further results leading to a resolution of this problem.

* See Proceedings of the London Mathematical Society, vol. 22 (1891), p. 67, where I have applied this extension of the scalar function to a problem of SYLVESTER's.

† See Encyclopädie der Mathematischen Wissenschaften, vol. 1, p. 170, where this theorem is ascribed to ED. WEYR, though reference is made to C. S. PEIRCE's prior statement of the theorem in another form. It was established by C. S. PEIRCE in 1870 for certain of the number systems given in BENJAMIN PEIRCE's Linear Associative Algebra, and enunciated as probably true for number systems in general. See Memoirs of the American Academy of Arts and Sciences, new series, vol. 9 (1870), p. 363; also Johns Hopkins University Circulars, no. 22 (1883), p. 87. C. S. PEIRCE gave a demonstration of the general theorem in the Proceedings of the American Academy of Arts and Sciences, vol. 10 (1875), p. 392, and subsequently in the American Journal of Mathematics, vol. 4 (1881), p. 221.

In the development of the theory of the scalar function given below I have not employed C. S. PEIRCE's theorem, or the conception of a number system as represented by a matrix. See p. 514 below.

‡ The arbitrary domain of rationality is not necessarily composed of real or ordinary complex numbers, e. g., the Galois fields are not. *The Editors.*

§ In its original form, this paper treated a special case of such systems: namely, that for which the constants of multiplication of the system were real and R included every real scalar,—the case of real hypercomplex number systems.

The nomenclature employed by PEIRCE will be followed, in general, in this paper. Thus the first factor A in any product AB is termed the *facient*, the second factor B, the *faciend*:*

if $AB = A = BA$, A is an *idemfactor* with respect to B;
if $AB = A$, A is *idemfacient* with respect to B;
if $BA = A$, A is *idemfaciend* with respect to B;
if $AB = 0 = BA$, A is a *nilfactor* with respect to B;
if $AB = 0$, A is *nilfacient* with respect to B;
if $BA = 0$, A is *nilfaciend* with respect to B;
if $A \neq 0$, and $A^2 = A$, A is *idempotent;*
if $A^m = 0$ for some positive integer m, A is *nilpotent;* †

and a number system which contains no idempotent number is a *nilpotent system.* ‡

A number system which contains one or more idempotent numbers is an *idempotent system.* §

The units of an idempotent system are *regular* with respect to the idempotent number I of the system when so chosen that each falls into one or other of PEIRCE's four groups, mentioned above, p. 510, with respect to I. §

A hypercomplex number A, satisfying the conditions fulfilled by the units of the kth group $(1 \leqq k \leqq 4)$ with respect to the idempotent number I, is said to *belong* to the kth group with respect to I.

If A belongs to the k-th group with respect to I, it is linear in the units of that group, and conversely. ‖ In particular, I belongs to the first group, and may be taken as one of the units of that group.

Following HAMILTON, the term *scalar* will be employed to denote a real or ordinary complex number.¶ If A is any given hypercomplex number, I shall designate by the term *polynomial in A* the hypercomplex number

$$c_1 A + c_2 A^2 + \cdots + c_p A^p$$

linear (with scalar coefficients) in positive integral powers of A. For any given number A of the system, there is at least one linear relation between the first $n+1$ power of A, and, therefore, a smallest integer $m \leqq n+1$ for which $A, A^2, \cdots, A^m$ are linearly related. This linear relation,

* These terms were suggested by HAMILTON, who employs the terms "factor" and "faciend" to denote A and B respectively, *Lectures on Quaternions*, p. 38.

† PEIRCE, loc. cit., p. 103–104.

‡ PEIRCE, loc. cit., p. 115.

§ HAWKES, loc. cit., pp. 314, 316.

‖ PEIRCE, loc. cit., p. 110. If $IA = A = AI$, A belongs to the first group with respect to I; if $IA = A$, $AI = 0$, A belongs to the second group with respect to I; etc. For the group of the non zero product of any two numbers or units belonging to the various groups, see PEIRCE, loc. cit., p. 111, and HAWKES, loc. cit., p. 316.

¶ *Lectures on Quaternions*, pp. 58, 664.

$$\Omega(A) \equiv A^m + p_1 A^{m-1} + \cdots + p_{m-2} A^2 + p_{m-1} A = 0,$$

I shall term the *fundamental equation of* A. Let

$$f(A) \equiv \sum_{q=1}^{p} c_q A^q \qquad (c_p \neq 0)$$

be any polynomial in A: *if* $f(A) = 0$, *then* $p \geqq m$ *and*

$$f(\lambda) \equiv \sum_{q=0}^{p-m} \gamma_q \lambda^q \Omega(\lambda)$$

for any scalar λ; *conversely, if*

$$f(\lambda) \equiv \sum_{q=0}^{p-m} c_q \lambda^q \Omega(\lambda),$$

then $f(A) = 0$.

The constants of multiplication of the units $e_1, e_2, \cdots, e_n$ of the number system will be denoted by γ_{ijk} for $i, j, k = 1, 2, \cdots n$, and thus

$$e_i e_j = \gamma_{ij1} e_1 + \gamma_{ij2} e_2 + \cdots + \gamma_{ijn} e_n;$$

and the domain of rationality of the constants γ_{ijk} will be denoted by $R(\gamma_{ijk})$. *The coefficients* $p_1, p_2, \cdots p_{m-1}$ *of the fundamental equation* $\Omega(A) = 0$ *of the hypercomplex number* $A = a_1 e_1 + \cdots + a_n e_n$ *are rational functions of* $a_1, a_2, \cdots a_n$ *for the domain* $R(\gamma_{ijk})$, being rational functions, for the domain $R(1)$, of $a_1, a_2, \cdots, a_n$ and of the constants γ_{ijk} of multiplication of the system.

Let R' denote the domain of rationality of any given aggregate of scalars. The totality of hypercomplex numbers

$$A = \sum_{i=1}^{n} a_i e_i,$$

for all possible sets of scalars $a_1, a_2, \cdots, a_n$ rational in R', will be said to constitute the *hypercomplex domain* $\mathbf{R}(R'; e_1, e_2, \cdots, e_n)$; and any such number A will be said to *belong to* $\mathbf{R}(R'; e_1, e_2, \cdots, e_n)$. *If* ρ *is any scalar rational in* R', *and both* A *and* B *belong to* $\mathbf{R}(R'; e_1, e_2, \cdots, e_n)$, *so also does* ρA *and* $A + B$; *moreover,* AB *belongs to this domain provided* R' *contains* $R(\gamma_{ijk})$, but, in general, not otherwise. Therefore, if $c_1, c_2, \cdots, c_p$ are rational in R' and A belongs to $\mathbf{R}(R'; e_1, e_2, \cdots, e_n)$, the polynomial

$$\sum_{q=1}^{p} c_q A^q$$

belongs to this domain if R' contains $R(\gamma_{ijk})$.

Let new units $e_1', e_2', \cdots, e_n'$ be introduced by the linear transformation T, and let γ_{ijk}' denote the constants of multiplication of the new system. Then, *if T is rational in R', the new units $e'_1, e'_2, \cdots, e'_n$ belong to* $\mathbf{R}(R'; e_1, e_2, \cdots, e_n)$, *and conversely;*

if T is rational in R', the hypercomplex domains $\mathbf{R}(R'; e_1, e_2, \cdots, e_n)$ *and* $\mathbf{R}(R'; e'_1, e'_2, \cdots, e'_n)$ *are identical, and conversely;* *

if T is rational in R', and R' contains $R(\gamma_{ijk})$, it also contains $R(\gamma'_{ijk})$.†

§ 1. *The scalar function of a hypercomplex number.*

Let

$$A = \sum_{i=1}^{n} a_i e_i$$

be any number of the given hypercomplex number system $e_1, e_2, \cdots, e_n$. I shall employ SA, in designation the *scalar* of A, to denote that function of the coefficients a and the constants γ_{ijk} of multiplication of the system ‡ defined as follows: §

$$SA = \frac{1}{n}\left(a_1 \sum_{j=1}^{n} \gamma_{1jj} + a_2 \sum_{j=1}^{n} \gamma_{2jj} + \cdots + a_n \sum_{j=1}^{n} \gamma_{njj}\right). \tag{1}$$

Whence, if ρ is any scalar,|| and

$$B = \sum_{i=1}^{n} b_i e_i$$

is any second number of the system, it follows that

$$S\rho A = \rho SA, \tag{2}$$

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* In order that T shall be rational in R' it is only necessary that $\mathfrak{R}(R'; e_1, e_2, e_n)$ shall contain $\mathfrak{R}(R'; e'_1, e'_2, \cdots, e'_n)$.

† In particular R' may be identical with $R(\gamma_{ijk})$, in which case, if T is rational in $R(\gamma_{ijk})$, this domain contains $R(\gamma'_{ijk})$; but, unless the coefficients of T are rational in $R(1)$, $R(\gamma'_{ijk})$ is, in general, not identical with $R(\gamma_{ijk})$. Thus, if

$$e_1 = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

e_1, e_2 constitute a number system, and $R(\gamma_{ijk})$ contains $\sqrt{2}$; and, thus, the transformation

$$e'_1 = \frac{1}{\sqrt{2}} \cdot e_1, \quad e'_2 = e_2$$

is rational in $R(\gamma_{ijk})$. But the constants of multiplication of the new system are rational in $R(1)$; and, therefore, $R(\gamma'_{ijk})$ is not identical with $R(\gamma_{ijk})$.

‡ See p. 513 above.

§ If $n = 4$, and e_1, e_2, e_3, e_4 are the quaternion units, this definition accords with the customary definition of the scalar function. Thus, if $e_4 = 1$ and e_1, e_2, e_3 constitute a system of mutually normal vectors, we have

$$\gamma_{4jj} = 1, \quad \gamma_{ijj} = 0 \qquad (i = 1, 2, 3; j = 1, 2, 3, 4)$$

and thus

$$S\sum_{i=1}^{4} a_i e_i = a_4$$

by the above definition.

|| See p. 512 above.

(3) $$S(A+B) = SA + SB.$$

In particular,

(4) $$SA = a_1 Se_1 + a_2 Se_2 + \cdots + a_n Se_n.$$

Moreover, if

$$\varepsilon = \sum_{i=1}^{n} a_i^{(0)} e_i$$

is a modulus of the system, in which case

$$\sum_{i=1}^{n} a_i^{(0)} e_i e_j = \epsilon e_j = e_j \qquad (j = 1, 2, \cdots, n),$$

we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i^{(0)} \gamma_{ijj} = n,$$

and therefore

(5) $$S\epsilon = 1.$$

Let R' denote the domain of rationality of any given aggregation of scalars that includes the constants γ_{ijk} of multiplication of the system. If A belongs to $\mathbf{R}(R'; e_1, e_2, \cdots, e_n)$,* SA is rational in R'. In particular, let R' include every real scalar, and let $e_1, e_2, \cdots, e_n$ be the units of a real hypercomplex number system † in which case the constants γ_{ijk} of multiplication of the system are real: then, if A is a number of the system, in which case $a_1, a_2, \cdots, a_n$ are real, SA is real.

By definition

(6) $$SAB = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{h=1}^{n} a_i b_j \gamma_{ijk} \gamma_{khh},$$

$$SBA = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{h=1}^{n} a_j b_i \gamma_{ijk} \gamma_{khh}.$$

But, since

$$e_i e_j \cdot e_h = e_i \cdot e_j e_h \qquad (i, j, h = 1, 2, \cdots, n),$$

we have

(6′) $$\sum_{k=1}^{n} \gamma_{ijk} \gamma_{khl} = \sum_{k=1}^{n} \gamma_{ikl} \gamma_{jhk} \qquad (i, j, h, l = 1, 2, \cdots, n);$$

therefore,

(7) $$SAB = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{h=1}^{n} a_i b_j \gamma_{ijk} \gamma_{khh}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{h=1}^{n} a_i b_j \gamma_{ikh} \gamma_{jhk}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{h=1}^{n} a_j b_i \gamma_{ikh} \gamma_{jhk} *$$

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* See p. 513 above.

† See note p. 511 above.

‡ By interchange of i and j, and of k and h.

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{h=1}^{n} a_j b_i \gamma_{ijk} \gamma_{khh} = SBA.$$

Let now $e_1', e_2', \cdots, e_n'$ be any second system of units of the number system obtained from $e_1, e_2, \cdots, e_n$ by the transformation T of non-zero determinant defined as follows:

(8) $$e_i' = \tau_{i1} e_1 + \tau_{i2} e_2 + \cdots + \tau_{in} e_n \qquad (i=1, 2, \cdots, n);$$

and let

(9) $$e_i' e_j' = \gamma_{ij1}' e_1' + \gamma_{ij2}' e_2 + \cdots + \gamma_{ijn}' e_n \qquad (i, j=1, 2, \cdots, n).$$

Let

$$a_1' e_1' + a_2' e_2' + \cdots + a_n' e_n' = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n,$$

$$b_1' e_1' + b_2' e_2' + \cdots + b_n' e_n' = b_1 e_2 + b_2 e_2 + \cdots + b_n e_n;$$

then

$$\left.\begin{aligned} a_k &= \tau_{1k} a_1' + \tau_{2k} a_2' + \cdots + \tau_{nk} a_n', \\ b_k &= \tau_{1k} b_1' + \tau_{2k} b_2' + \cdots + \tau_{nk} b_n', \end{aligned}\right\} \qquad (k=1, 2, \cdots, n),$$

that is, in CAYLEY's abbreviated notation *

$$(a_1, a_2, \cdots a_n) = (T \between a_1', a_2', \cdots a_n'),$$

$$(b_1, b_2, \cdots b) = (T \between b_1', b_2', \cdots b_n'),$$

where T is the matrix

$$\begin{bmatrix} \tau_{11} & \tau_{21} & \cdots & \tau_{n1} \\ \tau_{12} & \tau_{22} & \cdots & \tau_{n2} \\ \cdot & \cdot & \cdot & \cdot \\ \tau_{1n} & \tau_{2n} & \cdots & \tau_{nn} \end{bmatrix}.$$

Finally, let

$$\beta_1' e_1' + \beta_2' e_2' + \cdots + \beta_n' e_n' = (a_1' e_1' + a_2' e_2' + \cdots + a_n' e_n')(b_1' e_1' + b_2' e_2' + \cdots + b_n' e_n'),$$

$$\beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n = (a_1 e_1 + a_2 e_2 + \cdots + a_n e_n)(b_1 e_1 + b_2 e_2 + \cdots + b_n e_n).$$

Then

$$(\beta_1', \beta_2', \cdots, \beta_n') = \begin{bmatrix} \sum_{i=1}^{n} a_i' \gamma_{i11}' & \sum_{i=1}^{n} a_i' \gamma_{i21}' & \cdots & \sum_{i=1}^{n} a_i' \gamma_{in1}' \\ \sum_{i=1}^{n} a_i' \gamma_{i12}' & \sum_{i=1}^{n} a_i' \gamma_{i22} & \cdots & \sum_{i=1}^{n} a_i' \gamma_{in2} \\ \cdot & \cdot & \cdot & \cdot \\ \sum_{i=1}^{n} a_i' \gamma_{i1n}' & \sum_{i=1}^{n} a_i' \gamma_{i2n} & \cdots & \sum_{i=1}^{n} a_i' \gamma_{inn} \end{bmatrix} (b_1', b_2', \cdots, b_n'),$$

* *Memoir on Matrices*, Philosophical Transactions, vol. 148 (1858), p. 17.

$$(\beta_1, \beta_2, \cdots, \beta_n) = \begin{bmatrix} \sum_{i=1}^n a_i \gamma_{i11} & \sum_{i=1}^n a_i \gamma_{i21} & \cdots & \sum_{i=1}^n a_i \gamma_{in1} \\ \sum_{i=1}^n a_i \gamma_{i12} & \sum_{i=1}^n a_i \gamma_{i22} & \cdots & \sum_{i=1}^n a_i \gamma_{in2} \\ \cdot & \cdot & \cdot & \cdot \\ \sum_{i=1}^n a_i \gamma_{i1n} & \sum_{i=1}^n a_i \gamma_{i2n} & \cdots & \sum_{i=1}^n a_{inn} \end{bmatrix} (b_1, b_2, \cdots, b_n);$$

moreover,

$$\beta'_1 e'_1 + \beta'_2 e'_2 + \cdots + \beta'_n e'_n = \beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n,$$

and, therefore,

$$(\beta_1, \beta_2, \cdots, \beta_n) = (T)(\beta'_1, \beta'_2, \cdots, \beta'_n).$$

Wherefore,

$$(10)\quad \begin{bmatrix} \sum_{i=1}^n a'_i \gamma'_{i11} & \sum_{i=1}^n a'_i \gamma'_{i21} & \cdots & \sum_{i=1}^n a'_i \gamma'_{in1} \\ \sum_{i=1}^n a'_i \gamma'_{i12} & \sum_{i=1}^n a'_i \gamma'_{i22} & \cdots & \sum_{i=1}^n a'_i \gamma'_{in2} \\ \cdot & \cdot & \cdot & \cdot \\ \sum_{i=1}^n a'_i \gamma'_{i1n} & \sum_{i=1}^n a'_i \gamma'_{i2n} & \cdots & \sum_{i=1}^n a'_i \gamma'_{inn} \end{bmatrix} = T^{-1} \begin{bmatrix} \sum_{i=1}^n a_i \gamma_{i11} & \sum_{i=1}^n a_i \gamma_{i21} & \cdots & \sum_{i=1}^n a_i \gamma_{n1} \\ \sum_{i=1}^n a_i \gamma_{i12} & \sum_{i=1}^n a_i \gamma_{i22} & \cdots & \sum_{i=1}^n a_i \gamma_{in2} \\ \cdot & \cdot & \cdot & \cdot \\ \sum_{i=1}^n a_i \gamma_{i1n} & \sum_{i=1}^n a_i \gamma_{i2n} & \cdots & \sum_{i=1}^n a_i \gamma_{inn} \end{bmatrix} T.$$

Whence it follows that the characteristic equations of these two matrices are identical, that is,

$$(11)\quad \begin{vmatrix} \lambda - \sum_{i=1}^n a'_i \gamma'_{i11} & -\sum_{i=1}^n a'_i \gamma'_{i21} & \cdots & -\sum_{i=1}^n a'_i \gamma'_{in1} \\ -\sum_{i=1}^n a'_i \gamma'_{i12} & \lambda - \sum_{i=1}^n a'_i \gamma'_{i22} & \cdots & -\sum_{i=1}^n a'_i \gamma'_{in2} \\ \cdot & \cdot & \cdot & \cdot \\ -\sum_{i=1}^n a'_i \gamma'_{i1n} & -\sum_{i=1}^n a'_i \gamma'_{i2n} & \cdots & \lambda - \sum_{i=1}^n a'_i \gamma'_{inn} \end{vmatrix}$$

$$(11) \qquad \equiv \begin{vmatrix} \lambda - \sum_{i=1}^{n} a_i \gamma_{i11} & -\sum_{i=1}^{n} a_i \gamma_{i21} & \cdots & -\sum_{i=1}^{n} a_i \gamma_{in1} \\ -\sum_{i=1}^{n} a_i \gamma_{i12} & \lambda - \sum_{i=1}^{n} a_i \gamma_{i22} & \cdots & -\sum_{i=1}^{n} a_i \gamma_{in2} \\ \cdot & \cdot & \cdot & \cdot \\ -\sum_{i=1}^{n} a_i \gamma_{i1n} & -\sum_{i=1}^{n} a_i \gamma_{i2n} & \cdots & \lambda - \sum_{i=1}^{n} a_i \gamma_{inn} \end{vmatrix}.$$

Therefore, in particular,

$$(12) \qquad \sum_{i=1}^{n} \sum_{j=1}^{n} a_i' \gamma_{ijj}' = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \gamma_{ijj};$$

and, thus, SA is an invariant to any transformation of the units of the number system.

Let

$$I = \sum_{i=1}^{n} a_i^{(0)} e_i$$

be any idempotent number* of the system $e_1, e_2, \cdots, e_n$; and, let

$$(13) \qquad e_i^{(1)} = Ie_i, \qquad e_i^{(2)} = e_i - Ie_i \qquad (i=1, 2, \cdots, n).$$

We have

$$(14) \qquad \begin{aligned} Ie_i^{(1)} &= I^2 e_i = Ie_i = e_i^{(1)}, \\ Ie_i^{(2)} &= Ie_i - I^2 e_i = Ie_i - Ie_i = 0 \end{aligned} \qquad (i=1, 2, \cdots, n).$$

Thus, for $1 \leqq i \leqq n$, the numbers $e_i^{(1)}, e_i^{(2)}$ are, respectively, idemfaciend and nilfaciend † with respect to I. Further, any number linear in the numbers $e_i^{(1)}$ is idemfaciend with respect to I, and any number linear in the numbers $e_i^{(2)}$ is nilfaciend with respect to I. Since

$$(15) \qquad e_i = e_i^{(1)} + e_i^{(2)} \qquad (i=1, 2, \cdots, n),$$

every number of the system can be expressed linearly in the $2n$ numbers of (13). Let n_1 of the numbers $e_i^{(1)}$, as $e_1^{(1)}, e_2^{(1)}, \cdots, e_{n_1}^{(1)}$ be independent; and let n_2 of the numbers $e_i^{(2)}$, as $e_1^{(2)}, e^{(2)}, \cdots, e_{n_2}^{(2)}$ be independent. Then since, for $k = 1, 2$ and $i = 1, 2, \cdots, n$, each of the numbers $e_i^{(k)}$ (and, therefore, any n independent numbers of the system) can be expressed in terms of the above $n_1 + n_2$ hypercomplex numbers, and the latter are independent,‡ we have $n_1 + n_2 = n$. We may now

* See p. 512 above.

† See p. 512 above.

‡ For, if $c_1^{(1)} e_1^{(1)} + \cdots + c_{n_1}^{(1)} e_{n_1}^{(1)} + c_1^{(2)} e_1^{(2)} + \cdots + c_{n_2}^{(2)} e_{n_2}^{(2)} = 0$, then

$$c_1^{(1)} e_1^{(1)} + \cdots + c_{n_1}^{(1)} e_{n_1}^{(1)} = I(c_1^{(1)} e_1^{(1)} + \cdots + c_{n_1}^{(1)} e_{n_1}^{(1)} + c_1^{(2)} e_1^{(2)} + \cdots + c_{n_2}^{(2)} e_{n_2}^{(2)}) = 0;$$

and, therefore, $c_1^{(1)} = \cdots = c_{n_1}^{(1)} = c_1^{(2)} = \cdots = c_{n_2}^{(2)} = 0$.

show that any number idemfaciend with respect to I is linear in $e_1^{(1)}, e_2^{(1)}, \cdots, e_{n_1}^{(1)}$; whence it follows that I is expressible in these hypercomplex numbers. Further, that any number nilfaciend to I is linear in $e_1^{(2)}, e_2^{(2)}, \cdots, e_{n_2}^{(2)}$. It may be that $n_2 = 0$; but $n_1 \geqq 1$. For n_1 is the number of independent hypercomplex numbers in the aggregate Ie_1 for $i = 1, 2, \cdots, n$; and the n numbers of this aggregate cannot all be zero, that is, $n_1 \neq 0$, otherwise,

$$I = I^2 = I \sum_{i=1}^{n} a_i^{(0)} e_i = \sum_{i=1}^{n} a_i^{(0)} Ie_i = 0.$$

Of the numbers $e_i^{(1)}$, $n_1 - 1$ are independent of I. Let $e_1^{(1)}, e_2^{(1)}, \cdots, e_{n_1-1}^{(1)}$ be independent of I. Then the n hypercomplex numbers $I, e_i^{(1)}$ $(i=1, 2, \cdots n_1-1)$, and $e_i^{(2)}$ $(i = 1, 2, \cdots, n_2)$ are independent. Let

$$(16) \qquad \begin{aligned} e_1' &= I, \\ e_{1+i}' &= e_i^{(1)} \qquad (i = 1, 2, \cdots, n_1 - 1), \\ e_{n_1+i}' &= e_i^{(2)} \qquad (i = 1, 2, \cdots, n_2); \end{aligned}$$

and let γ_{ijk}' denote the constants of multiplication of this new system of units, so that

$$e_i' e_j' = \sum_{k=1}^{n} \gamma_{ijk}' e_k'.$$

Since

$$(17) \qquad \begin{aligned} e_1' e_j' &= e_j' \qquad (j = 1, 2, \cdots, n_1), \\ e_1' e_j' &= 0 \qquad (j = n_1 + 1, n_1 + 2, \cdots, n), \end{aligned}$$

we have

$$(18) \qquad \begin{aligned} \gamma_{1jj}' &= 1 \qquad (j = 1, 2, \cdots, n_1), \\ \gamma_{1jj}' &= 0 \qquad (j = n_1 + 1, n_1 + 2, \cdots, n); \end{aligned}$$

and, therefore, since SI is invariant to any transformation of the units of the system,

$$(19) \qquad SI = Se_1' = \frac{1}{n} \sum_{i=1}^{n} \gamma_{1jj}' = \frac{n_1}{n}.$$

If I belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, then, since each of the original system of units belongs to this domain, it follows that each of the hyper-complex numbers $e_i^{(k)}$, for $k = 1, 2$ and $i = 1, 2, \cdots, n_k$, belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$.* Therefore, if I belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, $n_1 = nSI$ is the number of independent hypercomplex numbers belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$ which are idemfaciend to I; and $n_2 = n(1 - SI)$ is the number of independent hypercomplex numbers belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$ which are nilfaciend to I.

Let N be any nilpotent† number, other than zero, of the system $e_1, e_2, \cdots, e_n$.

* See p. 513 above.

† See p. 512 above.

Thus, let $N^{m-1} \neq 0$ and $N^m = 0$. For any positive integer $p < m$, there is at least one number of the system, as N^{m-p}, to which N^p, but no lower power of N, is nilfacient;* and any such number is independent of the numbers to which N^q is nilfacient for $q < p$, since, if N^q is nilfacient to A_1, A_2, etc., it is nilfacient to any number linear in A_1, A_2, etc. If for $1 < p \leqq m$, N^p, but no lower power of N, is nilfacient to A_1', A_2', etc., and if these hypercomplex numbers are independent of each other and of all hypercomplex numbers to which N^q is nilfacient for $q < p$, we then cannot have

$$N^{p'}(c_1 A_1' + c_2 A_2' + \cdots) = 0,$$

for $p' < p$, unless the c's are all zero; otherwise, there is a linear relation between the A''s and the hypercomplex numbers to which $N^{p'}$ is nilfacient. Let the system contain just $n_1 \geqq 1$ independent numbers, as

$$A_1^{(1)}, A_2^{(1)}, \cdots, A_{n_1}^{(1)},$$

to which N is nilfacient. Then, if $NB = 0$, we have

$$B = \sum_{j=1}^{n_1} c_j A_j^{(1)}.$$

If $m > 2$, let the system contain just $n_2 \geqq 1$ numbers, as

$$A_1^{(2)}, A_2^{(2)}, \cdots, A_{n_2}^{(2)},$$

to which N^2 is nilfacient, but N is not nilfacient, and which are independent, moreover, of each other and the $A^{(1)}$'s † (and, therefore, of numbers to which N is nilfacient). Then, if $NB \neq 0$, and $N^2 B = 0$, we have

$$B = \sum_{j=1}^{n_1} c_j^{(1)} A_j^{(1)} + \sum_{j=1}^{n_2} c_j^{(2)} A_j^{(2)},$$

where the c's with index 2 are not all zero. If $m > 3$, let the system contain just $n_3 \geqq 1$ numbers, as

$$A_1^{(3)}, A_2^{(3)}, \cdots, A_{n_3}^{(3)},$$

to which N^3, but no lower power of N, is nilfacient, and which are independent moreover, of each other and of the $A^{(1)}$'s and $A^{(2)}$'s (and, therefore, of numbers to which N^2 is nilfacient). Then, as before, if $N^2 B \neq 0$, $N^3 B = 0$, we have

$$B = \sum_{j=1}^{n_1} c_j^{(1)} A_j^{(1)} + \sum_{j=1}^{n_2} c_j^{(2)} A_j^{(2)} + \sum_{j=1}^{n_3} c_j^{(3)} A_j^{(3)},$$

---

* See p. 512 above.

† By this I mean, not only that the $A^{(2)}$'s are independent of each other, and, severally of the $A^{(1)}$'s, but that, if

$$\sum_{j=1}^{n_1} c_j^{(1)} A_j^{(1)} + \sum_{j=1}^{n_2} c_j^{(2)} A_j^{(2)} = 0,$$

the coefficients c are then all zero.

where the c's with index 3 are not all zero. Etc., etc. Finally, let the system contain just n_{m-1} numbers, as

$$A_1^{(m-1)},\ A_2^{(m-1)},\ \cdots,\ A_{n_{m-1}}^{(m-1)},$$

to which N^{m-1}, but no lower power of N, is nilfacient, and which are independent, moreover, of each other and of the $A^{(1)}$'s, $A^{(2)}$'s, $\cdots$, $A^{(m-2)}$'s (and, therefore, of numbers to which N^{m-2} is nilfacient). Then, if $N^{m-2}B \neq 0$, $N^{m-1}B = 0$, we have

$$B = \sum_{j=1}^{n_1} c_j^{(1)} A_j^{(1)} + \sum_{j=1}^{n_2} c_j^{(2)} A_j^{(2)} + \cdots + \sum_{j=1}^{n_{m-1}} c_j^{(m-1)} A_j^{(m-1)},$$

where the c's with index $m-1$ are not all zero. We may take N as one of the $A^{(m-1)}$: thus let $A_{n_{m-1}}^{(m-1)} = N$. Let now the system contain just $n_m \geqq 0$ numbers, as

$$A_1^{(m)},\ A_2^{(m)},\ \cdots,\ A_{nm}^{(m)},$$

independant of each other and of the $A^{(1)}$'s, $A^{(2)}$'s etc., and $A^{(m-1)}$'s (and, therefore, of numbers to which $N^{\pi-1}$ is nilfacient). Then if $N^{m-1}B \neq 0$, we have

$$B = \sum_{j=1}^{n_1} c_j^{(1)} A_j^{(1)} + \sum_{j=1}^{n_2} c_j^{(2)} A_j^{(2)} + \cdots + \sum_{j=1}^{n_m} c_j^{(m)} A_j^{(m)},$$

where the c's with index m are not all zero. By definition,

$$NA_i^{(1)} = 0 \qquad (i=1,\ 2,\ \cdots,\ n_1); \tag{20}$$

and, by what precedes, for $1 < p \leqq m$,

$$NA_i^{(p)} = \sum_{j=1}^{n_1} \gamma_{ij}^{(1)} A_j^{(1)} + \sum_{j=1}^{n_2} \gamma_{ij}^{(2)} A_j^{(2)} + \cdots + \sum_{j=1}^{n_{p-1}} \gamma_{ij}^{(p-1)} A_j^{(p-1)} \quad (i=1,\ 2,\ \cdots,\ n_p), \tag{21}$$

since N^{p-1} is nilfacient to $NA_i^{(p)}$.* Moreover, every number of the system $e_1, e_2, \cdots, e_n$ is linear in the $A^{(1)}$'s, $A^{(2)}$'s, $\cdots$, $A^{(m)}$'s, since with respect to

* Let $1 < p \leqq m$. Since, by supposition, the $A^{(p)}$'s are independent of each other and of all numbers to which N^{p-1} is nilfacient we cannot have

$$N^{p-1}(c_1 A_1^{(p)} + c_2 A_2^{(p)} + \cdots + c_{n_p} A_{n_p}^{(p)}) = 0,$$

unless the c's are all zero. Therefore,

$$N^{p-2}(c_1 NA_1^{(p)} + c_2 NA_2^{(p)} + \cdots + c_{n_p} NA_{n_p}^{(p)}) \neq 0,$$

unless the c's are all zero, that is, the numbers $NA_i^{(p)}$ ($i = 1,\ 2,\ \cdots,\ n_p$) are independent of each other and of all numbers to which N^{p-2} is nilfacient; moreover,

$$N^{p-1} \cdot NA_i^{(p)} = N^p A_i^{(p)} = 0 \qquad (i = 1, 2, \ldots, n_p)$$

But, by supposition, there are but n_{p-1} numbers to which N^{p-1} is nilfacient, and which are independent of each other and of all numbers to which N^{p-2} is nilfacient. Therefore, $n_p \leqq n_{p-1}$. Whence it follows that we may put $NA_i^{(p)} = A_i^{(p-1)}$ ($i = 1,\ 2,\ \cdots,\ n_p$).

every number B of the system either $N^{m-1}B \neq 0$, or $N^p B = 0$ for $1 \leqq p < m$. Therefore, new units e_1', e_2', $\cdots$, e_n' may be introduced by the transformation,

$$
(22) \qquad
\begin{aligned}
e_i' &= A_i^{(1)} && (i = 1, 2, \cdots, n_1),\\
e_{n_1+i}' &= A_i^{(2)} && (i = 1, 2, \cdots, n_2),\\
&\cdots\cdots\\
e_{n_1+n_2+\cdots+n_{m-1}+i}' &= A_i^{(m)} && (i = 1, 2, \cdots, n_m).
\end{aligned}
$$

Since

$$
(23) \qquad N = A_{n_{m-1}}^{(m-1)} = e_{n_1+n_2+\cdots+n_{m-1}}',
$$

if γ_{ijk}' denote the constants of multiplication of the new units, we have, by (20) and (21),

$$
(24) \qquad \gamma_{n_1+n_2+\cdots n_{m-1},jj}' = 0 \qquad (j = 1, 2, \cdots, n).
$$

Therefore, since SA is invariant to any transformation of the units of the system,

$$
(25) \qquad SN = SA_{n_{m-1}}^{(m-1)} = Se_{n_1+n_2+\cdots+n_{m-1}}' = \sum_{j=1}^{n} \gamma_{n_1+n_2+\cdots+n_{m-1},jj}' = 0.
$$

That is, the scalar function of any nilpotent number A of the system is equal to zero. But if A is nilpotent, so also are N^2, N^3, etc. Therefore, if N is nilpotent, $SN^p = 0$ for any positive integer p.

Let the number system contain n independent numbers, A_1, A_2, $\cdots$, A_n, such that

$$
SA_1 = SA_2 = \cdots = SA_n = 0.
$$

For any number A of the system, we have

$$
A = c_1 A_1 + c_2 A_2 + \cdots + c_n A_n,
$$

and, therefore, by (2) and (3),

$$
SA = c_1 SA_1 + c_2 SA_2 + \cdots + c_n SA_n = 0.
$$

In this case the system is nilpotent;* that is, contains no idempotent number, since, if I is idempotent, $SI \neq 0$. In particular, if

$$
Se_1 = Se_2 = \cdots = Se_n = 0,
$$

that is, if

$$
\sum_{j=1}^{n} \gamma_{ijj} = 0 \qquad (i = 1, 2, \cdots, n),
$$

the system is nilpotent.

These results may be summarized in the following theorem:

THEOREM (1). *Let*

$$
A = \sum_{i=1}^{n} a_i e_i
$$

* See p. 512 above.

be any number of the hypercomplex number system $e_1, e_2, \cdots, e_n$; *let* R' *denote any scalar domain of rationality including* $R(\gamma_{ijk})$;* *and let*

$$SA \equiv \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \gamma_{ijj}.$$

Then SA *is invariant to any transformation of the units of the system; and, if* ρ *is any scalar and* B *is any second number of the system,*

$$S\rho A = \rho SA,$$

$$S(A+B) = SA + SB,$$

$$SAB = SBA.$$

If ϵ *is a modulus of the system,* $S\epsilon = 1$. *If* A *belongs to* $\mathfrak{R}(R'; e_1, e_2, \cdots e_n)$, SA *is rational in* R'; *and if, moreover,* A *is idempotent,* $nSA \neq 0$ *is the number of independent hypercomplex numbers of the system, belonging to* $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, *that are idemfaciend to* A, *and* $n(1 - SA) \geqq 0$ *is the number of independent hypercomplex numbers of the system, belonging to* $\mathfrak{R}(R'; e_1, e_2, \cdots)$, *that are nilfaciend to* A. *If the system contains any* n *independent numbers* $A_1, A_2, \cdots, A_n$ *such that*

$$SA_1 = SA_2 = \cdots SA_n = 0,$$

in particular, if

$$Se_1 = Se_2 = \cdots = Se_n = 0,$$

the system is nilpotent. Finally, if A *is nilpotent,* $SA^p = 0$ *for any positive integer* p; *and, therefore, if the system contains* n *independent nilpotent numbers* $A_1, A_2, \cdots, A_n$, *in particular, if each of the units* $e_1, e_2, \cdots, e_n$ *is nilpotent, the system is nilpotent.*

Let the fundamental equation of A † be

$$\Omega(A) \equiv A^m + p_1 A^{m-1} + \cdots + p_{m-1} A = 0.$$

By the aid of this equation we can express A^p for $p \geqq m$ linearly in $A, A^2, \cdots, A^{m-1}$, and consequently, any polynomial in A ‡ linearly in the first $m-1$ powers of A. Therefore, by (2) and (3), *if the fundamental equation of* A *is of order* m *and*

$$SA = SA^2 = \cdots = SA^{m-1} = 0,$$

the scalar of any polynomial

$$\sum_{q=1}^{p} c_q A^q$$

in A *is equal to zero.*

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*See p. 513 above.

† See p. 513 above.

‡ See p. 512 above.

§ 2. *The idempotent number* $\mathfrak{f}(A)$.

Let A be any number of the hypercomplex number system $e_1, e_2, \cdots, e_n$; and let

(26) $$\Omega(A) \equiv A^m + p_1 A^{m-1} + \cdots + p_{m-2}A^2 + p_{m-1}A = 0$$

be the fundamental equation of A. Either

$$p_1 = p_2 = \cdots = p_{m-1} = 0$$

and $A^m = 0$, or the roots of the equation

(27) $$\Omega(\lambda) \equiv \lambda^m + p_1\lambda^{m-1} + \cdots + p_{m-2}\lambda^2 + p_{m-1}\lambda = 0$$

are not all zero. Let $A^m \neq 0$; and let zero be a root of multiplicity $\nu < m$ of equation (27). Let the distinct roots, other than zero, of this equation be $\lambda_1, \lambda_2, \cdots, \lambda_r$, respectively, of multiplicity $\nu_1, \nu_2, \cdots, \nu_r$; and let $p_1^{(\nu)}, p_2^{(\nu)}, \cdots, p_{m-\nu}^{(\nu)}$, with alternate negative and positive sign, denote the simple symmetric functions of the νth powers of the non-zero roots of $\Omega(\lambda) = 0$, $\lambda_1^\nu, \lambda_2^\nu, \cdots, \lambda_r^\nu$ being counted, respectively, $\nu_1, \nu_2, \cdots, \nu_r$ times. Then, since

$$(\lambda^\nu)^{m-\nu+1} + p_1^{(\nu)}(\lambda^\nu)^{m-\nu} + \cdots + p_{m-\nu-1}^{(\nu)}(\lambda^\nu)^2 + p_{m-\nu}^{(\nu)}(\lambda^\nu)$$
$$\equiv \lambda^\nu(\lambda^\nu - \lambda_1^\nu)^{\nu_1}(\lambda^\nu - \lambda_2^\nu)^{\nu_2} \cdots (\lambda^\nu - \lambda_r^\nu)^{\nu_r}$$

contains $\Omega(\lambda)$ as a factor, it follows that

$$(A^\nu)^{m-\nu+1} + p_1^{(\nu)}(A^\nu)^{m-\nu} + \cdots + p_{m-\nu-1}^{(\nu)}(A^\nu)^2 + p_{m-\nu}^{(\nu)}(A^\nu) = 0,^*$$

where

$$p_{m-\nu}^{(\nu)} = \pm(\lambda_1^{\nu_1}\lambda_2^{\nu_2}\cdots\lambda_r^{\nu_r})^\nu \neq 0.$$

Let now

(28) $$\mathfrak{f}(A) \equiv \frac{1}{-p_{m-\nu}^{(\nu)}}[(A^\nu)^{m-\nu} + p_1^{(\nu)}(A^\nu)^{m-\nu-1} + \cdots + p_{m-\nu-1}^{(\nu)}(A^\nu)].$$

We then have

$$A^\nu\mathfrak{f}(A) - A^\nu = \frac{1}{-p_{m-\nu}^{(\nu)}}[(A^\nu)^{m-\nu+1} + p_1^{(\nu)}(A^\nu)^{m-\nu} + \cdots + p_{m-\nu-1}^{(\nu)}(A^\nu)^2$$
$$+ p_{m-\nu}^{(\nu)}(A^\nu)] = 0.$$

Therefore,

(29) $$A^\nu\mathfrak{f}(A) = A^\nu = \mathfrak{f}(A)A^\nu;$$

and, thus, for any positive integer p,

(30) $$A^{\nu+p}\mathfrak{f}(A) = A^{\nu+p} = \mathfrak{f}(A)A^{\nu+p}.$$

In particular, for any positive integer q,

(31) $$(A^\nu)^q\mathfrak{f}(A) = (A^\nu)^q = \mathfrak{f}(A)(A^\nu)^q;$$

and, therefore,

(32) $$\mathfrak{f}(A)\mathfrak{f}(A) = \mathfrak{f}(A).$$

---

* See p. 513 above.

The polynomial $\mathfrak{f}(A) \neq 0$; for, otherwise, by (29) $A^\nu = 0$, and, therefore, $A^m = 0$, which is contrary to supposition. By (31), the hypercomplex number $\mathfrak{f}(A)$ is a modulus of the number system whose units are the independent powers of A^ν. We have

$$(33) \qquad \mathfrak{f}(\lambda) \equiv 1 - \left(\frac{\lambda^\nu - \lambda_1^\nu}{-\lambda_1^\nu}\right)^{\nu_1}\left(\frac{\lambda^\nu - \lambda_2^\nu}{-\lambda_2^\nu}\right)^{\nu_2}\cdots\left(\frac{\lambda^\nu - \lambda_r^\nu}{-\lambda_r^\nu}\right)^{\nu_r};$$

in particular, if $\nu = 1$,

$$(33a) \qquad \mathfrak{f}(\lambda) \equiv \frac{1}{-p_{m-1}}(\lambda^{m-1} + p_1\lambda^{m-2} + \cdots + p_{m-2}\lambda).$$

Since the coefficients of the polynomial $\mathfrak{f}(A)$ in A are rational functions of $p_1, p_2, \cdots, p_{m-1}$ for the domain $R(1)$, they are rational functions in the same domain of $a_1, a_2, \cdots, a_n$ and of the constants γ_{ijk} of multiplication of the number system;* and, thus, are rational functions of $a_1, a_2, \cdots, a_n$ in the domain $R(\gamma_{ijk})$. Wherefore, if A belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, so also does $\mathfrak{f}(A)$, provided R' includes $R(\gamma_{ijk})$.† In particular, if $e_1, e_2, \cdots, e_n$ are the units of a real hypercomplex number system, and A is a number of this system, so also is $\mathfrak{f}(A)$.

Let R' include $R(\gamma_{ijk})$. Each of the units $e_1, e_2, \cdots e_n$ belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots e_n)$. Therefore, either one, at least, of the units, as e_k, is not nilpotent, in which case, if $A = e_k$ there is an idempotent number $\mathfrak{f}(A)$ belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$,—or each of the units is nilpotent, in which case by theorem (1) the system is nilpotent.

We have, therefore, the following theorems, of which theorem (3) is theorem I of the *Introduction.*

THEOREM 2. *If $A = a_1e_1 + \cdots + a_ne_n$ is any number of any given hypercomplex number system $e_1, e_2, \cdots, e_n$, either $A^m = 0$, for some positive integer $m \leqq n + 1$, or there is an idempotent polynomial $\mathfrak{f}(A)$ in A whose coefficients are rational functions in the domain R (1) of $a_1, a_2, \cdots, a_n$ and of the constants γ_{ijk} of multiplication.*

THEOREM 3. *In any hypercomplex number system there is an idempotent number, or every number of the system is nilpotent. Therefore, in a nilpotent system every number is nilpotent.* ‡

THEOREM 4. *Let R' denote the domain of rationality of any aggregate of scalars including the constants γ_{ijk} of multiplication of any given hypercomplex number system $e_1, e_2, \cdots, e_n$: either there is an idempotent number*

* See p. 513 above.

† See p. 513 above.

‡ PIERCE, loc. cit., p. 113.

belonging to $\mathfrak{R}(R';\, e_1, e_2, \cdots e_n)$, *or the number system is nilpotent. In particular, in any real hypercomplex number system there is an idempotent number, or every number is nilpotent; and, therefore in a real nilpotent hypercomplex number system every number is nilpotent.*

By supposition, we have

$$(34)\qquad \Omega(\lambda) \equiv \lambda^{\nu}(\lambda-\lambda_1)^{\nu_1}(\lambda-\lambda_2)^{\nu_2}\cdots(\lambda-\lambda_r)^{\nu_r}.$$

Let

$$(35)\qquad W(\lambda) \equiv \lambda(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_r).$$

Then, *if any polynomial* $\sum_{q=1}^{q=p} c_q A^q$ *in* A *is nilpotent*, $\sum_{q=1}^{q=p} c_q\lambda^q$ *contains* $W(\lambda)$. Conversely, if $\sum_{q=1}^{q=p}\lambda^q$ contains $W(\lambda)$, $\sum_{q=1}^{q=p} c_q A^q$ is nilpotent, since then some power of $\sum_{q=1}^{q=p} c_q\lambda^q$ contains $\Omega(\lambda)$. By (30) and (32), we have

$$(36)\quad A^{\nu}[A-\lambda_1\mathfrak{f}(A)]^{\nu_1}[A-\lambda_2\mathfrak{f}(A)]^{\nu_2}\cdots[A-\lambda_r\mathfrak{f}(A)]^{\nu_r}=\Omega(A)=0;$$

and, therefore, if m' is the greatest of the integers $\nu_1, \nu_2, \cdots, \nu_r$,

$$(37)\quad A^{\nu}\{[A-\lambda_1\mathfrak{f}(A)][A-\lambda_2\mathfrak{f}(A)]\cdots[A-\lambda_r\mathfrak{f}(A)]\}^{m'}=0.$$

Further, if

$$A^{\nu}\{[A-\rho_1\mathfrak{f}(A)][A-\rho_2\mathfrak{f}(A)]\cdots[A-\rho_{r'}\mathfrak{f}(A)]\}^{p}=0,$$

then $r' \geqq r$, and r of the scalars $\rho_1, \rho_2, \cdots, \rho_{r'}$ are equal, respectively, to $\lambda_1, \lambda_2, \cdots, \lambda_r$. For, from the last equation, it follows that

$$A^{\nu+r'}-\sum_{i=1}^{r'}\rho_i A^{\nu+r'-1}+\cdots+(-1)^{r'}\rho_1\rho_2\cdots\rho_{r'}A^{\nu}$$
$$=A^{\nu}[A-\rho_1\mathfrak{f}(A)][A-\rho_2\mathfrak{f}(A)]\cdots[A-\rho_{r'}\mathfrak{f}(A)],$$

by (30), is nilpotent; and, therefore,

$$\lambda^{\nu+r'}-\sum_{i=1}^{r'}\rho_i\lambda^{\nu+r'-1}+\cdots+(-1)^{r'}\rho_1\rho_2\cdots\rho_{r'}\lambda^{\nu}$$

contains $W(\lambda)$, which is impossible unless $r' \geqq r$ and the scalars $\rho_1, \rho_2, \cdots, \rho_{r'}$, comprise $\lambda_1, \lambda_2, \cdots, \lambda_r$.

Let

$$f(A)=\sum_{q=1}^{p} c_q A^q$$

be any polynomial in A such that

$$A^{\nu} f(A)=0.$$

If $f(A)$ is not nilpotent, then by theorem (2), there is an idempotent polynomial

$$I=\sum_{q=1}^{p'}\gamma_q[f(A)]^q$$

in $f(A)$, and thus

$$A^{\nu}I=A^{\nu}\sum_{q=1}^{p'}\gamma_q[f(A)]^q=0;$$

whence, since $\mathfrak{f}(A)$ is a polynomial in A^ν, it follows that

$$\mathfrak{f}(A)I = 0.$$

But, by (30), $\mathfrak{f}(A)$ is an idemfactor* of the νth power of any polynomial in A; and, therefore,

$$I = I^\nu = \mathfrak{f}(A)I^\nu = 0,$$

which is impossible. *Therefore, if*

$$A^\nu \sum_{q=1}^{p} c_q A^q = 0,$$

then

$$\sum_{q=1}^{p} c_q A^q$$

is nilpotent.† Whence, by (37), it follows that

$$(38) \qquad \{[A - \lambda_1 \mathfrak{f}(A)][A - \lambda_2 \mathfrak{f}(A)] \cdots [A - \lambda_r \mathfrak{f}(A)]\}^{m'} = 0$$

for some positive integer m'. Further, *if*

$$\{[(A - \rho_1 \mathfrak{f}(A)][A - \rho_2 \mathfrak{f}(A)] \cdots [A - \rho_{r'} \mathfrak{f}(A)]\}^p = 0,$$

then $r' \geqq r$ *and* r *of the scalars* $\rho_1, \rho_2, \cdots, \rho_{r'}$ *are equal, respectively, to* $\lambda_1, \lambda_2, \cdots, \lambda_r$. For, from the last equation, it follows that

$$A^\nu \{(A - \rho_1 \mathfrak{f}(A)][A - \rho_2 \mathfrak{f}(A)] \cdots [A - \rho_{r'} \mathfrak{f}(A)]\}^p = 0,$$

which, as proved above, is impossible unless $r' \geqq r$ and the scalars $\rho_1, \rho_2, \cdots, \rho_{r'}$ comprise $\lambda_1, \lambda_2, \cdots, \lambda_r$.‡

If $r \geqq 2$, the number system contains at least two independent idempotent

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* See p. 512 above.

† Similarly, if $A^{\nu'} f(A)$ is nilpotent, where $\nu' \leqq \nu$ and

$$f(A) = \sum_{q=1}^{p} c_q A^q,$$

then $A^\nu f(A)$ is nilpotent; and, therefore, $\mathfrak{f}(A) f(A)$, and thus $[f(A)]^\nu = [\mathfrak{f}(A) f(A)]^\nu$ is nilpotent, that is $f(A)$ is nilpotent. Let $\nu' > \nu$, and let $A^{\nu'} f(A)$ be nilpotent; then, by what has just be proved, $A^{\nu' - \nu} f(A)$ is nilpotent. Therefore, if $\nu' - \nu \leqq \nu$, $f(A)$ is nilpotent. On the other hand, if $\nu' - \nu > \nu$, then $A^{\nu' - 2\nu} f(A)$ is nilpotent; and, proceeding thus, we may show ultimately that $f(A)$ is nilpotent. In particular, if $A^{\nu'} f(A) = 0$, $f(A)$ is nilpotent.

‡ These results may be established as follows. For $1 \leqq i \leqq r$, the polynomial $\lambda - \lambda_i \mathfrak{f}(\lambda)$ contains both λ and $\lambda - \lambda_i$; whereas, $\lambda - \rho \mathfrak{f}(\lambda)$ contains $\lambda - \lambda_i$ only if $\rho = \lambda_i$. Therefore,

$$\prod_{i=1}^{r'} [\lambda - \rho_i \mathfrak{f}(\lambda)]$$

contains $W(\lambda)$ if $r' \geqq r$ and if the scalars $\rho_1, \rho_2, \cdots, \rho_{r'}$ comprise $\lambda_1, \lambda_2, \cdots, \lambda_r$, but not otherwise. Whence follow the theorems given above.

numbers. For then, by what has just been proved, neither $A - \lambda_r \mathfrak{f}(A)$ nor

$$\prod_{i=1}^{r-1} [A - \lambda_i \mathfrak{f}(A)]$$

is nilpotent; and, therefore, by theorem (2), there is an idempotent polynomial I in $A - \lambda_r \mathfrak{f}(A)$, and an idempotent polynomial J in

$$\prod_{i=1}^{r-1} (A - \lambda_i \mathfrak{f}(A)];$$

and, by (38)

$$IJ = I^{m'} J^{m'} = (IJ)^{m'} = 0,$$

which is impossible if $J = \rho I$.*

Let A belong to the domain $\mathfrak{R}\ (R';\ e_1, e_2, \cdots, e_n)$, where R' contains $R(\gamma_{ijk})$; and let the factors of $\Omega(\lambda)$ rationally irreducible with respect to R' be

$$\lambda,\ \Omega_1(\lambda),\ \Omega_2(\lambda),\ \cdots,\ \Omega_s(\lambda),$$

so that

$$\Omega(\lambda) \equiv \lambda^{\nu} [\Omega_1(\lambda)]^{\nu_1'} [\Omega_2(\lambda)]^{\nu_2'} \cdots [\Omega_s(\lambda)]^{\nu_s'}. \tag{39}$$

For $1 \leqq i \leqq s$, the roots of $\Omega_i(\lambda) = 0$ are all distinct. Let

$$\begin{aligned} \Omega_i(\lambda) &\equiv \lambda^{r_i} + q_1^{(i)} \lambda^{r_i - 1} + \cdots + q_{r_i - 1}^{(i)} \lambda + q_{r_i}^{(i)} \\ &\equiv (\lambda - \lambda_1^{(i)})(\lambda - \lambda_2^{(i)}) \cdots (\lambda - \lambda_{r_i}^{(i)}) \end{aligned} \qquad (i = 1, 2, \cdots, s). \tag{40}$$

Then, for $1 \leqq i \leqq s$, $q_1^{(i)}, q_2^{(i)}, \cdots, q_{r_i}^{(i)}$ are rational in R'; and, therefore, since $\mathfrak{f}(A)$ belongs to $\mathfrak{R}\ (R';\ e_1, e_2, e_n)$, so also does

$$\omega_i(A) \equiv [A - \lambda_1^{(i)} \mathfrak{f}(A)]\,[A - \lambda_2^{(i)} \mathfrak{f}(A)] \cdots [A - \lambda_{r_i}^{(i)} \mathfrak{f}(A)] \qquad (i = 1, 2, \cdots, s). \tag{41}$$

By (38),

$$\begin{aligned} &[\omega_1(A)\, \omega_2(A) \cdots \omega_s(A)]^{m'} \\ &= \left(\prod_{i=1}^{s} \{ [A - \lambda_1^{(i)} \mathfrak{f}(A)]\,[A - \lambda_2^{(i)} \mathfrak{f}(A)]\,[A - \lambda_{r_i}^{(i)} \mathfrak{f}(A)] \} \right)^{m'} = 0. \end{aligned} \tag{42}$$

Moreover, since no two of the equations $\Omega_1(\lambda) = 0$, $\Omega_2(\lambda) = 0$, etc., have a root in common, and zero is a root of neither of these equations,

$$\prod_{i=1}^{s'} \omega_i(A) = \prod_{i=1}^{s'} \{ [A - \lambda_1^{(i)} \mathfrak{f}(A)]\,[A - \lambda_2^{(i)} \mathfrak{f}(A)] \cdots [A - \lambda_{r_i}^{(i)} \mathfrak{f}(A)] \},$$

is not nilpotent if $s' < s$. Therefore if $s \geqq 2$, neither $\omega_s(A)$ nor

$$\prod_{i=1}^{s-1} \omega_i(A)$$

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* For, if $IJ = 0$ and $J = \rho I$, then $\rho I = \rho I^2 = IJ = 0$, and thus $\rho = 0$; therefore, $J = 0$, which is contrary to supposition, since J is idempotent.

is nilpotent; and consequently, by theorem (2), there is an idempotent polynomial I in $\omega_s(A)$, and an idempotent polynomial J in

$$\prod_{i=1}^{s-1} \omega_i(A).$$

But, by (42)

$$IJ = I^{m'} J^{m'} = (IJ)^{m'} = 0,$$

which is impossible if $J = \rho I$. Since $\omega_i(A)$ belongs to $\mathfrak{R}(R';\, e_1, e_2, \cdots, e_n)$, so also do I and J. Therefore, if $s \geqq 2$, the number system contains at least two independent idempotent numbers belonging to $\mathfrak{R}(R';\, e_1, e_2, \cdots, e_n)$.

Let now the number system $e_1, e_2, \cdots, e_n$ contain but one idempotent number I. Then by what has just been proved, if $A^m \neq 0$,

$$\Omega(\lambda) \equiv \lambda^{\nu}(\lambda - \alpha)^{\nu_1},$$

where $\alpha \neq 0$. Further, $\mathfrak{f}(A) = I$; and, therefore, by (38),

$$A - \alpha I = A - \alpha \mathfrak{f}(A)$$

is nilpotent. Therefore,

$$A = \alpha I + N$$

where $N = A - \alpha I = A - \alpha \mathfrak{f}(A)$ is nilpotent. If, moreover,

$$A = \rho I + N',$$

when N' is nilpotent, then, by theorem (1),

$$\rho S I = S(\rho I + N') = SA = S(\alpha I + N) = \alpha S I;$$

and, therefore, since $SI \neq 0$, $\rho = \alpha$ and, thus, $N = N'$. If A is nilpotent, and

$$A = \rho I + N$$

where N is nilpotent, by theorem (1),

$$\rho S I = S(\rho I + N) = SA = 0;$$

and, therefore, $\rho = 0$. Let R' include $R(\gamma_{ijk})$. At least one of the units $e_1, e_2, \cdots, e_n$ is not nilpotent; otherwise, by theorem (1), the system is nilpotent. Therefore, there is at least one non-nilpotent number A belonging to $\mathfrak{R}[R(\gamma_{ijk});\, e_1, e_2, \cdots, e_n]$; and, therefore, since $I = \mathfrak{f}(A)$, I belongs to $\mathfrak{R}[R(\gamma_{ijk});\, e_1, e_2, \cdots, e_n]$. If A belongs to $\mathfrak{R}(R';\, e_1, e_2, \cdots, e_n)$, and $A^m \neq 0$, then α is rational in R'; and, therefore, $N = A - \alpha I$ belongs to $\mathfrak{R}(R',\, e_1, e_2, \cdots, e_n)$. Whence follows

THEOREM 5. *Let the number system $e_1, e_2, \cdots, e_n$ contain one and only idempotent number I; and let R' denote the domain of rationality of any aggregate of scalars including the constants γ_{ijk} of multiplication of the sys-*

tem. Then I belongs to $\mathbf{R}(R';\ e_1, e_2, \cdots, e_n)$. Moreover, any number A of the system is separable in one and only one way into the sum $\alpha I + N$ of a scalar multiple of I and a nilpotent number N, which is commutative with A. If A is not nilpotent, and $\Omega(A)$ is the fundamental equation of A, then

$$\Omega(\lambda) \equiv \lambda^{\nu}(\lambda - \alpha)^{\nu_1}.$$

Finally, if A belongs to $\mathbf{R}(R';\ e_1, e_2, \cdots, e_n)$, N belongs to this domain, and α is rational in R'.

Let

$$e_i = \alpha_i I + N_i \qquad (i=1, 2, \cdots, n),$$

where $N_i (i = 1, 2, \cdots, n)$ is nilpotent. Then, by the preceding theorem, I and $N_i (1 \leqq i \leqq n)$ belong to $\mathbf{R}[R(\gamma_{ijk});\ e_1, e_2, \cdots, e_n]$, and $\alpha_i (1 \leqq i \leqq n)$ is rational in $R(\gamma_{ijk})$. The idempotent number I and $n-1$ of the units are independent. Thus let I and $e_1, e_2, \cdots, e_{n-1}$ be independent. Then I and $N_1, N_2, \cdots, N_{n-1}$ are independent; and, therefore, we may substitue

$$e'_n = I, \qquad e'_i = N_i = e_i - \alpha_i I \qquad (i=1, 2, \cdots, n-1),$$

for the original units. Since each of the new units belongs to

$$\mathbf{R}[R(\gamma_{ijk});\ e_1, e_2, \cdots, e_n],$$

the transformation is rational in $R(\gamma_{ijk})$. Wherefore,

THEOREM 6. *If the number system $e_1, e_2, \cdots, e_n$ contains but one idempotent number we may substitute, by a transformation rational in $R(\gamma_{ijk})$, new units $e'_1, e'_2, \cdots, e'_n$ such that*

$$e'^2_n = e'_n, \qquad e'^{m_i} = 0 * \qquad (i=1, 2, \cdots, n-1).$$

Let the number system $e_1, e_2, \cdots, e_n$ contain but one idempotent number I belonging to $\mathbf{R}(R';\ e_1, e_2, \cdots, e_n)$, where $\mathbf{R}'$ includes $R(\gamma_{ijk})$; and let A be any non-nilpotent number of the system belonging to this domain. Then, by what precedes,

$$\Omega(\lambda) \equiv \lambda^{\nu}[\Omega_1(\lambda)]^{\nu_1},$$

where

$$\Omega_1(\lambda) \equiv \lambda^{r_1} + q_1^{(1)}\lambda^{r_1 - 1} + \cdots + q_{r_1 - 1}^{(1)}\lambda + q_{r_1}^{(1)}$$

is rationally irreducible in R'. Since A belongs to $\mathbf{R}(R';\ e_1, e_2, \cdots, e_n)$, so also does $\mathfrak{f}(A)$; and, therefore, $\mathfrak{f}(A) = I$. By (29)

$$A^{\nu}[A^{r_1} + q_1^{(1)}A^{r_1 - 1} + \cdots + q_{r_1 - 1}^{(1)}A + q_{r_1}^{(1)}\mathfrak{f}(A)]^{\nu_1} = \Omega(A) = 0;$$

and therefore, by the theorem given on p. 527,

* This theorem includes the first part of theorem IV of the *Introduction*.

$$(A^{r_1} + q_1^{(1)} A^{r_1 - 1} + \cdots + q_{r_1-1}^{(1)} A + q_{r_1}^{(1)} I)^{m'}$$
$$= [A^{r_1} + q_1^{(1)} A^{r_1-1} + \cdots + q_{r_1-1}^{(1)} A + q_{r_1}^{(1)} \mathfrak{f}(A)]^{m'} = 0,$$

for some positive integer m'. Whence follows

THEOREM 7. *Let the hypercomplex number system $e_1, e_2, \cdots, e_n$ contain but one idempotent number I belonging to $\mathbf{R}(R'; e_1, e_2, \cdots, e_n)$, where R' denotes the domain of rationality of any aggregate of scalars including the constants γ_{ijk} of multiplication of the number system. Then, if A is any non-nilpotent number of the system belonging to $\mathbf{R}(R'; e_1, e_2, \cdots, e_n)$, and $\Omega(A) = 0$ is the fundamental equation of A,*

$$\Omega(\lambda) \equiv \lambda^{\nu} [\Omega_1(\lambda)]^{\nu_1'},$$

where $\Omega_1(\lambda)$ is rationally irreducible in R'. Moreover, if

$$\Omega_1(\lambda) \equiv \lambda^{r_1} + q_1^{(r)} \lambda^{r_1-1} + \cdots + q_{r_1-1}^{(1)} \lambda + q_{r_1}^{(1)},$$

then

$$(A^{r_1} + q_1^{(1)} A^{r_1-1} + \cdots + q_{r_1-1}^{(1)} A + q_{r_1}^{(1)} I)^{m'} = 0,$$

for some positive integer m'.

Let now $e_1, e_2, \cdots e_n$ be the units of any given number system containing at least one nilpotent number; and let A be any nilpotent number, other than zero, of the system,— thus let $A^{m-1} \neq 0$, $A^m = 0$. Then, by theorem (1), $SA^p = 0$ for any positive integer p; and, therefore,

$$SA^{m-1} = SA^{m-2} = \cdots = SA = 0.$$

Conversely, let m be the order of the fundamental equation of A, and let A satisfy the above conditions. Then A is nilpotent. For, otherwise, by theorem (2), there is an idempotent polynomial $\mathfrak{f}(A)$ in A; but, by the theorem given p. 523, it follows from the above equations that $SA^p = 0$ for any positive integer p, and thus $S\mathfrak{f}(A) = 0$, which is impossible since $\mathfrak{f}(A)$ is idempotent. Therefore,

THEOREM 8. *If m is the order of the fundamental equation of any number $A \neq 0$ of a given hypercomplex number system, and*

$$SA = SA^2 = \cdots = SA^{m-1} = 0,$$

a fortiori, if $SA^p = 0$ for any positive integer p, then A is nilpotent. Conversely, if A is nilpotent $SA^p = 0$ for any positive integer p.

§ 3. *Classification of the units of an idempotent hypercomplex number system.**

Let R' denote the domain of rationality of any aggregate of scalars comprising the constants γ_{ijk} of multiplication of the idempotent number system

* See p. 512 above.

$e_1, e_2, \cdots, e_n$. By theorem (4) there is at least one idempotent number belonging to the domain $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$. Let

$$I = \sum_{i=1}^{n} a_i^{(0)} e_i$$

be any idempotent number belonging to this domain; and, for $i = 1, 2, \cdots, n$, let

$$(43)\qquad \begin{aligned} e_i^{(1)} &= Ie_iI, \\ e_i^{(2)} &= Ie_i - Ie_iI, \\ e_i^{(3)} &= e_iI - Ie_iI, \\ e_i^{(4)} &= e_i - Ie_i - e_iI + Ie_iI. \end{aligned}$$

We have, for $i = 1, 2, \cdots, n$,

$$(44)\qquad \begin{aligned} &Ie_i^{(1)} = e_i^{(1)} = e_i^{(1)}I, \\ &Ie_i^{(2)} = e_i^{(2)}, \qquad e_i^{(2)}I = 0, \\ &Ie_i^{(3)} = 0, \qquad e_i^{(3)}I = e_i^{(3)}, \\ &Ie_i^{(4)} = 0 = e_i^{(4)}I; \end{aligned}$$

and thus, for $1 \leqq i \leqq n$, the numbers $e_i^{(1)}, e_i^{(2)}, e_i^{(3)}, e_i^{(4)}$ are, respectively, in the first, second, third, and fourth of PEIRCE's groups (mentioned in theorem II of the *Introduction*) with respect to the idempotent number I as the basis.* Further, any number linear in the $e^{(k)}$'s $(k = 1, 2, 3, 4)$ belongs to the kth groups with respect to I.† Since

$$(45)\qquad e_i = e_i^{(1)} + e_i^{(2)} + e_i^{(3)} + e_i^{(4)} \qquad (i=1, 2, \cdots, n),$$

every number of the system can be expressed linearly in the $4n$ numbers of (43).

Let n_1 of the numbers $e_i^{(1)}$ of the first group with respect to I, as $e_1^{(1)}, e_2^{(1)}, \cdots, e_{n_1}^{(1)}$, be independent; n_2 of the numbers $e_i^{(2)}$ of the second group, as $e_1^{(2)}, e_2^{(2)}, \cdots, e_{n_2}^{(2)}$; n_3 of the numbers $e_i^{(3)}$ of the third group, as $e_1^{(3)}, e_2^{(3)}, \cdots, e_{n_3}^{(3)}$; and, finally, let n_4 of the numbers $e_i^{(4)}$ of the fourth group, as $e_1^{(4)}, e_2^{(4)}, \cdots, e_{n_4}^{(4)}$, be independent. Then since each of the numbers $e_i^{(k)}$, for $k = 1, 2, 3, 4$ and $i = 1, 2, \cdots, n$ (and, therefore, any n independent numbers of the system) can be expressed in terms of the above $n_1 + n_2 + n_3 + n_4$ hypercomplex numbers, and the latter are independent,‡ we have $n_1 + n_2 + n_3 + n_4 = n$. We may now show that any number of the system belonging to the kth group with respect to $I\,(1 \leqq k \leqq 4)$ is linear in $e_1^{(k)}, e_2^{(k)}, \cdots, e_{n_k}^{(k)}$. Whence it follows that I is linear in

* See p. 509 above.

† See p. 512 above.

‡ There can be no linear relation between numbers belonging to different groups with respect to I. Cf. the third footnote on p. 518.

$e_1^{(1)}, e_2^{(1)}, \cdots, e_{n_1}^{(1)}$. We may, therefore, substitute I for one of these numbers. It may be that either n_2, n_3, or n_4 is zero; but $n_1 \geqq 1$. For n_1 is the number of independent hypercomplex numbers in the aggregate Ie_iI for $i = 1, 2, \cdots, n$; and these n numbers cannot all be zero, that is $n_1 \neq 0$, otherwise,

$$I = I^3 = I \sum_{i=1}^{n} a_i^{(0)} e_i \cdot I = \sum_{i=1}^{n} a_i^{(0)} Ie_iI = 0.$$

Since both I and the units e_1, e_2, etc., belong to $\mathfrak{R}\,(R';\, e_1, e_2, \cdots, e_n)$, so also do the numbers $e_i^{(k)}$. Therefore, the transformation from the original system of units to the new system of units,

$$(46)\qquad \begin{aligned} e_i' &= e_i^{(1)} & (i &= 1, 2, \cdots, n_1), \\ e_{n_1+i}' &= e_i^{(2)} & (i &= 1, 2, \cdots, n_2), \\ e_{n_1+n_2+i}' &= e_i^{(3)} & (i &= 1, 2, \cdots, n_3), \\ e_{n_1+n_2+n_3+i}' &= e_i^{(4)} & (i &= 1, 2, \cdots, n_4), \end{aligned}$$

is rational in R'; and, therefore, the domain $\mathfrak{R}\,(R';\, e_1', e_2', \cdots, e_n')$ is identical with the domain $\mathfrak{R}\,(R';\, e_1, e_2, \cdots, e_n)$.* The same is true if we substitute I for one of the units of the first group.

Let us now suppose that the first group with respect to I contains a second idempotent number J belonging to $\mathfrak{R}\,(R';\, e_1, e_2, \cdots e_n)$. We may then by a transformation rational in R' substitute for the second system of units, regular† with respect to I as the basis, a third system of units regular with respect to J as the basis, each of the units of the third system belonging to $\mathfrak{R}\,(R';\, e_1', e_2', \cdots, e_n')$, and therefore, to $\mathfrak{R}\,(R';\, e_1, e_2 \cdots e_n)$. The number of units of the third system in the first group with respect to J is the number of independent hypercomplex numbers in the aggregate $Je_i^{(k)}J$ for $k = 1, 2, 3, 4$ and $i = 1, 2, \cdots, n_k$. Since $IJ = J = JI$, we have

$$Je_i^{(k)}J = JI \cdot e_i^{(k)} \cdot IJ = J \cdot Ie_i^{(k)}I \cdot J = 0$$

for $k > 1$, $i = 1, 2, \cdots, n_k$; and, therefore, the number of units of the third system in the first group with respect to J cannot exceed n_1. But, if

$$I - J = c_1 e_1^{(1)} + c_2 e_2^{(1)} + \cdots + c_{n_1} e_{n_1}^{(1)},$$

then

$$\sum_{i=1}^{n_1} c_i Je_i^{(1)}J = J \sum_{i=1}^{n_1} c_i e_i^{(1)} \cdot J = J(I - J)J = J - J = 0;$$

and, therefore, the number of units of the third system in the first group with respect to J is less than n_1. Thus ultimately, by repeated transformations each

* See p. 514 above.

† See p. 512 above.

rational in R' we can feel free the first group of any idempotent number belonging to $\mathbf{R}\ (R'\ ;\ e_1, e_2, \cdots, e_n)$ other than the basis. In the system of units which we shall thus ultimately obtain, we may take the basis as one of the units of the first group. We have, therefore, the following theorem.

THEOREM (9). *Let R' denote the domain of rationality of any aggregate of scalars including the constants γ_{ijk} of multiplication of the idempotent number system $e_1, e_2, \cdots, e_n$. Then, there is at least one idempotent number I belonging to $\mathbf{R}\ (R'\ ;\ e_1, e_2, \cdots, e_n)$; and we may, by a transformation rational in R', introduce new units $e'_1, e'_2, \cdots, e'_n$ regular with respect to I as the basis, each of the new units belonging to $\mathbf{R}\ (R'\ ;\ e_1, e_2, \cdots, e_n)$, and this domain being identical with $\mathbf{R}\ (R'\ ;\ e'_1, e'_2, \cdots, e'_n)$. We may take I as one of the units $e'_1, e'_2, \cdots, e'_n$. If the first group with respect to I of the system of units $e'_1, e'_2, \cdots, e'_n$ contains a second idempotent number I' belonging to $\mathbf{R}\ (R'\ ;\ e'_1, e'_2, \cdots, e'_n)$, we may by a second transformation rational in R' introduce a third system of units $e''_1, e''_2, \cdots, e''_n$ (of which we may take I' to be one) regular with respect to I' as the basis; each of these units will then belong to $\mathbf{R}\ (R'\ ;\ e'_1, e'_2, \cdots, e'_n)$, this domain being identical with $\mathbf{R}\ (R'\ ;\ e''_1, e''_2, \cdots, e''_n)$; and the number of units in the first group with respect to I' of the system $e''_1, e''_2, \cdots, e''_n$ will be at least one less than the number of units of the second system in the first group with respect to I. Therefore, we may introduce, by a transformation rational in R', new units $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_n$ regular with respect to an idempotent number $\bar{I}$ belonging to $\mathbf{R}\ (R'\ ;\ e_1, e_2, \cdots, e_n)$, and such that the first group with respect to $\bar{I}$ contains no second idempotent number; the domain $\mathbf{R}\ (R'\ ;\ \bar{e}_1, \bar{e}_2, \cdots, \bar{e}_n)$ will then be identical with $\mathbf{R}\ (R'\ ;\ e_1, e_2, \cdots, e_n)$. We may take $\bar{I}$ as one of the units of the final system.*

After the substitution, by a transformation rational in R', of a new system of units, $e_i^{(k)} (k = 1, 2, 3, 4$ and $i = 1, 2, \cdots, n_k)$ regular with respect to an idempotent number I belonging to $\mathbf{R}\ (R';\ e_1, e_2, \cdots, e_n)$, let us suppose that the fourth group with respect to I contains an idempotent number J belonging to $\mathbf{R}\ (R';\ e_1, e_2, \cdots, e_n)$. By a transformation rational in R' we may introduce a third system of units regular with respect to J. The units of the third system in the first groups with respect to J are, then, the independent hypercomplex numbers in the aggregate $Je_i^{(k)}J$ for $k = 1, 2, 3, 4$ and $i = 1, 2, \cdots, n_k$. But, since $IJ = JI = 0$, we have

$$I \cdot Je_i^{(k)}J = 0 = Je_i^{(k)}J \cdot I \quad (k = 1, 2, 3, 4;\ i = 1, 2, \cdots, n_k); \tag{47}$$

and, therefore, the units of the third system in the first group with respect to J are in the fourth group with respect to I. The units of the third system in the second group with respect to J are the independent hypercomplex numbers in the aggregate $Je_i^{(k)} - Je_i^{(k)}J$ for $k = 1, 2, 3, 4$ and $i = 1, 2, \cdots, n_k$. But, by (44),

(48) $$Je_i^{(1)} - Je_i^{(1)}J = J \cdot Ie_i^{(1)} - J \cdot Ie_i^{(1)} \cdot J = 0 \qquad (i = 1, 2, \cdots, n_1),$$

(49) $$Je_i^{(2)} - Je_i^{(2)}J = J \cdot Ie_i^{(2)} \cdot J - J \cdot Ie_i^{(2)} \cdot J = 0 \qquad (i = 1, 2, \cdots, n_2);$$

and also

$$Je_i^{(3)}J = J \cdot e_i^{(3)}I \cdot J = 0 \qquad (i = 1, 2, \cdots, n_3),$$

whence follows

(50) $$(Je_i^{(3)} - Je_i^{(3)}J)I = Je_i^{(3)} \cdot I = Je_i^{(3)} = Je_i^{(3)} - Je_i^{(3)}J \qquad (i = 1, 2, \cdots, n_3).$$

Moreover, by (44), we have

(51) $$(Je_i^{(4)} - Je_i^{(4)}J)I = Je_i^{(4)}I = 0 \qquad (i - 1, 2, \cdots, n_4).$$

Finally,

(52) $$I(Je_i^{(k)} - Je_i^{(k)}J) = 0 \qquad (k = 3, 4;\ i = 1, 2, 3, 4).$$

Therefore, by (48) to (52), the units of the third system in the second group with respect to J are each, severally, in one of PEIRCE's four groups with respect to I,—namely, in either the third or fourth group with respect to I. Similarly, we may show that each of the units of the third system in the third group with respect to J is in either the second or fourth group with respect to I; and that each of the units of the third system in the fourth group with respect to J is in either the first, second, third, or fourth group with respect to I. Since $IJ = JI = 0$, we may take both I and J as units of the third system. We have, therefore, the following theorem:

THEOREM. 10. *Let R' denote the domain of rationality of any aggregate of scalars including the constants of multiplication of the idempotent number system $e_1, e_2, \cdots, e_n$; and let the units of the system be regular with respect to an idempotent number I belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$. Then, if the fourth group with respect to I contains an idempotent number belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, we may, by a transformation rational in R', introduce new units $e'_1, e'_2, \cdots, e'_n$ which shall be regular both with respect to I and with respect to J. Of this system we may take both I and J as units.*

When R' includes every scalar real and imaginary, this theorem reduces to the theorem on which HAWKES bases his method for regularizing a number system with respect to each of the idempotent numbers chosen as units.*

§ 4. *Nilpotent hypercomplex number system.*

Let $e_1, e_2, \cdots, e_n$ be the units of a nilpotent hypercomplex number system. Then, by theorem (3), *every number of the system is nilpotent.* Further, by theorem (8), *if A is any number of the system,* $SA = 0$; in particular,

(53) $$Se_1 = Se_2 = \cdots = Se_n = 0.$$

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* HAWKES, loc. cit., p. 317.

Conversely, if the scalar of each of any n independent numbers of any number system in n units is zero, the system is nilpotent *. Whence, or directly from theorem (3), it follows that *every subsystem of a nilpotent system is nilpotent.*

In consequence of a well-known theorem of SCHEFFERS', relating to non-quaternion number systems,† it follows that, in a nilpotent numbert system, new units $e'_1, e'_2, \cdots, e'_n$ can be so selected that

$$(54) \qquad e'_i e'_j = \sum_{k=h}^{n} \gamma'_{ijk} e'_k \qquad (i, j = 1, 2, \cdots n;\ h > i;\ h > j),$$

that is, such that the constants γ'_{ijk} of multiplication of the new units satisfy the conditions

$$(55) \qquad \gamma'_{ijk} = 0 \ddagger \qquad (i, j = 1, 2, \cdots, n;\ k \leqq i;\ k \leqq j).$$

The transformation by which this choice of units is effected can be taken rational in $\mathbf{R}[R(\gamma_{ijk});\ e_1, e_2, \cdots, e_n]$. For this is true for $n = 1$, in which case $e_1^2 = 0$. Again for $n = 2$, if there is a number $A = a_1 e_1 + a_2 e_2$ belonging to $\mathbf{R}[R(\gamma_{ijk});\ e_1, e, \cdots, e_n]$ whose square is not zero, we may put

$$e'_1 = A, \qquad e'_2 = A^2,$$

when we have

$$e'^2_1 = e'_2, \qquad e'_1 e'_2 = e'_2 e'_1 = e'^2_2 = 0 .\S$$

If there is no number belonging to $\mathbf{R}[R(\gamma_{ijk});\ e_1, e, \cdots, e_n]$ whose square is not zero, we have $e_1 e_2 + e_2 e_1 = 0$;|| and, if $e_1 e_2 \neq 0$, we may put

$$e'_1 = e_1, \qquad e'_2 = e_1 e_2,$$

when we have

$$e'^2_1 = 0, \qquad e'_1 e'_2 = e'_2 e'_1 = e'^2_2 = 0 ;\P$$

whereas, if $e_1 e_2 = 0$ (in which case $e_2 e_1 = 0$), the identical transformation,

$$e'_1 = e_1, \qquad e'_2 = e_2$$

satisfies the requirements. Therefore, for $n = 1$ and $n = 2$, new units can be

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* Theorem (1).

† *Encyclopädie der Mathematischen Wissenschaften*, vol. 1, p. 181.

‡ HAWKES, loc. cit., p. 323.

§ The non-vanishing powers of a nilpotent number are independent, PEIRCE, loc. cit., p. 114. Therefore, $A^3 = 0$.

|| For in the case supposed $e_1^2 = 0$, $e_2^2 = 0$, $(e_1 + e_2)^2 = 0$; whence follows $e_1 e_2 + e_2 e_1 = 0$.

¶ We have $e'^2_1 = 0$, $e'^2_2 = 0$, and $e'_1 e'_2 = e_1^2 e_2 = 0$ by supposition. If

$$e'_2 e'_1 = e_1 e_2 \cdot e_1 = c_1 e'_1 + c_2 e'_2 = c_1 e_1 + c_2 e_1 e_2,$$

then since

$$0 = e_1 e_2 \cdot e_1^2 = c_1 e_1^2 + c_2 e_1 e_2 \cdot e_1 = c_2 e_1 e_2 \cdot e_1,$$

either $e'_2 e'_1 = e_1 e_2 \cdot e_1 = 0$, or we have $c_2 = 0$, in which case $e_1 e_2 \cdot e_1 = c_1 e_1$, and, therefore,

$$0 = e_1 e_2 \cdot e_1 e_2 = c_1 e_1 e_2,$$

that is $c_1 = 0$.

chosen to accord with (55) by a transformation rational in $R(\gamma_{ijk})$. Finally, we may show that, if new units can be selected to accord with (55) by a transformation rational in $R(\gamma_{ijk})$ for every nilpotent system in less than n units, the same is true for the given system in n units.

In demonstrating this theorem, I shall assume that, for every nilpotent number system in less than n units, new units can be obtained by a transformation rational in $R(\gamma_{ijk})$ to accord with (55): first, I show by the aid of theorems (i) to (iii) below, irrespective of the rationality of the transformation for a system with less than n units, and without recourse to the theory of groups, that in any given nilpotent system with n units there are numbers (one or more) nilfacient and nilfaciend to every number of the system; second, that certain of these numbers belong to $\mathfrak{R}[R(\gamma_{ijk});\ e_1, e_2, \cdots, e_n]$, follows from simple considerations; third, from the existence of such numbers belonging to $\mathfrak{R}[R(\gamma_{ijk});\ e_1, e_2, \cdots, e_n]$, the lemma given at the end of this section, and our assumption, it follows that the theorem assumed for less than n units is true also of the given nilpotent system with n units.

Let now $A = \sum_{i=1}^{i=n} a_i e_i \neq 0$ be any number of the nilpotent system $e_1, e_2, \cdots, e_n$. Then, since A is nilpotent, it is nilfacient to at least one number of the system. Let A be nilfacient to just m independent numbers, as $A_1, A_2, \cdots, A_m$, of the system. Then, in the first place, A is nilfacient to any number linear in $A_1, A_2, \cdots, A_m$. Secondly, these numbers form a subsystem by themselves. For, if $m = n$, $e_1, e_2, \cdots, e_n$ can be expressed linearly in $A_1, A_2, \cdots, A_n$; on the other hand, if $m < n$, and, for $1 \leqq i \leqq m$, $1 \leqq j \leqq m$, we have

$$A_i A_j = \sum_{h=1}^{m} c_h A_h + B,$$

where B is independent of $A_1, A_2, \cdots, A_m$, then

$$0 = AA_i \cdot A_j = A \cdot A_i A_j = \sum_{h=1}^{m} c_h AA_h + AB = AB,$$

which is contrary to supposition.* The system $A_1, A_2, \cdots, A_m$ being identical with $e_1, e_2, \cdots, e_n$, or a subsystem of the latter, is nilpotent. We may now interchange the order of multiplication and the terms nilfacient and nilfaciend, when we have

THEOREM (i). If A is nilfacient (nilfaciend) to m hypercomplex numbers, $A_1, A_2, \cdots, A_m$, it is nilfacient (nilfaciend) to any number linear in $A_1, A_2, \cdots, A_m$.

THEOREM (ii). Any number $A \neq 0$ of a nilpotent system $e_1, e_2, \cdots, e_n$ is

*CARTAN, Annales de la Faculté des Sciences de Toulouse, vol. 12 (1898).

nilfacient (nilfaciend) to just m independent hypercomplex numbers, where $1 \leqq m \leqq n$, and these numbers form a nilpotent system by themselves. *

Let $A \neq 0$ be nilfacient or nilfaciend to just $m < n$ independent numbers of the system $e_1, e_2, \cdots, e_n$, as $A_1, A_2, \cdots, A_m$. Then by theorem (ii) $A_1, A_2, \cdots, A_m$ form by themselves a nilpotent subsystem; and, since $m < n$, by our assumption, this subsystem contains m independent numbers $A_1', A_2', \cdots, A_m'$ such that

(56) $$A_i' A_j' = \sum_{k=h}^{m} \alpha_{ijk} A_k' \qquad (i, j = 1, 2, \cdots, m;\ h > i;\ h > j).$$

I shall now show that there is a number B independent of $A_1', A_2', \cdots, A_m'$ such that

(57) $$BA_i' = b_{i1} A_1' + b_{i2} A_2' + \cdots + b_{im} A_m' \qquad (i = 1, 2, \cdots, m).$$

First, if there is any number B_m such that $B = B_m A_m'$ is independent of the A''s, then since, by (56),

$$BA_i' = B_m A_m' \cdot A_i' = B_m A_m' A_i' = 0 \qquad (i = 1, 2, \cdots, m),$$

there is a number B independent of the A''s satisfying equations (57). We may, therefore, assume that the product of any number of the system $e_1, e_2, \cdots, e_n$ multiplied by A_m' as post factor is linear in the A''s. Second, if there is any number B_{m-1} such that $B = B_{m-1} A_{m-1}'$ is independent of the A''s, then since, by (56),

$$BA_i' = B_{m-1} A_{m-1}' \cdot A_i' = B_{m-1} \cdot A_{m-1}' A_i' = \alpha_{m-1\, im} B_{m-1} A_m' \quad (i = 1, 2, \cdots, m-1),$$

and both BA_m' and $B_{m-1} A_m'$ are linear in the A''s by supposition, it follows that there is a number B independent of the A''s satisfying equations (57). We may, therefore, assume that the product of any number of the system $e_1, e_2, \cdots, e_n$ multiplied by A_i' $(i = m-1, m)$ as post factor is linear in the A''s. Proceeding thus, we find either a number B independent of the A''s satisfying equations (57); or that the product of any number of the system by A_i' $(i = 2, 3, \cdots, m)$ as post factor is linear in the A''s. We may, therefore, assume that the last is true. Finally, if there is any number B_1 such that $B = B_1 A_1'$ is independent of the A''s, then since by (56),

$$BA_i' = B_1 A_1' \cdot A_i' = B_1 \cdot A_1' A_i' = \sum_{k=h}^{m} \alpha_{1ik} B_1 A_k' \quad (i = 1, 2, \cdots, m;\ 2 \leqq h > i),$$

and since $B_1 A_k'$ for $k \geqq 2$ is linear in the A''s by supposition, there is a number B independent of the A''s satisfying equations (57). If there is no such number B_1, then the product of any number of the system by A_i' $(i = 1, 2, \cdots, m)$ as post factor is linear in the A''s; and, therefore since $m < n$, there is a num-

* CARTAN. See preceding note.

ber B independent of the A''s satisfying equations (57). Similarly, we may show that there is a number C independent of the A''s such that

$$(58)\qquad A'_i C = c_{i1} A'_1 + c_{i2} A'_2 + \cdots + c_{im} A'_m \qquad (i=1, 2, \cdots, m).$$

Since B is nilpotent, some power of B is linear in the A''s. Similarly, some power of C is linear in the A''s. Let B^p and C^q, respectively, be the highest powers of B and C independent of the A''s; then

$$(59)\qquad B^{p+1} = \beta_1 A'_1 + \beta_2 A'_2 + \cdots + \beta_m A'_m,$$

$$(60)\qquad C^{q+1} = \gamma_1 A'_1 + \gamma_2 A'_2 + \cdots + \gamma_m A'_m.$$

From (57) and (58), we derive

$$(61)\qquad B^p A'_i = b^{(p)}_{i1} A'_1 + b^{(p)}_{i2} A'_2 + \cdots + b^{(p)}_{im} A'_m \qquad (i=1, 2, \cdots, m),$$

$$(62)\qquad A'_i C^q = c^{(q)}_{i1} A'_1 + c^{(q)}_{i2} A'_2 + \cdots + c^{(q)}_{im} A'_m \qquad (i=1, 2, \cdots, m),$$

Therefore, by (56), (59), and (61),

$$(63)\qquad \begin{aligned} A'_m B^p \cdot B &= A'_m \cdot B^{p+1} = 0, \\ A'_m B^p \cdot A'_i &= A'_m \cdot B^p A'_i = 0 \end{aligned} \qquad (i=1, 2, \cdots, m);$$

and, by (56), (60), and (62),

$$(64)\qquad \begin{aligned} C \cdot C^q A'_m &= C^{q+1} \cdot A'_m = 0, \\ A'_i \cdot C^q A'_m &= A'_i C^q \cdot A'_m = 0 \end{aligned} \qquad (i=1, 2, \cdots, m).$$

Let us now assume, first, that $A^{(1)} \equiv A'_m B^p \neq 0$.* This number is then, by (63), nilfacient to at least $m+1$ independent numbers $A'_1, A'_2, \cdots, A'_m, B$; and is, moreover, nilfaciend to $A'_1, A'_2, \cdots, A'_m$. If $A^{(1)} = A'_m B^p$ is nilfaciend to some number independent of the A''s, it is, then, also nilfaciend to at least $m+1$ independent numbers. On the other hand, if $A^{(1)} = A'_m B^p$ is nilfaciend to no number independent of the A''s, then, since C^q is independent of the A''s, $A^{(2)} \equiv C^q A'_m B^p \neq 0$. But in this case, the number $A^{(2)} = C^q A'_m B^p$ is nilfaciend to at least $m+1$ independent numbers, since by (64),

$$\begin{aligned} C \cdot C^q A'_m B^p &= C^{q+1} A'_m \cdot B^p = 0, \\ A'_i \cdot C^q A'_m B^p &= A'_i C^q A'_m \cdot B^p = 0 \end{aligned} \qquad (i=1, 2, \cdots, m);$$

and is, moreover, nilfacient to at least $m+1$ independent numbers, namely, $A'_1, A'_2, \cdots, A'_m, B$. Second, let $A'_m B^p = 0$, then since B^p is independent of the A''s, $A^{(3)} \equiv A'_m$ is nilfacient to at least $m+1$ independent numbers. If

* Since A'_m may be nilfacient (or nilfaciend) to numbers of the system other than $A'_1, A'_2, \cdots, A'_m$, it may happen $A'_m B^p = 0$ (or $C^q A'_m = 0$).

now $A^{(3)} = A'_m$ is nilfaciend to some number independent of the A''s, it is, then, also nilfaciend to at least $m+1$ independent numbers. On the other hand, if $A^{(3)} = A'_m$ is nilfaciend to no number independent of the A''s, then, since C^q is independent of the A''s, $A^{(4)} \equiv C^q A'_m \neq 0$. But in this case, the number $A^{(4)} = C^q A'_m$, which is nilfacient to at least $m+1$ independent numbers (namely, those to which A'_m is nilfacient), is also nilfaciend to at least $m+1$ independent numbers, since by (64),

$$\left.\begin{aligned} C \cdot C^q A'_m &= C^{q+1} \cdot A'_m = 0, \\ A'_i \cdot C^q A'_m &= A_i C^q \cdot A'_m = 0 \end{aligned}\right\} \qquad (i = 1, 2, \cdots, m).$$

Whence follows

THEOREM (iii). If the number $A \neq 0$ of a nilpotent system is nilfacient (nilfaciend) to just $m < n$ independent numbers, there is a number A' nilfacient to at least $m+1$ independent numbers and nilfaciend to at least $m+1$ independent numbers.

Let now $A \neq 0$ be any given number of the system, then, by theorem (ii), A is nilfacient to at least one number and nilfaciend to at least one number of the system. Therefore, if A is not both nilfacient and nilfaciend to n independent numbers, it is either nilfacient or nilfaciend to just $m < n$ independent numbers, in which case, by the last theorem, there is a number A' nilfacient to at least $m+1$ independent numbers and nilfaciend to at least $m+1$ independent numbers. Therefore, if A' is not nilfacient and nilfaciend to n independent numbers, it is either nilfacient or nilfaciend to just $m' < n$ independent numbers. But, since $m' \geqq m+1$, proceeding thus, we must ultimately obtain a number both nilfacient and nilfaciend to n independent numbers, and, therefore, nilfacient and nilfaciend to every number of the system.

The conditions necessary and sufficient that $A = \sum_{i=1}^{i=n} a_i e_i$ shall be nilfacient and nilfaciend to every number of the system are, by theorem (i),

$$\sum_{i=1}^{n}\sum_{k=1}^{n} a_i \gamma_{ijk} e_k = \sum_{i=1}^{n} a_i e_i e_j = A e_j = 0 \qquad (j = 1, 2, \cdots, n),$$

that is,

$$a_1 \gamma_{1jk} + a_2 \gamma_{2jk} + \cdots + a_n \gamma_{njk} = 0 \qquad (j, k = 1, 2, \cdots, n),$$

and

$$\sum_{i=1}^{n}\sum_{k=1}^{n} a_i \gamma_{1ik} e_k = \sum_{i=1}^{n} a_i e_j e_i = e_j A = 0 \qquad (j = 1, 2, \cdots, n),$$

that is,

$$a_1 \gamma_{j1k} + a_2 \gamma_{j2k} + \cdots + a_n \gamma_{jnk} = 0 \qquad (j, k = 1, 2, \cdots, n).$$

Since there is at least one such number A, there is at least one such number A belonging to $\mathbf{R}[R(\gamma_{ijk});\ e_1, e_2, \cdots e_n]$. We may, therefore, by a transformation rational in $R(\gamma_{ijk})$, introduce new units $e'_1, e'_2, \cdots, e'_n$ such that

$$e'_n e'_i = 0 = e'_i e'_n \qquad (i = 1, 2, \cdots, n),$$

the constants γ'_{ijk} of multiplication of the new units being rational in $R(\gamma_{ijk})$. By the lemma given below, there is, then, a nilpotent number system $\bar{e}_1, \bar{e}_2, \cdots \bar{e}_{n-1}$ such that

$$\bar{e}_i\bar{e}_j = \sum_{k=1}^{n-1}\gamma'_{ijk}\bar{e}_k \qquad (i, j = 1, 2, \cdots, n-1).$$

By our assumption, we may introduce into this number system, by a transformation

$$\bar{e}'_i = \sum_{k=1}^{n-1}\tau_{ik}\bar{e}_k \qquad (i = 1, 2, \cdots, n-1)$$

rational in $R(\gamma'_{ijk})$, and, therefore, rational in $R(\gamma_{ijk})$, new units $\bar{e}'_1, \bar{e}'_2, \cdots, \bar{e}'_{n-1}$ to accord with (55), that is, such that

$$\bar{e}'_i\bar{e}'_j = \sum_{k=h}^{n-1}\gamma''_{ijk}\bar{e}'_k \quad (i, j = 1, 2, \cdots, n-1;\ h > i;\ h > j).$$

When, by the lemma, if

$$e''_i = \sum_{k=1}^{n-1}\tau_{ik}e'_k \qquad (i = 1, 2, \cdots, n-1),$$

$$e''_n = e'_n,$$

we have

$$e''_i e''_j = \sum_{k=h}^{n-1}\gamma''_{ijk}e''_k + \gamma''_{ijn}e''_n \quad (i, j = 1, 2, \cdots, n;\ h > i;\ h > j),$$

$$e''_n e''_i = 0 = e''_i e''_n \qquad (i = 1, 2, \cdots, n).$$

This transformation is also rational in $R(\gamma'_{ijk})$ and, therefore, in $R(\gamma_{ijk})$; and, thus, by two successive transformations, each rational in $R(\gamma_{ijk})$, we may obtain a system of units whose multiplication table accords with (55).*

THEOREM 11. *In any nilpotent number system $e_1, e_2, \cdots, e_n$ we may substitute, by a transformation rational in $R(\gamma_{ijk})$, new units $e'_1, e'_2, \cdots, e'_n$ such that*

$$e'_i e'_j = \sum_{k=h}^{n}\gamma_{ijk}e'_k \quad (i, j = 1, 2, \cdots, n;\ h > i;\ h > j).$$

LEMMA. Let $e_1, e_2, \cdots, e_n$ be any number system for which

$$e_n e_i = 0 = e_i e_n \qquad (i = 1, 2, \cdots, n).$$

Then, denoting by γ_{ijk} the constants of multiplication of this system, there is a number system $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{n-1}$ in $n-1$ units such that

$$\bar{e}_i\bar{e}_j = \sum_{k=1}^{n-1}\gamma_{ijk}\bar{e}_k \qquad (i, j = 1, 2, \cdots, n-1),$$

* Since the constants of multiplication of the system $e''_1, e''_2, \cdots, e''_n$ are rational functions of the coefficients of the transformation for the domain $R(\gamma'_{ijk})$, they are rational in the domain $R(\gamma_{ijk})$.

which is nilpotent if the system $e_1, e_2, \cdots, e_n$ is nilpotent. Moreover, if we introduce new units $e'_1, e'_2, \cdots, e'_n$ and $\bar{e}'_1, \bar{e}'_2, \bar{e}'_{n-1}$ into the respective systems $e_1, e_2, \cdots, e_n$ and $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{n-1}$ by transformations

$$e'_i = \sum_{k=1}^{n-1} \tau_{ik} e_k, \qquad e'_n = e_n \qquad (i=1, 2, \cdots, n-1),$$

and

$$\bar{e}'_i = \sum_{k=1}^{n-1} \tau_{ik} \bar{e}_k \qquad (i=1, 2, \cdots, n-1);$$

and, if

$$e'_i e'_j = \sum_{k=1}^{n} \gamma'_{ijk} e'_k \qquad (i, j=1, 2, \cdots, n),$$

$$\bar{e}'_i \bar{e}'_j = \sum_{1}^{n-1} \bar{\gamma}_{ijk} \bar{e}'_k \qquad (i, j=1, 2, \cdots, n-1),$$

then

$$\gamma'_{ijk} = \bar{\gamma}_{ijk} \qquad (i, j, k=1, 2, \cdots, n-1).$$

For let $e_1, e_2, \cdots, e_n$ be the units of any hypercomplex number system such that

$$\gamma_{nij} = 0 = \gamma_{inj} \qquad (i, j=1, 2, \cdots, n),$$

that is, such that $e_n e_i = 0 = e_i e_n$ for $i = 1, 2, \cdots, n$. Then

$$\sum_{h=1}^{n-1} \gamma_{ihl}\gamma_{jkh} = \sum_{h=1}^{n-1} \gamma_{ihl}\gamma_{jkh} + \gamma_{inl}\gamma_{jkn} = \sum_{h=1}^{n} \gamma_{ihl}\gamma_{jkh} = \sum_{h=1}^{n} \gamma_{ijh}\gamma_{hkl}$$

$$= \sum_{h=1}^{n-1} \gamma_{ijh}\gamma_{hkl} + \gamma_{ijn}\gamma_{nkl} = \sum_{h=1}^{n-1} \gamma_{ijh}\gamma_{hkl} \quad (i, j, k, l=1, 2, \cdots, n-1),$$

which is the condition necessary and sufficient that a number system $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{n-1}$ shall exist such that

$$\bar{e}_i \bar{e}_j = \sum_{h=1}^{n-1} \gamma_{ijh} \bar{e}_h\,\text{*} \qquad (i, j=1, 2, \cdots, n-1).$$

Let

$$(a_1 e_1 + a_2 e_2 + \cdots + a_n e_n)(b_1 e_1 + b_2 e_2 + \cdots + b_n e_n) = (c_1 e_1 + c_2 e_2 + \cdots + c_n e_n),$$

$$(a_1 \bar{e}_1 + \cdots + a_{n-1} \bar{e}_{n-1})(b_1 \bar{e}_1 + \cdots + b_{n-1} \bar{e}_{n-1}) = (\bar{c}_1 \bar{e}_1 + \cdots + \bar{c}_{n-1} \bar{e}_{n-1}).$$

Then, since

$$\bar{c}_k = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_i b_j \gamma_{ijk} \qquad (k=1, 2, \cdots, n-1),$$

and

$$c_k = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i b_j \gamma_{ijk} = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_i b_j \gamma_{ijk} \quad (k=1, 2, \cdots, n-1),$$

we have

$$\bar{c}_k = c_k \qquad (k=1, 2, \cdots, n-1).$$

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* See p. 515 above.

Let now $\bar{e}'_1, \bar{e}'_2, \cdots, \bar{e}'_{n-1}$ and $e'_1, e'_2, \cdots, e'_n$ be new systems of units of the respective number systems $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{n-1}$ and $e_1, e_2, \cdots, e_n$, obtained, respectively, by the linear transformations $\overline{T}$ and T of non-zero determinant defined, respectively, as follows,

$$\bar{e}'_i = \tau_{i1}\bar{e}_1 + \tau_{i2}\bar{e}_2 + \cdots + \tau_{in-1}\bar{e}_{n-1} \qquad (i=1, 2, \cdots, n-1),$$

and

$$e'_i = \tau_{i1} e_1 + \tau_{i2} e_2 + \cdots + \tau_{in-1} e_{n-1}, \qquad e'_n = e_n \quad (i=1, 2, \cdots, n-1),$$

and let

$$\bar{e}'_i \bar{e}'_j = \sum_{h=1}^{n-1} \bar{\gamma}'_{ijh} \bar{e}'_h = \sum_{h=1}^{n-1} \sum_{k=1}^{n-1} \bar{\gamma}'_{ijh} \tau_{hk} \bar{e}_k \quad (i, j=1, 2, \cdots, n-1),$$

$$e'_i e'_j = \sum_{h=1}^{n} \gamma'_{ijh} e'_h = \sum_{h=1}^{n-1} \sum_{k=1}^{n-1} \gamma'_{ijh} \tau_{hk} e_k + \gamma'_{ijn} e_n \quad (i, j=1, 2, \cdots, n-1).$$

Then, by what has just been proved,

$$\sum_{h=1}^{n-1} \bar{\gamma}'_{ijh} \tau_{hk} = \sum_{h=1}^{n-1} \gamma'_{ijh} \tau_{hk} \qquad (i, j, k=1, 2, \cdots, n-1)$$

and therefore

$$\bar{\gamma}'_{ijk} = \gamma'_{ijk} \qquad (i, j=1, 2, n-1);$$

for, otherwise, the determinant of $\overline{T}$ is zero, which is contrary to supposition.

The number systems $e_1, e_2, \cdots, e_n$ and $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{n-1}$ have the same number of independent idempotent numbers. For, if

(65) $$(a_1^{(k)} e_1 + \cdots + a_n^{(k)} e_n)^2 = a_1^{(k)} e_1 + \cdots + a_n^{(k)} e_n \quad (k=1, 2, \cdots, m),$$

then

(66) $$a_n^{(k)} = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_i^{(k)} a_j^{(k)} \gamma_{ijn} \qquad (k=1, 2, \cdots, m),$$

and

(67) $$(a_1^{(k)} \bar{e}_1 + \cdots + a_{n-1}^{(k)} \bar{e}_{n-1})^2 = a_1^{(k)} \bar{e}_1 + \cdots + a_{n-1}^{(k)} \bar{e}_{n-1} \quad (k=1, 2, \cdots, m).$$

Conversely, from (66) and (67) we obtain (65). Moreover, if

$$\left\| \begin{matrix} a_1^{(1)} & a_2^{(1)} & \cdots & a_n^{(1)} \\ a_1^{(2)} & a_2^{(2)} & \cdots & a_n^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{(m)} & a_2^{(m)} & \cdots & a_n^{(m)} \end{matrix} \right\| = 0,$$

then

$$\left\| \begin{matrix} a_1^{(1)} & a_2^{(1)} & \cdots & a_{n-1}^{(1)} \\ a_1^{(2)} & a_2^{(2)} & \cdots & a_{n-1}^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{(m)} & a_2^{(m)} & \cdots & a_{n-1}^{(m)} \end{matrix} \right\| = 0;$$

and, conversely, by (66). Whence it follows that, if $e_1, e_2, \cdots, e_n$ is nilpotent, so also is $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{n-1}$; and conversely.*

§ 5. *Hypercomplex number system with but one idempotent number.*

Let $e_1, e_2, \cdots, e_n$ be the units of a hypercomplex number system with one and only one idempotent number

$$I = \sum_{i=1}^{n} a_i^{(0)} e_i,$$

in particular, $e_1, e_2, \cdots, e_n$ may be the units of the first group with respect to I of a number system in $n' \geqq n$ units; and let R' denote the domain of rationality of any aggregate of scalars including the constants γ_{ijk} of multiplication of the number system. Then, by theorem (5), I belongs to $\mathbf{R}(R'; e_1, e_2, \cdots, e_n)$. Let

$$A = \sum_{i=1}^{n} a_i e_i$$

be any number of the system, and let

$$C = IAI - IA;$$

then

$$C^2 = 0, \qquad IC = C, \qquad CI = 0.$$

If $C \neq 0$, there is a second idempotent number in the system, namely, $I + \rho C$ for any scalar $\rho \neq 0$. For, we have

$$(I + \rho C)^2 = I + \rho(IC + CI) + \rho^2 C^2 = I + \rho C;$$

and, if $I + \rho C = \sigma I$, then

$$(1 - \sigma)I = [(1 - \sigma)I + \rho C] I = 0;$$

thus, $\sigma = 1$ and, therefore, $\rho = 0$, which is contrary to supposition. Therefore,

$$IAI - IA = 0.$$

Similarly, we may show that

$$IAI - AI = 0.$$

Whence it follows that

(68) $$IA = IAI = AI;$$

that is, every number of the system is commutative with I.

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* Or, otherwise, as follows. We have

$$\sum_{j=1}^{n} \gamma_{ijj} = \sum_{j=1}^{n-1} \gamma_{ijj}, \qquad \sum_{j=1}^{n} \gamma_{njj} = 0 \qquad (i = 1, 2, \ldots, n-1).$$

Therefore, if $Se_i = 0$ ($i = 1, 2, \cdots n$), then $S\bar{e}_i = 0$ ($i = 1, 2, \cdots n - 1$); and, conversely. See theorem (11).

By theorem (5) any number A of the system is separable in one and only one way into the sum of a scalar multiple of I and a nilpotent number. Let

$$A = \alpha I + N \tag{69}$$

where N is nilpotent. Since N is commutative with I, it is also commutative with A.*

By theorem (1),

$$SA = S(\alpha I + N) = \alpha SI, \tag{70}$$

where $SI \neq 0$. Therefore, if $SA = 0$, $\alpha = 0$, and $A = N$ is nilpotent; conversely, if A is nilpotent, $SA = 0$ by theorem (8).†

Since $IN = NI$, IN is nilpotent. Therefore,

$$SIA = S \cdot I(\alpha I + N) = S(\alpha I + IN) = \alpha SI = SA. \tag{71}$$

Let now A be nilpotent. Then

$$SIA = SA = 0,$$

and IA is nilpotent; and, therefore, there is a smallest positive integer p for which $(IA)^p = 0$. Let B be any second number of the system. If $p = 1$, then

$$SAB = S(I \cdot AB) = S(IA \cdot B) = 0.$$

On the other hand, if $p > 1$, and

$$AB = \rho I + N',$$

where $N'^{m'} = 0$, then

$$IA \cdot B = I \cdot AB = \rho I + IN',$$

and, therefore,

$$\rho(IA)^{p-1} + (IA)^{p-1}N' = \rho(IA)^{p-1}I + (IA)^{p-1}IN' = (IA)^{p-1}(\rho I + IN')$$
$$= (IA)^p B = 0,$$

*See theorem (5). Further, if A is not nilpotent, $N = A - \alpha I = A - \alpha \mathfrak{f}(A)$; whereas, on the other hand, if A is nilpotent $N = A$, see p. 529. Therefore, in either case, N is a polynomial in A.

† Again, since A is separable in but one way into the sum $\alpha I + N$ of a scalar multiple α of I and a nilpotent number N, A is nilpotent if and only if $\alpha = 0$, and, therefore, if and only if $SA = 0$.

Let $\Omega(A) = 0$ be the fundamental equation of the number A of any given idempotent number system; and let m_0 be the greatest number of distinct roots, other than zero, of $\Omega(\lambda) = 0$ for any number A of the system. Then the condition necessary and sufficient that an arbitrarily given number A of the system shall be nilpotent is $SA^p = 0$ for $p = 1, 2, \cdots, m_0$. If the given idempotent system contains but one idempotent number I belonging to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, where R' contains $R(\gamma_{ijk})$ and A belongs to $\mathfrak{R}(R'; e_1, e_2, \cdots, e_n)$, $\Omega(\lambda) \equiv \lambda^\nu [\Omega_1(\lambda)]^{\nu_1}$, where $\Omega_1(\lambda)$ is irreducible with respect to R'; and, therefore, if m_0 is the highest degree of $\Omega_1(\lambda)$ for any number A of the system, the condition necessary and sufficient that A shall be nilpotent is $SA^p = 0$ for $p = 1, 2, \cdots, m_0$.

that is,

$$(IA)^{p-1}N' = -\rho(IA)^{p-1};$$

whence follows

$$(-\rho)^{m'}(IA)^{p-1} = (-\rho)^{m'-1}(IA)^{p-1}N' = \cdots = -\rho(IA)^{p-1}N'^{m'-1}$$
$$= (IA)^{p-1}N'^{m'} = 0;$$

and, thus, $\rho = 0$. Therefore, also for $p > 1$

$$SAB = \rho SI = 0.$$

Whence it follows that AB is nilpotent if A is nilpotent; and, since $SBA = SAB$, BA is also nilpotent if A is nilpotent.

If $SA = 0$, $SB = 0$, then

$$S(A + B) = SA + SB = 0;$$

and, therefore, the sum of two nilpotent numbers is nilpotent.

Let A and B be any two numbers of the system. Since

$$S(SI\cdot A - SA\cdot I) = SA\cdot SI - SA\cdot SI = 0,$$

and, therefore, $SI\cdot A - SA\cdot I$ is nilpotent, by what precedes,

$$SI\cdot SAB - SA\cdot SB = SI\cdot SAB - SA\cdot SIB = S(SI\cdot A - SA\cdot I)B = 0,$$

that is,

$$(72)\qquad SI\cdot SAB = SA\cdot SB.$$

Let

$$N^m = (A - \alpha I)^m = 0.$$

Then, if $SA \neq 0$, and thus $\alpha \neq 0$,

$$\left(I + \frac{N}{\alpha}\right)\left[I - \frac{N}{\alpha} + \left(\frac{N}{\alpha}\right)^2 - \cdots + (-1)^{m-1}\left(\frac{N}{\alpha}\right)^{m-1}\right]I$$
$$= I + (-1)^m\left(\frac{N}{\alpha}\right)^m I = I;$$

and, therefore, if

$$(73)\qquad A^{(1)} \equiv \frac{1}{\alpha}\left[I + \left(I - \frac{A}{\alpha}\right) + \left(I - \frac{A}{\alpha}\right)^2 + \cdots + \left(I - \frac{A}{\alpha}\right)^{m-1}\right]I$$

we have

$$(74)\qquad A^{(1)}A = I = AA^{(1)}.$$

If A belongs to $\mathbf{R}\ (R';\ e_1, e_2, \cdots, e_n)$, where R' contains $R\ (\gamma_{ijk})$, then N belongs to this domain and α is rational in R' by theorem (5); and, therefore, in this case, $A^{(1)}$ belongs to $\mathbf{R}\ (R';\ e_1, e_2, \cdots, e_n)$.

By theorem (6) the number system contains $n - 1$ independent nilpotent numbers which belong to $\mathfrak{R}(R';\, e_1, e_2, \cdots, e_n)$, where R' contains $R(\gamma_{ijk})$. The system cannot contain n independent nilpotent numbers $A_1, A_2, \cdots, A_n$. For, we should then have

$$I = c_1 A_1 + c_2 A_3 + \cdots + c_n A_n;$$

and, therefore

$$SI = c_1 SA_1 + c_2 SA_2 + \cdots + c_n SA_n = 0,$$

which is impossible.* We may take as units of our system $e'_n = I$ and any $n-1$ independent nilpotent numbers $e'_1, e'_2, \cdots, e'_{n-1}$ belonging to $\mathfrak{R}(R';\, e_1, e_2, \cdots, e_n)$. The numbers $e'_1, e'_2, \cdots, e'_{n-1}$ form a nilpotent system by themselves. For, let

$$e'_i e'_j = \gamma'_{ij1} e'_1 + \gamma'_{ij2} e'_2 + \cdots + \gamma'_{ijn-1} e'_{n-1} + \gamma'_{ijn} e'_n \quad (i, j = 1, 2, \cdots, n-1).$$

Then, by (72),

$$0 = \frac{1}{SI} Se'_i \cdot Se'_j = Se'_i e'_j = \sum_{k=1}^{n} \gamma'_{ijk} Se'_k = \gamma_{ijn} SI \qquad (i, j = 1, 2, \cdots, n-1);$$

and, therefore,

$$\gamma_{ijn} = 0 \qquad (i, j = 1, 2, \cdots, n).$$

Moreover, since $Se'_i = 0$ for $i = 1, 2, \cdots, n-1$, the system $e'_1, e'_2, \cdots, e'_{n-1}$ is nilpotent.

By theorem (11) we may substitute for the units of this nilpotent subsystem $n-1$ units $e''_1, e''_2, \cdots, e''_{n-1}$ such that

$$e''_i e''_j = \sum_{k=h}^{n-1} \gamma''_{ijk} e''_k \quad (i, j = 1, 2, \cdots, n-1;\ h > i;\ h > j).$$

We have now the following theorem, the latter part of which includes theorem IV of the *Introduction.*

THEOREM 12. *Let the hypercomplex number system* $e_1, e_2, \cdots, e_n$ *contain one and only one idempotent number* I*; and let* A *be any number of the system. Then* $IA = AI$*; and the condition necessary and sufficient that* A *shall be nilpotent is* $SA = 0$. *If* B *is any second number of the system, we have*

$$SI \cdot SAB = SA \cdot SB.$$

Furthermore, the sum of two nilpotent numbers is nilpotent, and any product is nilpotent of which one factor is nilpotent.† *Finally, if* R' *denotes the*

*This suffices to demonstrate the theorem proved in the next paragraph. For let $A_1, A_2, \cdots, A_{n-1}$ be any $n-1$ independent nilpotent numbers of the system. Then, since $A_i A_j$ $(1 \leqq i \leqq n-1,\ 1 \leqq j \leqq n-1)$ is nilpotent, it is linear in $A_1, A_2, \cdots, A_{n-1}$; otherwise, there are n independent nilpotent numbers in the system.

† For any given hypercomplex number system containing but one idempotent number belonging to $\mathfrak{R}(R';\, e_1, e_2, \cdots, e_n)$, where R' contains $R(\gamma_{ijk})$, it is also true, for numbers belonging to this hypercomplex domain, that both the sum of two nilpotent numbers, and any product of which one factor is nilpotent, are nilpotent.

domain of rationality of any aggregate of scalars including the constants γ_{ijk} *of multiplication of the system, there are systems of just* $n-1$ *independent nilpotent numbers belonging to* $\mathbf{R}\,(R';\ e_1, e_2, \cdots, e_n)$*; and by a transformation rational in* R' *we may substitute new units* $e'_n = I$, *and* $n-1$ *nilpotent numbers* $e'_1, e'_2, \cdots, e'_{n-1}$ *such that*

$$e'_i e'_j = \sum_{k=h}^{n-1} \gamma'_{ijk} e'_k \qquad (i, j = 1, 2, \cdots, n-1;\ h > i;\ h > j),$$

$$e'_n e'_i = e'_i = e'_i e'_n \qquad (i = 1, 2, \cdots, n),$$

the nilpotent numbers $e'_1, e'_2, \cdots, e'_{n-1}$ *thus forming a nilpotent subsystem.*

It follows as a corollary of this theorem that, in a number system regular with respect to an idempotent number I and containing no second idempotent number in the first group, no product of a number in the second group as facient and a number in the third group as faciend can contain a term involving I unless the fourth group contains one or more indempotent numbers.

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