THE DOUBLY PERIODIC SOLUTIONS OF POISSON'S EQUATION IN TWO INDEPENDENT VARIABLES*

BY

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The only doubly periodic solution † of LAPLACE's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is $u = c$, where c is a constant. For if u be such a solution, and v the conjugate potential to u, then $u + iv$ would be a complex analytic function which has a value under a fixed finite limit for all values of x, y. But, as is well known, such a function is necessarily a constant.

It is the object of this paper to investigate, by the use of methods analogous to those of the potential theory, the doubly periodic solutions of POISSON's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \tag{1}$$

where $f(x, y)$ is continuous and periodic in x and in y with the periods a and b respectively.‡ A "doubly periodic GREEN's function," G, will be formed from known functions, and the desired solution of (1) found by quadrature from G and f.

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† A function u will be called a solution of the differential equation within a region Ω, provided that u satisfies the differential equation at every point within Ω. This definition requires the existence of the second derivatives of u at every point in Ω, and therefore the continuity of the first derivatives. By a doubly periodic solution we shall mean a doubly periodic function which is a solution of the equation in the period rectangle, and therefore in the entire plane. Such a function, in particular, has a value less than a fixed finite number for all values of x, y.

‡ In a recent article (Journal de Mathématiques, ser. 5, vol. 10 (1904), p. 445) I have considered by a different method the existence of periodic solutions of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda A(x, y) u = f(x, y),$$

where λ is a parameter.

§ 1. *A doubly periodic Green's function and its law of reciprocity.*

Let $\mathfrak{R}$ denote the real part of the term before which it is written, and consider the function

$$\mathfrak{R} \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)}; \; z = x + iy, \; \zeta = \xi + i\eta, \; \gamma = \alpha + i\beta,$$

where $\sigma(z)$ is the Sigma function of WEIERSTRASS, formed with the periods a, ib; and ξ, γ are two points in the interior of the period rectangle Ω bounded by the lines $x = 0, y = a, x = 0, y = b$. This function is a solution of Laplace's equation within Ω, except at the points (ξ, η) and (α, β), and has the form

$$\log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} + g,$$

where g is a solution of LAPLACE'S equation throughout Ω.

Since for any integers m, n, the function σ obeys the law

$$\sigma(z + ma + inb) = (-1)^{mn+m+n} e^{(m\eta_1 + n\eta_3)(2z + ma + inb)} \sigma(z),$$

where η_1, η_3 are certain complex constants,* we have

$$(2) \qquad \mathfrak{R} \log \frac{\sigma(z + ma + inb - \zeta)}{\sigma(z + ma + inb - \gamma)} = \mathfrak{R} \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)} = m \mathfrak{R}\, 2\eta_1(\zeta - \gamma) - n \mathfrak{R}\, 2\eta_3(\zeta - \gamma).$$

Define a real function V by the equation

$$V(x, y, \xi, \eta, \alpha, \beta) = \mathfrak{R} \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)} + \frac{x}{a} \mathfrak{R}\, 2\eta_1(\zeta - \gamma) + \frac{y}{b} \mathfrak{R}\, 2\eta_3(\zeta - \gamma).$$

Since the last two terms are linear in x and y, V has the form

$$V(x, y, \xi, \eta, \alpha, \beta) = \log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} + S(x, y, \xi, \eta, \alpha, \beta),$$

where S is a known solution of

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} = 0$$

within Ω. Furthermore V is doubly periodic in x, y with the periods a, b, since from (2) the equation results:

*See *e. g.*, BURKHARDT, *Elliptische Functionen*, p. 53.

$$V(x+ma,\, y+nb,\, \xi,\, \eta,\, a,\, \beta) = \Re \log \frac{\sigma(z+ma+inb-\zeta)}{\sigma(z+ma+inb-\gamma)}$$

$$+ \frac{x+ma}{a} \Re\, 2\eta_1(\zeta-\gamma) + \frac{y+nb}{b} \Re\, 2\eta_3(\zeta-\gamma)$$

$$= \Re \log \frac{\sigma(z-\zeta)}{\sigma(z-\gamma)} + \frac{x}{a} \Re\, 2\eta_1(\zeta-\gamma) + \frac{y}{b} \Re 2\eta_3(\zeta-\gamma)$$

$$= V(x,\, y,\, \xi,\, \eta,\, a,\, \beta)^*.$$

We shall call the function

$$G(x,\, y,\, \xi,\, \eta,\, a,\, \beta) = V(x,\, y,\, \xi,\, \eta,\, a,\, \beta) - S(a,\, \beta,\, \xi,\, \eta,\, a,\, \beta)$$

the doubly periodic Green's function for the periods a, b. This function has the following characteristics:

1°. Except at (ξ, η) and (a, β), G is, within Ω, a solution of the equation

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial^2 y} = 0.$$

2°. G has the form

$$G = \log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-a)^2 + (y-\beta)^2}} + R(x, y, \xi, \eta, a, \beta),$$

where R is a known function, which, with respect to the variables x, y, is a solution of LAPLACE's equation within Ω, and which satisfies for all values of ξ, η in Ω, the equation

$$R(a,\, \beta,\, \xi,\, \eta,\, a,\, \beta) = 0.$$

3°. G is doubly periodic in x, y with the periods a, b.

The functions G and R obey the following laws of reciprocity:

$$G(x,\, y,\, \xi,\, \eta,\, \alpha,\, \beta) + \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} = G(\xi,\, \eta,\, x,\, y,\, \alpha,\, \beta)$$

$$+ \log \frac{1}{\sqrt{(\xi-\alpha)^2 + (\eta-\beta)^2}},$$

$$R(x,\, y,\, \xi,\, \eta,\, \alpha,\, \beta) = R(\xi,\, \eta,\, x,\, y,\, \alpha,\, \beta).$$

* In the same manner may be formed a doubly periodic function

$$V(x,\, y,\, \xi_1,\, \eta_1,\, \cdots,\, \xi_n,\, \eta_n) = \sum_{i=1}^{n} c_i \log \sqrt{(x-\xi_i)^2 + (y-\eta_i)^2} + S(x,\, y,\, \xi_1,\, \eta_1,\, \cdots,\, \xi_n,\, \eta_n)$$

with any number of logarithmic singularities, where S is a solution of LAPLACE's equation in Ω with respect to the variables x, y, provided that $c_1 + c_2 + \cdots + c_n = 0$.

To prove these laws apply GREEN's theorem,

$$\int\int (v\Delta u - u\Delta v)dxdy = \int \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) ds,$$

where n is the outward drawn normal, to the region Ω' formed by excluding from Ω the circles $c(\xi, \eta)$, $c(\xi', \eta')$, $c(\alpha, \beta)$ of radius r about the points (ξ, η), (ξ', η'), (α, β) respectively, and choose

$$u = G(x, y, \xi, \eta, \alpha, \beta), \qquad v = G(x, y, \xi', \eta', \alpha, \beta).$$

Since u and v are, within Ω', solutions of LAPLACE's equation the double integral over Ω' is zero. Furthermore, since u and v are doubly periodic in x, y, each assumes equal values at opposite points of the bounding lines of the rectangle Ω, while at these points the normal derivative of each assumes values numerically equal but opposite in sign. Therefore the line integral over the sides of the rectangle Ω is zero, and we have, replacing ds by $rd\theta$,

$$\int_{c(\xi,\eta)} \frac{1}{r} G(x, y, \xi', \eta', \alpha, \beta) rd\theta - \int_{c(\xi',\eta')} \frac{1}{r} G(x, y, \xi, \eta, \alpha, \beta) rd\theta$$
$$+ \int_{c(\alpha,\beta)} \frac{1}{r} \{ G(x, y, \xi, \eta, \alpha, \beta) - G(x, y, \xi', \eta', \alpha, \beta \} rd\theta + h = 0,$$

where

$$\lim_{r=0} h = 0.$$

But

$$G(x, y, \xi, \eta, \alpha, \beta) - G(x, y, \xi', \eta', \alpha, \beta) =$$
$$\log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} - \log \frac{1}{\sqrt{(x-\xi')^2 + (y-\eta')^2}}$$
$$+ R(x, y, \xi, \eta, \alpha, \beta) - R(x, y, \xi', \eta', \alpha, \beta),$$

and

$$R(\alpha, \beta, \xi, \eta, \alpha, \beta) = 0, \qquad R(\alpha, \beta, \xi', \eta', \alpha, \beta) = 0.$$

We obtain therefore in the limit $r = 0$, by well known methods,

$$G(\xi, \eta, \xi', \eta', \alpha, \beta) - G(\xi', \eta', \xi, \eta, \alpha, \beta)$$
$$+ \log \frac{1}{\sqrt{(\alpha-\xi)^2 + (\beta-\eta)^2}} - \log \frac{1}{\sqrt{(\alpha-\xi')^2 + (\beta-\eta')^2}} = 0,$$

or writing x, y for ξ', η',

$$G(x, y, \xi, \eta, \alpha, \beta) + \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}}$$
$$= G(\xi, \eta, x, y, \alpha, \beta) + \log \frac{1}{\sqrt{(\xi-\alpha)^2 + (\eta-\beta)^2}}.$$

which is the law of reciprocity for G. If we replace G in this equation by its expression from 2° we have immediately,

$$R(x, y, \xi, \eta, \alpha, \beta) = R(\xi, \eta, x, y, \alpha, \beta).$$

Thus R is symmetrical with respect to x, y and ξ, η and is therefore a solution of

$$\frac{\partial^2 R}{\partial \xi^2} + \frac{\partial^2 R}{\partial \eta^2} = 0$$

within Ω.

§ 2. *The doubly periodic solutions of*

(1) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y).$$

Suppose a doubly periodic solution u of equation (1) exists, for the periods a, b, where f is continuous and doubly periodic with the same periods. If a second solution of the same nature existed, the difference of the two would be a doubly periodic solution of LAPLACE's equation, and therefore a constant. It follows that a doubly periodic solution of (1) for the periods a, b is uniquely determined if its value at a fixed point is given.

Apply GREEN's theorem to the period rectangle Ω, choosing for u a doubly periodic solution of (1) and taking $v = 1$. Since the integral over the boundary vanishes, the following equation results:

$$\int_0^b \int_0^a \Delta u \, dx \, dy = \int_0^b \int_0^a f(x, y) \, dx \, dy = 0.$$

This equation is a necessary condition for the existence of a doubly periodic solution of (1). We shall now show that it is also sufficient. Consider the function

$$\begin{aligned} u(\xi, \eta) &= -\frac{1}{2\pi} \int_0^b \int_0^a G(x, y, \xi, \eta, \alpha, \beta) f(x', y) \, dx \, dy \\ &= -\frac{1}{2\pi} \int_0^b \int_0^a \log \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} f(x, y) \, dx \, dy \\ &\quad + \frac{1}{2\pi} \int_0^b \int_0^a \log \frac{1}{\sqrt{(x-\alpha)^2 + (y-\beta)^2}} f(x, y) \, dx \, dy \\ &\quad - \frac{1}{2\pi} \int_0^b \int_0^a R(x, y, \xi, \eta, \alpha, \beta) \, dx \, dy. \end{aligned}$$

Since

$$\frac{\partial^2 R}{\partial \xi^2} + \frac{\partial^2 R}{\partial \eta^2} = 0,$$

we see at once, from the potential theory, that $u(\xi, \eta)$ is a solution of

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} = f(\xi, \eta).$$

Furthermore, putting $\xi = \alpha$, $\eta = \beta$ we have

$$u(\alpha, \beta) = 0,$$

since

$$R(x, y, \alpha, \beta, \alpha, \beta) = R(\alpha, \beta, x, y, \alpha, \beta) = 0.$$

From the law of reciprocity for G we have

$$G(x, y, \xi + ma, \eta + nb) - G(x, y, \xi, \eta, \alpha, \beta)$$

$$= \log \frac{1}{\sqrt{(\xi + ma - \alpha)^2 + (\eta + nb - \beta)^2}} - \log \frac{1}{\sqrt{(\xi - \alpha)^2 + (\eta - \beta)^2}},$$

and therefore

$$u(\xi + ma, \eta + nb) - u(\xi, \eta)$$

$$= -\frac{1}{2\pi} \log \sqrt{\frac{(\xi - \alpha)^2 + (\eta - \beta)^2}{(\xi + ma - \alpha)^2 + (\eta + nb - \beta)^2}} \int_0^b \int_0^a f(x, y)\, dx dy.$$

Therefore, if

$$\int_0^b \int_0^a f(x, y)\, dx dy = 0,$$

the function u possesses the periods a, b. We have therefore proved the theorems:

The necessary and sufficient condition for the existence of a doubly periodic solution (periods a, b) of the equation

$$(1) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y),$$

where f is a continuous doubly periodic function with the periods a, b, is that f satisfy the equation

$$\int_0^b \int_0^a f(x, y)\, dx dy = 0.$$

If this condition is satisfied, then the doubly periodic solution of (1), *with periods a, b, which assumes the value C at $x = \alpha$, $y = \beta$ is uniquely determined, and is given by the formula:*

$$u(\xi, \eta) = -\frac{1}{2\pi} \int_0^b \int_0^a G(x, y, \xi, \eta, \alpha, \beta) f(x, y)\, dx dy + C,$$

where G is a known function, expressible in terms of Sigma functions.

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