ON DIFFERENTIAL INVARIANTS*

BY

JOSEPH EDMUND WRIGHT

Introduction.

In the consideration of differential equations there arise expressions, such as for instance, the JACOBI-POISSON alternant of two first order partial equations, which are in their nature invariantive to all contact transformations. An important problem immediately presents itself, namely, the obtaining of all invariants of this type, that is of all invariants, with respect to contact transformations, of differential expressions.

In this paper are obtained all such invariants of a restricted type.

The restrictions are the following:—

(1) The only expressions considered are: (*a*) expressions of the first order with m dependent and n independent variables; (*b*) expressions of the second order with one dependent variable.

(2) The invariants are only of the first order, that is to say, they involve only first derivatives of the differential expressions.

The variables assumed to occur in case (*a*) are

$$x_1, \cdots, x_n \text{ (the independent variables)},$$

$$z_1, \cdots, z_m \text{ (the dependent variables)},$$

$$p_k^i \equiv \frac{\partial z_i}{\partial x_k} \qquad (i=1, \cdots, m;\ k=1, \cdots, n),$$

$$p_{kl}^i \equiv \frac{\partial^2 z_i}{\partial x_k \partial x_l} \qquad (i=1, \cdots, m;\ k, l=1, \cdots, n),$$

and

$$\frac{\partial f_\lambda}{\partial x_k}, \quad \frac{\partial f_\lambda}{\partial z_i}, \quad \frac{\partial f_\lambda}{\partial p_k^i} \qquad (\lambda=1, \cdots, r;\ i=1, \cdots, m;\ k=1, \cdots, n),$$

where $f_\lambda(x, z, p_k^i)$, $(\lambda = 1, \cdots, r)$, are the differential expressions considered.

In addition the variables dx_k, dz_i, dp_k^i will be assumed to enter, subject to the conditions

$$dz_i = \sum_{k=1}^{n} p_k^i dx_k \qquad (i = 1 \cdots m),$$

$$dp_k^i = \sum_{l=1}^{n} p_{kl}^i dx_l \qquad (i=1, \cdots, m;\ k=1, \cdots, n).$$

* Presented to the Society April 29, 1905. Received for publication November 12, 1904.

In invariants corresponding to case (*b*) the variables are the same, with the exception that m is 1, and that there are additional variables

$$\frac{\partial f_\lambda}{\partial p_{kl}} \qquad (\lambda = 1, \cdots, r;\, k,\, l = 1, \cdots, n),$$

where $f_\lambda(x, z, p_k, p_{kl})$ are the differential expressions considered.

The method used is a modification of that given by LIE in his paper "Ueber Differentialinvarianten,"* in which invariants for certain simpler types of infinitesimal transformations are obtained. In this paper LIE shows (p. 566) that by the method there outlined, a series of invariants may be obtained satisfying a system of linear differential equations which form a complete system.

In the paper mentioned LIE suggests a problem connected with invariants of surfaces, the class desired being that which does not change owing to "deformation" of the surface.

ŻORAWSKI † attempted the solution of this question and found a class of such invariants.

FORSYTH, ‡ in 1903 attacked the more general question of invariants due to a purely arbitrary point transformation performed on the surface, and also obtained in this manner the differential invariants of space. In his paper certain modifications are made on the LIE-ŻORAWSKI method, one modification being that he sought for relative, as well as absolute, invariants. The method as modified by FORSYTH will be used here.

The invariant sought will, therefore, be such that if F denote its expression in the original, F_1 in the transformed variables, we shall have

$$F_1 = \Omega F,$$

where Ω is a function depending only on the transformation.

Now if the transformation were a general one in the variables considered, it is well known that Ω would be some power of the Jacobian of the transformation. But the transformation is not perfectly general.

In fact, in our most general case, the Jacobian of X_k, Z_i, P^i_k, P^i_{kl} with reference to x_k, z_i, p^i_k, p^i_{kl}, where capitals denote transformed variables, and where $\{k, l = 1, 2, \cdots n;\, i = 1, 2 \cdots m\}$ breaks up into two factors, the first of which is

$$\bar{J}_1 \equiv J\left(\frac{X_1 X_2 \cdots X_n Z_1 \cdots Z_m P'_1 P'_2 \cdots P^m_n}{x_1 x_2 \cdots p^m_n}\right),$$

and the second

$$J_2 \equiv J\cdot\left(\frac{\cdots P^i_{kl} \cdots}{\cdots p^i_{kl} \cdots}\right).$$

* LIE, Mathematische Annalen, vol. 24 (1884), pp. 537–578.

† ŻORAWSKI, Acta Mathematica, vol. 16 (1892–3), pp. 1–64.

‡ FORSYTH, Philosophical Transactions, ser. A., vol. 201 (1903), pp. 329–402, and ser. A., vol. 202 (1904), pp. 277–333.

Further, if the transformation is a point transformation, $\bar{J}_1$ breaks up into two factors,

$$J_0 \equiv J\left(\frac{X_1 X_2 \cdots X_n Z_1 \cdots Z_m}{x_1 x_2 \cdots x_n z_1 \cdots z_m}\right),$$

$$J_1 \equiv J\left(\frac{P_1' P_2' \cdots P_n^m}{p_1' p_2' \cdots p_n^m}\right).$$

Now if the number of dependent variables is greater than one, it may easily be shown that the most general contact transformation possible is an extended point transformation.

The discussion will be limited to those cases in which the factor Ω is of the form

$$J_0^{\mu_0} J_1^{\mu_1} J_2^{\mu_2} \cdots$$

when the number of dependent variables is greater than one, and

$$\bar{J}_1^{\mu_1} J_2^{\mu_2} \cdots$$

when there is only one dependent variable.

Now let F be an invariant of the type considered, and let an infinitesimal contact transformation be performed on F.

The condition for invariance is that, t being the parameter of the transformation,

$$F_1 = \Omega F, \qquad \text{or} \qquad \frac{dF}{dt} = F\frac{d\Omega}{dt}$$

when $d\phi/dt$ denotes the increment of ϕ due to the infinitesimal transformation and F_1 is F in the tranformed variables. Expressing the fact that this equation holds for all such transformations as considered, we obtain a complete system of linear differential equations, the solutions of which are the invariants desired.

In the course of the work the values of certain increments are required, and they will be given now, before we consider the various cases in detail.

The following notation is used throughout:—

$$\frac{dx_k}{dt} \equiv \xi_k, \qquad \frac{dz_i}{dt} \equiv \zeta_i, \qquad \frac{dp_k^i}{dt} \equiv \pi_k^i, \qquad \frac{dp_{kl}^i}{dt} \equiv \pi_{kl}^i, \qquad \theta_i \equiv \zeta_i - \sum_{k=1}^{n} p_k^i \xi_k,$$

$$\frac{\partial f}{\partial x_k} \equiv X_k, \qquad \frac{\partial f}{\partial z_i} \equiv Z_i,$$

$$\frac{\partial f}{\partial p_k^i} \equiv P_k^i, \qquad \frac{\partial f}{\partial p_{kl}^i} \equiv P_{kl}^i,$$

where f is used to denote one of the forms considered. If it is desired to specify any one of the f's particularly, the notation

$$f_\lambda, \qquad X_{\lambda,k}, \qquad P_{\lambda,k}^i, \qquad P_{\lambda,kl}^i,$$

etc., is used.

S denotes a summation taken over all the expressions f.

In addition,

$$dx_k \equiv a_k, \qquad dz_i \equiv b_i, \qquad dp^i_k \equiv c^i_k, \qquad dp^i_{kl} \equiv c^i_{kl},$$

$$\frac{d}{dx_k} \equiv \frac{\partial}{\partial x_k} + \sum_{\lambda=1}^{m} p^\lambda_k \frac{\partial}{\partial z_\lambda} + \sum_{\lambda,\, l} p^\lambda_{kl} \frac{\partial}{\partial p^\lambda_l}.$$

Using this notation we have the following increments:

$$\frac{dJ_0}{dt} \equiv \sum_{i=1}^{m} \frac{\partial \theta_i}{\partial z_i} + \sum_{k=1}^{n} \frac{d\xi_k}{dx_k},$$

$$\frac{dJ_1}{dt} \equiv n \sum_{i=1}^{m} \frac{\partial \theta_i}{\partial z_i} - m \sum_{k=1}^{n} \frac{d\xi_k}{dx_k},$$

$$\frac{dJ_2}{dt} \equiv \tfrac{1}{2} n(n+1) \sum_{i=1}^{m} \frac{\partial \theta_i}{\partial z_i} - m(n+1) \sum_{k=1}^{n} \frac{d\xi_k}{dx_k}$$

when m is greater than unity, and

$$\frac{d\bar{J}_1}{dt} \equiv (n+1) \frac{\partial \theta}{\partial z},$$

$$\frac{dJ_2}{dt} \equiv (n+1) \sum_{\lambda=1}^{n} \frac{d}{dx_h} \left(\frac{\partial \theta}{\partial p_h} \right) + \tfrac{1}{2} n(n+1) \frac{\partial \theta}{\partial z}$$

when m is unity.

The quantities π^i_k, π^i_{hk}, etc., are readily obtained in terms of the ξ's and ζ's and their derivatives by the method given in LIE-ENGEL, *Theorie der Transformationsgruppen*, vol. 1, p. 544, *et seq.*

In the case, however, where m is unity the increment of the variables in an extended infinitesimal contact transformation may be expressed in a particularly simple manner.

The theorem is as follows:

Let there be an extended infinitesimal contact transformation in the n independent variables $x_1, x_2, \cdots, x_n$ and the dependent variable z, and let $p_{hkl}\ldots$ denote $\partial^r z / \partial x_h \partial x_k \partial x_l \cdots$, where there are r letters $h, k, l, \cdots$. Then the increment of $p_{hkl}\ldots$ due to the transformation is $(d^r \theta / dx_h dx_k dx_l \cdots) \delta t$, where θ, with the usual notation, is equal to

$$\zeta - \sum_{l=1}^{n} p_l \xi_l,$$

and $(d^r \theta / dx_h dx_k dx_l \cdots)$ denotes a total differentiation of θ in which the terms containing the highest derivatives of z are omitted.

For example, in the case when there are two independent variables,

$$\pi_1 = \frac{\partial \theta}{\partial x_1} + p_1 \frac{\partial \theta}{\partial z}, \qquad \pi_2 = \frac{\partial \theta}{\partial x_2} + p_2 \frac{\partial \theta}{\partial z},$$

$$\pi_{11} = \frac{\partial^2 \theta}{\partial x_1^2} + 2p_1 \frac{\partial^2 \theta}{\partial z\, \partial x_1} + p_1^2 \frac{\partial^2 \theta}{\partial z^2}$$

$$+ 2p_{11}\left(\frac{\partial^2 \theta}{\partial x_1 \partial p_1} + p_1 \frac{\partial^2 \theta}{\partial z \partial p_1}\right) + 2p_{12}\left(\frac{\partial^2 \theta}{\partial x_1 \partial p_2} + p_1 \frac{\partial^2 \theta}{\partial z \partial p_2}\right)$$

$$+ p_{11}^2 \frac{\partial^2 \theta}{\partial p_1^2} + 2p_{11}p_{12} \frac{\partial^2 \theta}{\partial p_1 \partial p_2} + p_{12}^2 \frac{\partial^2 \theta}{\partial p_2^2} + p_{11} \frac{\partial \theta}{\partial z}, \text{ etc.}$$

This theorem is known to be true * for the increments of the first derivatives of z, and it may be easily proved for higher derivatives by induction.

The increments of the quantities $X_{\lambda, k}$, $Z_{\lambda, i}$, $P^i_{\lambda, k}$, $P^i_{\lambda, hk}$, owing to the infinitesimal contact transformation are determined by the method given by FORSYTH in his paper already quoted.†

Using this method, we have the results

$$-\frac{dX_\lambda}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial x_\lambda} X_k + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial x_\lambda} Z_i + \sum_{ik} \frac{\partial \pi^i_k}{\partial x_\lambda} P^i_k + \sum_{ihk} \frac{\partial \pi^i_{hk}}{\partial x_\lambda} P^i_{hk},$$

$$-\frac{dZ_\lambda}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial z_\lambda} X_k + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial z_\lambda} Z_i + \sum_{ik} \frac{\partial \pi^i_k}{\partial z_\lambda} P^i_k + \sum_{ihk} \frac{\partial \pi^i_{hk}}{\partial z_\lambda} P^i_{hk},$$

$$-\frac{dP^\lambda_\mu}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial p^\lambda_\mu} X_k + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial p^\lambda_\mu} Z_i + \sum_{ik} \frac{\partial \pi^i_k}{\partial p^\lambda_\mu} P^i_k + \sum_{ihk} \frac{\partial \pi^i_{hk}}{\partial p^\lambda_\mu} P^i_{hk},$$

$$-\frac{dP^\lambda_{\mu\nu}}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial p^\lambda_{\mu\nu}} X_k + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial p^\lambda_{\mu\nu}} Z_i + \sum_{ik} \frac{\partial \pi^i_k}{\partial p^\lambda_{\mu\nu}} P^i_k + \sum_{ihk} \frac{\partial \pi^i_{hk}}{\partial p^\lambda_{\mu\nu}} P^i_{hk}.$$

The increments of the quantities a_k, b_i, c^i_k, etc., are readily calculated, for the transformation changes x_k into $x_k + \xi_k \delta t$, and therefore dx becomes $dx_k + d\xi_k \delta t$.

Hence

$$\frac{da_k}{dt} = \sum_{l=1}^{n} \frac{\partial \xi_k}{\partial x_l} a_l + \sum_{\lambda=1}^{m} \frac{\partial \xi_k}{\partial z_\lambda} b_\lambda + \sum_{l,\lambda} \frac{\partial \xi_k}{\partial p^\lambda_l} c^\lambda_l .$$

Similarly

$$\frac{db_i}{dt} = \sum_{l=1}^{n} \frac{\partial \zeta_i}{\partial x_l} a_l + \sum_{\lambda=1}^{m} \frac{\partial \zeta_i}{\partial z_\lambda} b_\lambda + \sum_{l,\lambda} \frac{\partial \zeta_i}{\partial p^\lambda_l} c^\lambda_l ,$$

with similar expressions for the other increments of this type.

---

* See LIE-ENGEL, *Theorie der Transformationsgruppen*, vol. 2, p. 82, 252.

† FORSYTH, Philosophical Transactions, vol. 201, p. 337, 338.

§ 1.

We shall now proceed to the determination of invariants of expressions of the first order. The case in which there is only one dependent variable differs from the others in that the most general infinitesimal contact transformation is not an extended point transformation. This case will therefore be considered independently of the other.

Assuming F to be an invariant, and performing on it the infinitesimal transformation corresponding to a function

$$\theta \equiv \zeta - \sum_{k=1}^{n} p_k \xi_k,$$

we have the equation

$$\frac{dF}{dt} + \mu(n+1)\theta_z F = 0.$$

If the variables occurring in F are

$$x_1, \cdots, x_n, z, p_1, \cdots, p_n, X_{\lambda,1}, \cdots, X_{\lambda,n}, Z_\lambda, P_{\lambda,1}, \cdots, P_{\lambda,n} \quad (\lambda = 1 \cdots r),$$

this equation becomes on expansion,

$$\mu(n+1)\theta_z F + \left(\theta - \sum_{i=1}^{n} p_i \theta_{p_i}\right)\frac{\partial F}{\partial z} + \sum_{i=1}^{n}(\theta_{x_i} + p_i\theta_z)\frac{\partial F}{\partial p_i} - \sum_{i=1}^{n}\theta_{p_i}\frac{\partial F}{\partial_{x_i}}$$

$$- \mathbf{S}\sum_{i=1}^{n}\frac{\partial F}{\partial X_i}\left\{\sum_{\rho=1}^{n} X_\rho(-\theta_{p_\rho x_i}) + \left(\theta_{x_i} - \sum_{\rho=1}^{n} p_\rho \theta_{p_\rho x_i}\right)Z + \sum_{\rho=1}^{n}(\theta_{x_\rho x_i} + p_\rho\theta_{x_i z})P_\rho\right\}$$

$$- \mathbf{S}\frac{\partial F}{\partial Z}\left\{\sum_{\rho=1}^{n} X_\rho(-\theta_{p_\rho z}) + \left(\theta_z - \sum_{\rho=1}^{n} p_\rho\theta_{p_\rho z}\right)Z + \sum_{\rho=1}^{n}(\theta_{x_\rho z} + p_\rho\theta_{zz})P_\rho\right\}$$

$$- \mathbf{S}\sum_{i=1}^{n}\frac{\partial F}{\partial P_i}\left\{\sum_{\rho=1}^{n} X_\rho(-\theta_{p_\rho p_i}) + \sum_{\rho=1}^{n}(-p_\rho\theta_{p_\rho p_i})Z\right.$$

$$\left. + \sum_{\rho=1}^{n}(\theta_{p_i x_\rho} + p_\rho\theta_{z p_i})P_\rho + \theta_z P_i\right\} = 0.$$

Now θ is a perfectly arbitrary function of the variables $x_1 \ldots x_n$, z, $p_1 \ldots p_n$ and the above equation must be satisfied for all values of θ.

Hence we may equate to zero the coefficients of the derivatives of θ, and thus obtain the system of linear partial equations which F must satisfy.

The system is the following:

From θ

(1) $$F_z = 0.$$

From θ_{x_i},

(2) $$F_{p_i} - \mathbf{S}ZF_{X_i} = 0 \qquad (i = 1 \cdots n).$$

From θ_{p_i},

(3) $$F_{x_i} = 0 \qquad (i=1, \cdots, n)$$

From θ_z,

(4) $$\mu(n+1)F + \sum_{i=1}^{n} p_i F_{p_i} - \mathbf{S} Z F_Z - \mathbf{S}\sum_{i=1}^{n} P_i F_{P_i} = 0.$$

From $\theta_{x_i x_k}$,

(5) $$\mathbf{S} P_k F_{X_i} + \mathbf{S} P_i F_{X_k} = 0 \qquad (i=1, \cdots, n;\ k=1, \cdots, n).$$

From $\theta_{x_i z}$,

(6) $$\mathbf{S}\sum_{\rho=1}^{n} p_\rho P_\rho F_{X_i} + \mathbf{S} P_i F_Z = 0 \qquad (i=1, \cdots, n).$$

From $\theta_{x_i p_k}$,

(7) $$\mathbf{S}(X_k + p_k Z) F_{X_i} - \mathbf{S} P_i F_{P_k} = 0 \qquad (i=1, \cdots, n;\ k=1, \cdots, n).$$

From θ_{zz},

(8) $$\mathbf{S}\left(\sum_{\rho=1}^{n} p_\rho P_\rho\right) F_Z = 0.$$

From $\theta_{z p_i}$,

(9) $$\mathbf{S}(X_i + p_i Z) F_Z - \mathbf{S}\left(\sum_{\rho=1}^{n} p_\rho P_\rho\right) F_{P_i} = 0 \qquad (i=1, \cdots, n).$$

From $\theta_{p_i p_k}$,

(10) $$\mathbf{S}(X_k + p_k Z) F_{P_i} + \mathbf{S}(X_i + p_i Z) F_{P_k} = 0. \qquad (i, k=1, \cdots, n).$$

It follows from these equations that F must be a function of the variables

$$A_{1,1}, \cdots, A_{1,n}, A_{2,1}, \cdots, A_{2,n}, \cdots, Z_1, Z_2, \cdots, Z_r, P_{1,1}, \cdots, P_{r,n},$$

where

$$A_{i,k} \equiv X_{i,k} + p_k Z_i \qquad (i=1, \cdots, r;\ k=1, \cdots, n).$$

If we modify the system of equations by assuming F to be a function of these variables only, it becomes

(11) $$\mu(n+1)F = \mathbf{S} Z F_Z + \mathbf{S}\sum_{i=1}^{n} P_i F_{P_i},$$

(12) $$\mathbf{S} P_i F_{A_k} + \mathbf{S} P_k F_{A_i} = 0 \qquad (i, k=1, \cdots, n),$$

(13) $$\mathbf{S} P_i F_Z = 0 \qquad (i=1, \cdots, n),$$

(14) $$\mathbf{S} A_k F_{A_i} - \mathbf{S} P_i F_{P_k} = 0 \qquad (i, k=1, \cdots, n),$$

(15) $$\mathbf{S} A_i F_Z = 0 \qquad (i=1, \cdots, n),$$

(16) $$\mathbf{S} A_i F_{P_k} + \mathbf{S} A_k F_{P_i} = 0 \qquad (i, k=1, \cdots, n).$$

If we put aside for the present the first of these equations, the remaining equations form a complete system. The number of functionally independent solutions is, however, not immediately deducible, as some of the equations may depend algebraically on the others. In particular, equations (13) and (15) show that if the number r of expressions f considered is not greater than $2n$, and if all the determinants of order r of the matrix

$$\left\| \begin{matrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} & P_{1,1} & \cdots & P_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} & P_{2,1} & \cdots & P_{2,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ A_{r,1} & A_{r,2} & \cdots & A_{r,n} & P_{r,1} & \cdots & P_{r,n} \end{matrix} \right\|$$

do not vanish, F_Z must be zero.

Further, there is no necessity to consider more than $2n + 1$ expressions f, for if there were more than this number they could all be expressed by means of any $(2n + 1)$ of them.

Suppose that there is only one expression f, then F_{A_i}, F_Z, F_{P_i} $(i = 1, \cdots, n)$ are all zero, and therefore it follows that there is no invariant of the type sought of a single expression f.

Next let there be two expressions f. Then F_{Z_1}, F_{Z_2} are zero unless

$$\frac{A_{1,1}}{A_{2,1}} = \frac{A_{1,2}}{A_{2,2}} = \cdots = \frac{P_{1,i}}{P_{2,i}} = \cdots.$$

But these relations are equivalent to the conditions that any equation $\phi(x_i z p_i) = 0$ which lies in involution with $f_1 = a$, shall lie in involution with $f_2 = b$, where a and b are arbitrary constants.

Now corresponding to a particular value of a there are ∞^{2n-1} characteristic strips which go to build up the integrals of the equation $f_1 = a$.

Hence taking account of all values of a, all the surface elements (zx_ip_i) in space of $(n + 1)$ dimensions are arranged in ∞^{2n} characteristic strips.

The condition given above is easily seen to be the same as the condition that $f_2 = b$ determines the same system of ∞^{2n} characteristic strips.

Exactly similarly in the case of r expressions f, there are obtained ∞^{2n} $(2n - r + 1)$-fold manifolds which are common to the first $(r - 1)$ equations $f_1 = a_1, \cdots, f_{r-1} = a_{r-1}$, where $a_1, \cdots, a_{r-1}$ are arbitrary constants, and the conditions in virtue of which $F_{Z_1}, F_{Z_2}, \cdots, F_{Z_r}$ are not zero are the conditions that these manifolds should also satisfy $f_r = a_r$ for all values of the constant a_r.

We accordingly neglect this particular case, and then all the differential coefficients F_Z are zero, provided $r < 2n + 1$.

The system of equations which F satisfies is readily integrated, and it is seen that F must be a function of the POISSON alternants

$$[f_\lambda f_\mu] \qquad (\lambda, \mu = 1, 2 \cdots r),$$

and in addition must satisfy equation (11). This equation merely expresses the fact that F must be homogeneous of degree $\mu(n+1)$ in the quantities P, and F is therefore a homogeneous function of the quantities

$$[f_\lambda f_\mu] \qquad (\lambda, \mu = 1, 2 \cdots r).$$

There remains one case still to be considered, namely that in which we have $(2n+1)$ expressions f. In this case there exist invariants of the form $[f_\lambda f_\mu](\lambda, \mu = 1, 2, \cdots, 2n-1)$ and among these the only relations are those of the type $[f_\lambda f_\mu] + [f_\mu f_\lambda] = 0$.

We therefore have $n(2n+1)$ functionally independent solutions of our system of equations.

But returning to this system, we find that it consists of $2n^2 + 3n$ equations in addition to an equation which expresses a condition of homogeneity. There are $(2n+1)^2$ variables involved in the equations, and the equations are now algebraically independent. They therefore possess $(2n+1)^2 - (2n^2+3n)$ functionally independent solutions. Of this number, $(2n^2+n+1)$, $n(2n+1)$ are accounted for, and therefore one solution still remains to be discovered.

It is readily seen that this solution is

$$\begin{vmatrix} Z_1 & A_{1,1} & A_{1,2}, \cdots & P_{1,1} & P_{1,2}, \cdots & P_{1,n} \\ Z_2 & A_{2,1} & & \cdots & & P_{2,n} \\ \vdots & & & & & \\ Z_{2n+1} & A_{2n+1,1} & & & & P_{2n+1,n} \end{vmatrix} \equiv J.$$

If we substitute in (11) we see that, if F is J, μ is equal to unity.

Collecting results we see that *the only functionally independent relative invariants of our type, of r expressions f, are the alternants* $[f_\lambda f_\mu]$ *if r is less than* $2n+1$, *and if r is equal to* $(2n+1)$ *there is one additional invariant, the Jacobian of the forms with respect to the variables involved in them.*

It is well known that the alternants $[f_\lambda f_\mu]$ are all invariants of the forms f.

The theorem that these are the only invariants of the type sought, has been given by LIE,* who, however, merely suggests the method of proof. Further, LIE has apparently overlooked the additional invariant which arises in connection with $(2n+1)$ forms though he must have been perfectly familiar with the fact that the Jacobian is an invariant.

§ 2.

Let us now consider expressions involving one dependent variable, n independent variables, and the derivatives of the dependent variable with respect to the independent ones of the first and second orders.

The variables involved are now

$$z, x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n, \cdots, p_{ik} \qquad (i, k = 1, 2 \cdots n).$$

* LIE, Mathematische Annalen, vol. 24 (1884), p. 578, and Göttingen Nachrichten (1872), pp. 478–479.

The invariant sought will be a function of these, and of the first derivatives of the expression with respect to the variables involved in them.

Assume that the infinitesimal contact transformation is determined as before by a function θ, and that F is the invariant. Then the equation satisfied by F is

$$\frac{dF}{dt} + \left\{ \mu_0 (n+1) \frac{\partial \theta}{\partial z} + \mu_1 \left[\frac{n(n+1)}{2} \frac{\partial \theta}{\partial z} + (n+1) \sum_{h=1}^{m} \frac{d}{dx_h} \left(\frac{\partial \theta}{\partial p_h} \right) \right] \right\} F = 0.$$

Expand this, and it becomes

$$F_z \frac{dz}{dt} + \sum_{r=1}^{n} F_{x_r} \frac{dx_r}{dt} + \sum_{r=1}^{n} F_{p_r} \frac{dp_r}{dt} + \sum_{\alpha\beta} F_{p_{\alpha\beta}} \frac{dp_{\alpha\beta}}{dt} + \mathbf{S} F_Z \frac{dZ}{dt} + \mathbf{S} \sum_{r=1}^{n} F_{X_r} \frac{dX_r}{dt}$$

$$+ \mathbf{S} \sum_{r=1}^{n} F_{P_r} \frac{dP_r}{dt} + \mathbf{S} \sum_{\alpha\beta} F_{P_{\alpha\beta}} \frac{dP_{\alpha\beta}}{dt} + \left[\lambda \theta_z + \mu \sum_{r=1}^{n} \frac{d}{dx_r} (\theta_{p_r}) \right] F = 0,$$

where

$$\lambda = (n+1) \left(\mu_0 + \frac{n}{2} \mu_1 \right),$$

$$\mu = (n+1) \mu_1.$$

As before, F is an invariant to all contact transformations, and therefore, if we substitute in the above equation the values of

$$\frac{dz}{dt}, \frac{dx_r}{dt}, \cdots, \frac{dZ}{dt}, \cdots$$

in terms of θ and its derivatives, and if we then equate to zero the coefficients of the various derivatives of θ, we obtain a system of linear differential equations which F must satisfy.

If we equate to zero the coefficients of

$$\theta, \theta_{x_1}, \cdots, \theta_{x_n}, \theta_{x_1 x_1}, \cdots, \theta_{x_n x_n}, \theta_{p_1}, \cdots, \theta_{p_n},$$

we obtain the following system of equations:

$$F_z = 0, \qquad F_{x_i} = 0 \qquad (i = 1, 2, \cdots, n),$$

$$F_{p_i} - \mathbf{S} Z F_{X_i} = 0 \qquad (i = 1, 2, \cdots, n),$$

$$F_{p_{ii}} - \mathbf{S} P_i F_{X_i} = 0 \qquad (i = 1, 2, \cdots, n),$$

$$F_{p_{ik}} - \mathbf{S} P_i F_{X_k} - \mathbf{S} P_k F_{X_i} = 0 \qquad (i, k = 1, 2, \cdots, n, i \neq k).$$

We therefore introduce the variables

$$A_i = X_i + p_i Z + \sum_{k=1}^{n} p_{ik} P_k \qquad (i = 1, 2, \cdots, n),$$

and then these equations show that F is a function of

$$A_1, \cdots, A_n, Z, P_1, \cdots, P_n, P_{11}, P_{12}, P_{nn},$$

only.

The increments of the A's due to the transformation are readily found, and we have

$$-\frac{dA_i}{dt}=-\sum_{k=1}^{n}A_k\left(\frac{d\theta_{p_k}}{dx_i}\right)+\sum_{jk}\left\{\frac{d}{dx_i}\left(\frac{d^2\theta}{dx_j dx_k}\right)\right\}P_{jk},$$

where $(d^2\theta/dx_\alpha dx_\beta)$ denotes as before the differential coefficient of θ with respect to x_α and x_β, in which $z, p_r, p_{\alpha\beta}$ are taken to be functions of the x's, the terms containing third derivatives of z being omitted, and d/dx_i is written for

$$\frac{\partial}{\partial x_i}+p_i\frac{\partial}{\partial z}+\sum_{k=1}^{n}p_{ik}\frac{\partial}{\partial p_k}.$$

It is easy to show that

$$\frac{d}{dx_i}\left(\frac{d^2\theta}{dx_\alpha dx_\beta}\right)=\frac{d}{dx_\alpha}\left(\frac{d^2\theta}{dx_i dx_\beta}\right)=\frac{d}{dx_\beta}\left(\frac{d^2\theta}{dx_i dx_\alpha}\right)=\left(\left(\frac{d^3\theta}{dx_i dx_\alpha dx_\beta}\right)\right),$$

where the last expression denotes $d^3\theta/dx_i dx_\alpha dx_\beta$ with the terms containing third and fourth derivatives of z omitted.

The increments expressed in the variables A, Z, P are therefore as follows:

$$-\frac{dA_i}{dt}=-\sum_{k=1}^{n}A_k\frac{d\theta_{p_k}}{dx_i}+\sum_{jk}\left(\left(\frac{d^3\theta}{dx_i dx_j dx_k}\right)\right)P_{jk},$$

$$-\frac{dZ}{dt}=-\sum_{k=1}^{n}A_k\frac{\partial^2\theta}{\partial p_k\partial z}+Z\theta_z+\sum_{k=1}^{n}P_k\frac{d\theta_z}{dx_k}+\sum_{jk}\frac{\partial}{\partial z}\left(\frac{d^2\theta}{dx_j dx_k}\right)P_{jk},$$

$$-\frac{dP_i}{dt}=-\sum_{k=1}^{n}A_k\frac{\partial^2\theta}{\partial p_i\partial p_k}+\sum_{k=1}^{n}P_k\frac{d\theta_{p_i}}{dx_k}+P_i\theta_z+\sum_{jk}\frac{\partial}{\partial p_i}\left(\frac{d^2\theta}{dx_j dx_k}\right)P_{jk},$$

$$-\frac{dP_{hk}}{dt}=\sum_{\alpha\beta}\frac{\partial}{\partial p_{hk}}\left(\frac{d^2\theta}{dx_\alpha dx_\beta}\right)P_{\alpha\beta}.$$

If F is taken as a function of $A_1, \cdots, A_n, Z, P_1, \cdots, P_n, P_{11}, P_{12}, \cdots, P_{nn}$, the equation

$$\frac{dF}{dt}+\left[\lambda\theta_z+\mu\sum_k\frac{d\theta_{p_k}}{dx_k}\right]F=0$$

becomes

$$\begin{aligned}&\mathbf{S}\sum_{i=1}^{n}F_{A_i}\left\{\sum_{k=1}^{n}A_k\frac{d\theta_{p_k}}{dx_i}-\sum_{jk}P_{jk}\left(\left(\frac{d^3\theta}{dx_i dx_j dx_k}\right)\right)\right\}\\&+\mathbf{S}F_Z\left\{\sum_{k=1}^{n}A_k\theta_{p_k z}-Z\theta_z-\sum_{k=1}^{n}\frac{d\theta_z}{dx_k}P_k-\sum_{jk}P_{jk}\frac{\partial}{\partial z}\left(\frac{d^2\theta}{dx_j dx_k}\right)\right\}\\&+\mathbf{S}\sum_{i=1}^{m}F_{P_i}\left\{\sum_{r=1}^{n}A_r\theta_{p_i p_r}-\sum_{r=1}^{n}P_r\frac{d\theta_{p_i}}{dx_r}-P_i\theta_z+\sum_{jk}P_{jk}\frac{\partial}{\partial p_i}\left(\frac{d^2\theta}{dx_j dx_k}\right)\right\}\\&-\mathbf{S}\sum_{hk}F_{P_{hk}}\left\{\sum_{\alpha\beta}P_{\alpha\beta}\frac{\partial}{\partial p_{hk}}\left(\frac{d^2\theta}{dx_\alpha dx_\beta}\right)\right\}+\left[\lambda\theta_z+\mu\sum_{k=1}^{n}\frac{d\theta_{p_k}}{dx_k}\right]F=0.\end{aligned}$$

Equating coefficients of the derivatives of θ to zero, we obtain the following system of equations:

(17) $$\mathbf{S}\left[P_{jk}F_{A_i}+P_{ki}F_{A_j}+P_{ij}F_{A_k}\right]=0 \quad (i\neq j,\ j\neq k,\ k\neq i),$$

(18) $$\mathbf{S}\left[P_{ij}F_{A_i}+P_{ii}F_{A_j}\right]=0,$$

(19) $$\mathbf{S}\left[P_{ij}F_{Z}\right]=0,$$

(20) $$\mathbf{S}\left[P_{ij}F_{P_k}\right]=0,$$

(21) $$\mathbf{S}\left[A_iF_{A_i}-P_iF_{P_i}-P_{ii}F_{P_{ii}}-\sum_{j=1}^{n}P_{ij}F_{P_{ij}}\right]+\mu F=0,$$

(22) $$\mathbf{S}\left[A_iF_{A_j}-P_jF_{P_i}-\sum_{k=1}^{n}P_{kj}F_{P_{ki}}-P_{jj}F_{P_{ji}}\right]=0 \quad (i\neq j),$$

(23) $$\mathbf{S}\left[P_iF_Z\right]=0,$$

(24) $$\mathbf{S}\left[A_iF_Z\right]=0,$$

(25) $$\mathbf{S}\left[A_jF_{P_i}+A_iF_{P_j}\right]=0,$$

(26) $$\mathbf{S}\left[ZF_Z+\sum_{i=1}^{n}P_iF_{P_i}+\sum_{\alpha\beta}P_{\alpha\beta}F_{P_{\alpha\beta}}\right]=\lambda F \quad (i, j, k=1, 2, \cdots, n).$$

There now arise two cases to be considered. In the first case $P_{ij}=0$ $(i, j=1, 2, \cdots, n)$, and our expressions are therefore of the first order. In this case the equations become

$$\mathbf{S}\left[A_iF_{A_i}-P_iF_{P_i}\right]+\mu F=0,$$

$$\mathbf{S}\left[A_iF_{A_j}-P_jF_{P_i}\right]=0,$$

$$\mathbf{S}\left[P_iF_Z\right]=0,\ \mathbf{S}\left[A_iF_Z\right]=0,$$

$$\mathbf{S}\left[A_jF_{P_i}+A_iF_{P_j}\right]=0,$$

$$\mathbf{S}\left[ZF_Z+\sum_{i=1}^{n}P_iF_{P_i}\right]=\lambda F \quad (i, j=1, 2, \ldots n).$$

These equations are almost identical with the set $(11)\ldots(16)$; they must of course possess the same integrals as that set, together with others arising from the facts that equations of the type (12) have not now to be satisfied, and that μ, which is an arbitrary constant, has the particular value zero in the first set of equations.

It is easy to see that the integrals still to be found are functions of the variables A alone, and therefore they satisfy the equations

$$\mathbf{S}A_iF_{A_i}+\mu F=0,$$

$$\mathbf{S}A_iF_{A_j}=0 \quad (i, j=1, 2, \cdots, n).$$

Assuming that there are r expressions of the first order whose invariants we are seeking, the solutions of the above equations are easily seen to be the n-row determinants of the matrix,

$$\begin{Vmatrix} A_{1,1} & A_{2,1} & \cdots & A_{r,1} \\ A_{1,2} & & \cdots & \vdots \\ \vdots & & & \\ A_{1,n} & & \cdots & A_{r,n} \end{Vmatrix},$$

provided that r is greater than n.

There are no additional solutions if r is less than n.

These solutions are the Jacobians of sets of n of the forms.

We shall now consider the case in which the quantities P_{ij} are not all zero.

Before discussing the general case, we shall consider the case in which there are only two independent variables.

We shall take in order the cases in which there are one, two, three expressions of the second order whose invariants we are seeking.

In the case of one such expression,

$$F_{A_1} = F_{A_2} = 0 = F_Z = F_{P_1} = F_{P_2},$$

$$2P_{11}F_{P_{11}} + P_{12}F_{P_{12}} = \mu F,$$

$$P_{12}F_{P_{12}} + 2P_{22}F_{P_{22}} = \mu F,$$

$$2P_{11}F_{P_{12}} + P_{21}F_{P_{22}} = 0,$$

$$2P_{22}F_{P_{12}} + P_{21}F_{P_{11}} = 0.$$

These equations show that F must be a homogeneous function of the algebraic invariant of the binary form

$$(P_{11},\ \tfrac{1}{2}P_{12},\ P_{22}\between\ *\ )^2.$$

Hence $F \equiv \text{const.} \times (P_{12}^2 - 4P_{11}P_{22})^{\mu/2}$.

To interpret this invariant, suppose that

$$f(x_1\, x_2\, z\, p_1\, p_2\, p_{11}\, p_{12}\, p_{22}) = 0$$

is a differential equation of the second order.

Let $z = \phi(x_1 x_2)$ be some non-singular solution of this equation. We define two directions on this integral surface by means of the equation

$$P_{22}dx_1^2 - P_{12}dx_1dx_2 + P_{11}dx_2^2 = 0.$$

Along one of the curves thus determined on the particular integral surface, $x_1,\ x_2,\ z,\ p_1,\ p_2,\ p_{11},\ p_{12},\ p_{22}$ are functions of a single parameter while dz, dp_1, dp_2 are determined from the equations

$$dz = p_1 dx_1 + p_2 dx_2,$$

$$dp_1 = p_{11} dx_1 + p_{12} dx_2,$$

$$dp_2 = p_{12} dx_1 + p_{22} dx_2.$$

Also

$$\frac{df}{dx_1} + P_{11} \frac{dp_{11}}{dx_1} + P_{22} \frac{dp_{12}}{dx_2} = 0,$$

$$\frac{df}{dx_2} + P_{11} \frac{dp_{12}}{dx_1} + P_{22} \frac{dp_{22}}{dx_2} = 0.$$

The system of seven equations thus obtained are equivalent to six distinct relations, and they determine the "characteristics" of the given equation $f = 0$.

The fundamental property in connection with the curves obtained is that if two integral surfaces have contact of the first order along a characteristic, and if they have contact of the second order at any one point of this curve, they have contact of the second order all along the curve.*

We notice that the directions of the curves are given at every point by means of $z = \phi(x_1 x_2)$ and $P_{11} dx_2^2 - P_{12} dx_1 dx_2 + P_{22} dx_1^2 = 0$.

Now the transformation considered changes an integral surface into an integral surface, and also a characteristic upon an integral surface into a characteristic upon the transformed surface. We therefore expect $P_{11} dx_2^2 - \cdots = 0$ to be an invariant of the expression considered, and we also expect any function geometrically connected with it to be an invariant. $P_{12}^2 - 4P_{11} P_{22}$ was therefore *a priori* to be expected as an invariant.

We next consider the case of two expressions f_1 and f_2.

It is easy to see that the only invariants are the algebraic invariants of the two binary forms

$$(P_{1,11} \tfrac{1}{2} P_{1,12} P_{1,22} \between *)^2,$$

$$(P_{2,11} \tfrac{1}{2} P_{2,12} P_{2,22} \between *)^2,$$

unless the condition

$$I \equiv \begin{vmatrix} P_{1,12} & P_{2,12} \\ P_{1,11} & P_{2,11} \end{vmatrix} \times \begin{vmatrix} P_{1,22} & P_{2,22} \\ P_{1,12} & P_{2,12} \end{vmatrix} - \begin{vmatrix} P_{1,11} & P_{2,11} \\ P_{1,22} & P_{2,22} \end{vmatrix}^2 = 0$$

holds.

These invariants have an immediate interpretation from the theory of characteristics. I is itself one of the invariants above mentioned, and $I = 0$ is the condition that the two quadratic forms above mentioned, when equated to zero, have a common root. Therefore, unless the characteristics of $f_1 = 0$ have one direction common with those of $f_2 = 0$ at every point on a common integral surface, the two expressions have only three functionally independent invariants of

* GOURSAT, *Equations aux dérivées partielles du second ordre*, vol. 1, pp. 170 seq.

our type, namely those of two quadratic forms. Suppose now that I is zero. Let m be used to denote the common root mentioned above. Then

$$m = -\frac{P_{1,11}P_{2,22} - P_{2,11}P_{1,22}}{P_{1,12}P_{2,11} - P_{2,12}P_{1,11}} = -\frac{P_{1,22}P_{2,12} - P_{2,22}P_{1,12}}{P_{1,11}P_{2,22} - P_{2,11}P_{1,22}}.$$

Let H denote $A_{1,1}P_{2,11} - A_{2,1}P_{1,11}$ and let K denote $A_{1,2}P_{2,22} - A_{2,2}P_{1,22}$. Then it easily follows that $K + mH$ satisfies our system of equations provided it is zero, and further, this is the only additional solution the system can have.

We may verify that

$$\frac{d}{dt}(K + mH) = \left\{3\theta_z - 3\frac{d\theta_{p_1}}{dx_1} + m\frac{d\theta_{p_1}}{dx_2} + \frac{1}{m}\frac{d\theta_{p_2}}{dx_1} - \frac{d\theta_{p_2}}{dx_1}\right\} \times (K + mH).$$

Hence $K + mH$ is not an invariant of our type, although $K + mH = 0$ is an invariant relation.

These two equations

$$I = 0,$$

$$K + mH = 0$$

have an important signification in the theory of differential equations. They are* the conditions that the two equations $f_1 = 0, f_2 = 0$, form a system in involution, in other words, they are the conditions that the two equations have a system of common integrals depending on an infinite number of arbitrary constants.

We shall now consider invariants of three expressions f_1, f_2, f_3.

From the system of equations it follows that $F_z = 0$ unless all the 3-row determinants of the matrix

$$\left\| \begin{matrix} A_{1,1} & A_{1,2} & P_{1,1} & P_{1,2} & P_{1,11} & P_{1,12} & P_{1,22} \\ A_{2,1} & A_{2,2} & P_{2,1} & P_{2,2} & P_{2,11} & P_{2,12} & P_{2,22} \\ A_{3,1} & A_{3,2} & P_{3,1} & P_{3,2} & P_{3,11} & P_{3,12} & P_{3,22} \end{matrix} \right\|$$

are zero.

Also F_{p_λ} is zero unless all the 3-row determinants of the matrix

$$\left\| \begin{matrix} A_{1,1} & A_{1,2} & P_{1,11} & P_{1,12} & P_{1,22} \\ A_{2,1} & A_{2,2} & P_{2,11} & P_{2,12} & P_{2,22} \\ A_{3,1} & A_{3,2} & P_{3,11} & P_{3,12} & P_{3,22} \end{matrix} \right\|$$

are zero.

Assuming that the conditions mentioned are not satisfied we see that

$$F_z = 0, \qquad F_{p_1} = 0, \qquad F_{p_2} = 0.$$

---

* GOURSAT, *Équations aux dérivées partielles du second ordre*, vol. 2, p. 40 and p. 76.

The functionally independent solutions of our equations are then seen to be the algebraic invariants of the binary forms $(K, H \between *)$:

$$(P_{1,11},\ \tfrac{1}{2} P_{1,12},\ P_{1,22} \between *)^2,$$
$$(P_{2,11},\ \tfrac{1}{2} P_{2,12},\ P_{2,22} \between *)^2,$$
$$(P_{3,11},\ \tfrac{1}{2} P_{3,12},\ P_{3,22} \between *)^2,$$

where

$$H \equiv \begin{vmatrix} P_{1,11} & P_{2,11} & P_{3,11} \\ A_{1,1} & A_{2,1} & A_{3,1} \\ P_{1,22} & P_{2,22} & P_{3,22} \end{vmatrix} - \begin{vmatrix} A_{1,2} & \cdots \\ P_{1,12} & \cdots \\ P_{1,22} & \cdots \end{vmatrix},$$

$$K \equiv \begin{vmatrix} P_{1,11} & \cdots \\ A_{1,2} & \cdots \\ P_{1,22} & \cdots \end{vmatrix} - \begin{vmatrix} P_{1,11} & \cdots \\ P_{1,12} & \cdots \\ A_{1,1} & \cdots \end{vmatrix}.$$

It is important to notice the meaning of the equations $H = 0$, $K = 0$. They are in fact the conditions that the three equations $f_1 = 0$, $f_2 = 0$, $f_3 = 0$ have a common integral surface.*

The three quadratic binary forms and their invariants have as before, immediate interpretation from the theory of characteristics, but the linear form $K dx_2 - H dx_1$ has no such immediate interpretation.

As another example of invariants of this type we shall now consider the case in which there are two expressions, one of the second order and the other of the first.

The equations are readily solved, and the solutions are the algebraic invariants of the two forms

$$(P_{1,11},\ \tfrac{1}{2} P_{1,12},\ P_{1,22} \between *)^2,$$
$$(A_{2,2},\ - A_{2,1} \between *),$$

where $P_{2,11}$, $P_{2,12}$, $P_{2,22}$ are all zero. We know that

$$P_{1,11} dx_2^2 - P_{1,12} dx_1 dx_2 + P_{1,22} dx_1^2 = 0$$

is the equation for the directions of the characteristics of $f_1 = 0$. It seems, therefore, important to consider the meaning of

$$A_{2,2} dx_2 - (- A_{2,1}) dx_1,$$

or

$$A_{2,1} dx_1 + A_{2,2} dx_2.$$

But this is equal to df_2, provided

$$dz - p_1 dx_1 - p_2 dx_2 = 0, \quad dp_1 - p_{11} dx_1 - p_{12} dx_2 = 0, \quad dp_2 - p_{12} dx_1 - p_{22} dx_2 = 0.$$

* VALYI, Crelle's Journal, vol. 95 (1883) p. 100; GOURSAT, loc. cit., vol. 2, p. 199.

We thus have an interpretation of both invariants.

We now return to the general case, when there are n independent variables. The equations (17)—(26) possess all the invariants of our type as solutions.

Suppose that there are r expressions f. Then from (19), (23), (24), $F_Z = 0$ unless r is greater than $\frac{1}{2}n(n+1)+2n$, or unless all the r-row determinants of the matrix

$$\left\| \begin{matrix} P_{1,11} & P_{1,12} & \cdots & P_{1,nn} & P_{1,1} & \cdots & P_{1,n} & A_{1,1} & \cdots & A_{1,n} \\ \vdots & \vdots & & & & & & & & \\ P_{r,11} & P_{r,12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & A_{r,n} \end{matrix} \right\|$$

vanish.

Also the equations (17) (18) show that unless the determinants of another matrix vanish, $F_{A_i} = 0$ for all values of i.

Suppose that $F_Z, F_{A_1}, \ldots, F_{A_n}$ are all zero. There remains the system of equations for F

$$\mathbf{S}(P_{ij} F_{P_i}) = 0,$$

$$\mathbf{S}\left[P_i F_{P_i} + P_{ii} F_{P_{ii}} + \sum_{i=1}^{n} P_{ij} F_{P_{ij}}\right] - \mu F = 0,$$

$$\mathbf{S}\left[P_j F_{P_i} + \sum_{k=1}^{n} P_{kj} F_{P_{ki}} + P_{jj} F_{P_{ji}}\right] = 0,$$

$$\mathbf{S}\left[A_j F_{P_i} + A_i F_{P_j}\right] = 0,$$

$$\mathbf{S}\left[\sum_{i=1}^{n} P_i F_{P_i} + \sum_{\alpha\beta} P_{\alpha\beta} F_{P_{\alpha\beta}}\right] = \lambda F \qquad (i, j, k = 1, \cdots, n),$$

and the fourth of these equations shows that F_{P_i} is zero for all values of i. Hence the system becomes

$$\mathbf{S}\left[P_{ii} F_{P_{ii}} + \sum_{j=1}^{n} P_{ij} F_{P_{ij}}\right] - \mu F = 0,$$

$$\mathbf{S}\left[\sum_{k=1}^{n} P_{kj} F_{P_{ki}} + P_{jj} F_{P_{ji}}\right] = 0,$$

$$\mathbf{S}\left[\sum_{\alpha\beta} P_{\alpha\beta} F_{P_{\alpha\beta}}\right] = \lambda F \qquad (i, j, k = 1, \cdots, n).$$

Hence $\lambda = n\mu/2$, and the invariants required are the algebraic invariants of the system of quadratic forms

$$\sum_{\alpha\beta} P_{\alpha\beta} x_\alpha x_\beta,$$

where none of the magnitudes $P_{\alpha\beta}$ is repeated and $P_{\alpha\beta} = P_{\beta\alpha}$.

These quadratic forms are easily seen to be those which FORSYTH * calls

---

* FORSYTH, Philosophical Transactions, ser. A, vol. 191 (1898), p. 2.

"characteristic invariants" when there are only three independent variables. This name might with advantage be extended to the general case in which there are n independent variables.

We now return to the case in which $F_{A_1} F_{A_2}$, etc., are not all zero.

Suppose that there are three independent variables, and two expressions f. Then if F_{A_1}, etc., are not zero, all the 6-row determinants of the matrix

$$\left\| \begin{matrix} P_{1,11} & 0 & 0 & P_{1,12} & P_{1,22} & 0 & 0 & P_{1,31} & P_{1,33} & P_{1,23} \\ P_{2,11} & 0 & 0 & P_{2,12} & P_{2,22} & 0 & 0 & P_{2,31} & P_{2,33} & P_{2,23} \\ 0 & P_{1,22} & 0 & P_{1,11} & P_{1,12} & P_{1,23} & P_{1,33} & 0 & 0 & P_{1,31} \\ 0 & P_{2,22} & 0 & P_{2,11} & P_{2,12} & P_{2,23} & P_{2,33} & 0 & 0 & P_{2,31} \\ 0 & 0 & P_{1,33} & 0 & 0 & P_{1,22} & P_{1,23} & P_{1,11} & P_{1,31} & P_{1,12} \\ 0 & 0 & P_{2,33} & 0 & 0 & P_{2,22} & P_{2,23} & P_{2,11} & P_{2,31} & P_{2,12} \end{matrix} \right\|$$

must vanish.

Let S_i denote the characteristic invariant of f_i, then it is easily seen that if we construct the cubic forms

$$S_1 x_1, \ S_1 x_2, \ S_1 x_3, \ S_2 x_1, \ S_2 x_2, \ S_2 x_3,$$

the above matrix is the matrix of the coefficients.

Hence, if our conditions hold, the above six cubics must belong to a five fold linear system. Expressing this condition we see that $S_1 L_1 + S_2 L_2 \equiv 0$, where L_1 and L_2 are certain linear forms.

Hence either S_1 and S_2 both break up into linear factors, or S_2 is equivalent to S_1.

In the case in which there are three expressions f, the conditions give

$$S_1 L_1 + S_2 L_2 + S_3 L_3 \equiv 0,$$

where L_1, L_2, L_3 are linear. Hence, in general S_1, S_2, S_3, regarded as conics, have two common points.

The generalization is immediate, and the condition in order that invariants involving the magnitudes A, of r expressions f in n independent variables exist, are equivalent to the conditions that r linear forms $L_1, \cdots, L_r$ should exist such that

$$\sum_{i=1}^{r} S_i L_i \equiv 0$$

identically, when S_i is the characteristic invariant of f_i. It is readily seen that these conditions may be expressed by the vanishing of certain algebraic invariants of the r quadratic forms S.

An upper limit to the number of these conditions may readily be obtained.

This upper limit is $n(n+1)(n+2)/3! - nr + 1$. The number may fall below this in certain cases, for example, if $n = 3$ and $r = 2$ it is 4, whilst

$$\frac{n(n+1)(n+2)}{3!} - nr + 1 = 5.$$

If all the linear expressions L are equivalent, the conditions require that constants λ can be found such that $\sum \lambda S = 0$.

The number of conditions for this is readily seen to be

$$\frac{n(n+1)}{2} - r + 1.$$

This number is less than the previous one if $[n(n+1)/6 - r](n-1) > 0$, and the second conditions are all independent.

Hence, if r is $< \frac{1}{6}n(n+1)$ the imposition of $n(n+1)/2 - r + 1$ conditions is sufficient to cause the remainder to be satisfied.

Further consideration of this question will be omitted from the present paper.

Suppose the conditions in question to be satisfied. We then obtain a solution of the set of equations in F_A which is a determinant linear in the magnitudes A. Call this determinant Δ, then in a manner strictly analogous to the case when $n = 2$, it may be shown that $\Delta = 0$ is an invariant relation, provided the previous set of conditions holds.

If there are solutions of the system of equations considered which involve the magnitudes A, the derivatives F_{P_i} are not necessarily zero. From the equations of type (20), we see that, if $F_{P_i} \neq 0$, the matrix

$$\left\| \begin{matrix} P_{1,11} & \cdots & P_{1,ij} & \cdots & P_{1,nn} \\ \vdots & & \vdots & & \\ P_{r,11} & \cdots & P_{r,ij} & \cdots & P_{r,nn} \end{matrix} \right\|$$

must have all its r-row determinants zero.

In addition, all the equations of type (25),

$$\mathbf{S}(A_j F_{P_i} + A_i F_{P_j}) = 0,$$

must be satisfied.

Now take any one of the determinants of the above matrix, replace one of its columns by the magnitudes $P_{1,i}, P_{2,i}, \cdots, P_{r,i}$. Call the determinant then formed Δ_i. Call the similar determinant with A_i instead of P_i, M_i. Then P_i only enters through Δ_i, and the equations (25) become

$$M_j F_{\Delta_i} + M_i F_{\Delta_j} = 0 \qquad (i, j = 1, \cdots, r).$$

Hence if an invariant contains P_i, we see from the case when $i = j$, that M_i must be zero. If $M_i = 0$ and $F_{\Delta_i} \neq 0$, we see that $M_j = 0$ $(j = 1, \cdots, r)$; and if $F_{\Delta_i} = 0$ and $M_i \neq 0$, we see that $F_{\Delta_j} = 0$ $(j = 1, 2, \cdots, r)$.

Hence either none of the magnitudes $P_i(i = 1, \cdots, r)$ occur in any invariant, or all the magnitudes $M_i(i = 1, \cdots, r)$ are zero.

It is easy to see that if all the given conditions are satisfied, then the magnitudes Δ_i satisfy the remaining equations, and therefore these magnitudes Δ_i are invariants.

There is no invariant involving Z unless all the conditions given in connection with the magnitude P_i are satisfied and, in addition,

$$\Delta_1 = \Delta_2 = \cdots = \Delta_r = 0.$$

If all these conditions are satisfied, there is an invariant involving Z given by replacing any column in any r-row determinant of the matrix

$$\left\| \begin{matrix} P_{1,11} & \cdots & P_{1,ij} & \cdots & P_{1,nn} \\ \vdots & & \vdots & & \vdots \\ P_{r,11} & \cdots & P_{r,ij} & \cdots & P_{r,nn} \end{matrix} \right\|$$

by $Z_1, Z_2, \cdots, Z_r$.

§ 3.

We have not as yet considered invariants which involve the magnitudes dx_i, dr, dp_i, $dp_{ij}(i, j = 1, \cdots, n)$.

It is clear that invariants of this type do exist. For example, it is easy to verify that

$$\sum_{\alpha, \beta} P_{\alpha\beta}\, dx_\alpha dx_\beta$$

is such an invariant.

The work is somewhat simplified if we take as variables

$$\begin{aligned} dx_i &\equiv a_i, \\ dz - \sum_i p_i dx_i &\equiv u, \\ dp_i - \sum_j p_{ij} dx_j &\equiv v_i, \\ dp_{ij} &\equiv c_{ij}. \end{aligned} \qquad (i, j = 1, \cdots, n).$$

The increments of these magnitudes are readily obtained. We have

$$-\frac{da_i}{dt} = \sum_k \frac{d\theta_{p_i}}{dx_k} a_k + \frac{\partial \theta_{p_i}}{\partial z} u + \sum_k \frac{\partial^2 \theta}{\partial p_i \partial p_k} v_k, \qquad \frac{du}{dt} = \theta_z u,$$

$$\frac{dv_i}{dt} = \frac{d\theta_z}{dx_i} u + \sum_k \frac{d\theta_{p_k}}{dx_i} v_k + \theta_z v_i,$$

$$\frac{dc_{ij}}{dt} = \sum_k \left(\left(\frac{d^3\theta}{dx_i dx_j dx_k} \right) \right) a_k + \left(\frac{d^2\theta_z}{dx_i dx_j} \right) u$$

$$+ \sum_k \left(\frac{d^2\theta_{p_k}}{dx_i dx_j} \right) v_k + \frac{d\theta_z}{dx_j} v_i + \frac{d\theta_z}{dx_i} v_j + \sum_{\alpha\beta} \frac{\partial}{\partial p_{\alpha\beta}} \left(\frac{d^2\theta}{dx_i dx_j} \right) c_{\alpha\beta}.$$

If F is an invariant, the equation which it satisfies is similar to the one given in the previous section, but it contains the additional terms

$$-\sum_i F_{a_i}\left\{\sum_k \frac{d\theta_{p_i}}{dx_k}a_k + \frac{\partial\theta_{p_i}}{\partial z}u + \sum_k \frac{\partial^2\theta}{\partial p_i \partial p_k}v_k\right\}$$

$$+ F_u\theta_z u + F_{v_i}\left\{\frac{d\theta_z}{dx_i}u + \theta_z v_i + \sum_k \frac{d\theta_{p_k}}{dx_i}v_k\right\}$$

$$+\sum_{ij}\left\{\sum_k\left(\left(\frac{d^3\theta}{dx_i dx_j dx_k}\right)\right)a_k + \left(\frac{d^2\theta_z}{dx_i dx_j}\right)u\right.$$

$$\left. + \sum_k\left(\frac{d^2\theta_{p_k}}{dx_i dx_j}\right)v_k + \frac{d\theta_z}{dx_j}v_i + \frac{d\theta_z}{dx_i}v_j + \sum_{\alpha\beta}\frac{\partial}{\partial p_{\alpha\beta}}\left(\frac{d^2\theta}{dx_i dx_j}\right)c_{\alpha\beta}\right\}F_{c_{ij}}.$$

The equations for F are now

$$(27)\qquad \mathbf{S}[P_{jk}F_{A_i} + P_{ki}F_{A_j} + P_{ij}F_{A_k}] - a_k F_{c_{ij}} - a_i F_{c_{jk}} - a_j F_{c_{ik}} = 0 \qquad (i \neq j,\ j \neq k,\ k \neq i),$$

$$(28)\qquad \mathbf{S}[P_{ij}F_{A_i} + P_{ii}F_{A_j}] - a_j F_{c_{ii}} - a_i F_{c_{ij}} = 0,$$

$$(29)\qquad \mathbf{S}P_{ij}F_Z - uF_{c_{ij}} = 0,$$

$$(30)\qquad \mathbf{S}P_{ij}F_{P_k} - v_k F_{c_{ij}} = 0,$$

$$(31)\qquad \mathbf{S}\left[A_i F_{A_i} - P_i F_{P_i} - P_{ii}F_{P_{ii}} - \sum_{j=1}^{n} P_{ij}F_{P_{ij}}\right] + \mu F - a_i F_{a_i} + v_i F_{v_i} + c_{ii}F_{c_{ii}} + \sum_{j=1}^{m} c_{ij}F_{c_{ij}} = 0,$$

$$(32)\qquad \mathbf{S}\left[A_i F_{A_j} - P_j F_{P_i} - \sum_{k=1}^{n} P_{kj}F_{P_{ki}} - P_{jj}F_{P_{ji}}\right] - a_j F_{a_i} + v_i F_{v_j} + c_{ij}F_{c_{jj}} + \sum_{k=1}^{m} c_{ki}F_{c_{kj}} = 0,$$

$$(33)\qquad \mathbf{S}[P_i F_Z] - uF_{v_i} = 0,$$

$$(34)\qquad \mathbf{S}A_i F_Z - uF_{a_i} = 0,$$

$$(35)\qquad \mathbf{S}[A_j F_{P_i} + A_i F_{P_j}] - v_j F_{a_i} - v_i F_{a_j} = 0,$$

$$(36)\qquad \mathbf{S}\left[ZF_Z + \sum_{i=1}^{n} P_i F_{P_i} + \sum_{\alpha\beta} P_{\alpha\beta}F_{P_{\alpha\beta}}\right] - \left(\sum_{\alpha\beta} c_{\alpha\beta}F_{c_{\alpha\beta}} + uF_u + \sum_{i=1}^{n} v_i F_{v_i}\right) = \lambda F \qquad (i, j, k, \alpha, \beta, = 1, \cdots, n).$$

We first consider the particular case when all the original expressions f are of the first order.

In this case $P_{ij} = 0\,(i, j = 1, \cdots, n)$, and it may be readily seen that F does not in general involve any of the magnitudes $c_{\alpha\beta}$.

We have therefore the reduced system of equations for F:

$$(37)\qquad \mathbf{S}\,[A_i F_{A_i} - P_i F_{P_i}] + uF - a_i F_{a_i} + v_i F_{v_i} = 0,$$

$$(38)\qquad \mathbf{S}\,[A_i F_{A_j} - P_j F_{P_i}] + v_i F_{v_j} - a_j F_{a_i} = 0,$$

$$(39)\qquad \mathbf{S} P_i F_Z - u F_{v_i} = 0,$$

$$(40)\qquad \mathbf{S} A_i F_Z - u F_{a_i} = 0,$$

$$(41)\qquad \mathbf{S}\,[A_j F_{P_i} + A_i F_{P_j}] - v_j F_{a_i} - v_i F_{a_j} = 0,$$

$$(42)\qquad \mathbf{S}\left[ZF_Z + \sum_{i=1}^{n} P_i F_{P_i}\right] - uF_u - \sum_{i=1}^{n} v_i F_{v_i} = \lambda F \qquad (i, j = 1, \cdots, n).$$

From (39) and (40) we deduce that F must be a function of u, W, P_i, A_i, where

$$W = uZ + \sum a_i A_i + \sum v_i P_i;$$

and the equations for F are now

$$(43)\qquad \mathbf{S}\,[A_i F_{A_i} - P_i F_{P_i}] + \mu F = 0,$$

$$(44)\qquad \mathbf{S}\,[A_i F_{A_j} - P_j F_{P_i}] = 0,$$

$$(45)\qquad \mathbf{S}\,[A_j F_{P_i} + A_i F_{P_j}] = 0,$$

$$(46)\qquad \mathbf{S}\left[\sum_{i=1}^{n} P_i F_{P_i}\right] + uF_u = \lambda F \qquad (i, j, = 1, \cdots, n).$$

Hence the quantities W are absolute invariants, and in addition there are the invariants given in the previous section. Also u is an invariant.

The additional invariants obtained are readily interpreted. The function

$$u \equiv dz - \sum p_i dx_i,$$

is an invariant arising in connection with the contact transformation itself, and W may easily be shown to be

$$df \equiv \frac{\partial f}{\partial z} dz + \sum_i \frac{\partial f}{\partial x_i} dx_i + \sum_i \frac{\partial f}{\partial p_i} dp_i.$$

Suppose that $u = 0$. Then unless the r-row determinants of the matrix

$$\left\| \begin{matrix} A_{1,1} & \cdots & A_{1,n} & P_{1,1} & \cdots & P_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{r,1} & & A_{r,n} & P_{r,n} & \cdots & P_{r,n} \end{matrix} \right\|$$

are all zero, and F_Z is zero.

If we write $v_i = A_{r+1,\,i}$ and $a_i = -P_{r+1,\,i}$, the set of equations becomes the same as that given previously in which there were $(r+1)$ expressions and in which the variables dz, dx_i, dp_i, did not occur, the last equation only being different. The additional solutions are therefore $[f_i, f_{r+1}]$ $(i=1, \cdots, r)$, which are df_1, df_2, $\cdots$, df_r subject to the condition that $u=0$.

We now consider expressions F of the second order. Suppose first that $u \neq 0$ and that $v_1, \cdots, v_n$ are not all zero.

Write L for

$$uZ + \sum_i a_i A_i + \sum_i v_i P_i + \sum_{\alpha\beta} c_{\alpha\beta} P_{\alpha\beta},$$

then F is readily shown to be a function of the variables u, L, v_i, P_{ij}, A_i $(i, j = 1, \cdots, n)$, which satisfies the system of equations

$$\mathbf{S}[P_{jk} F_{A_i} + P_{ki} F_{A_j} + P_{ij} F_{A_k}] = 0 \quad (j \neq k, k \neq i, i \neq j),$$

$$\mathbf{S}[P_{ij} F_{A_i} + P_{ii} F_{A_j}] = 0,$$

$$\mathbf{S}\left[A_i F_{A_i} - P_{ii} F_{P_{ii}} - \sum_{j=1}^n P_{ij} F_{P_{ij}}\right] + \mu F + v_i F_{v_i} = 0,$$

$$\mathbf{S}\left[A_i F_{A_j} - P_{jj} F_{P_{ji}} - \sum_{k=1}^n P_{kj} F_{P_{ki}}\right] + v_i F_{v_j} = 0,$$

$$\mathbf{S}\left(\sum_{\alpha\beta} P_{\alpha\beta} F_{P_{\alpha\beta}}\right) + uF_u + \sum_{i=1}^n v_i F_{v_i} = \lambda F.$$

The magnitudes L are therefore invariants. In addition u is an invariant, and the remaining invariants are solutions of the given system.

L is readily shown to be

$$df \equiv \sum_i \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial z} dz + \sum_i \frac{\partial f}{\partial p_i} dp_i + \sum_{\alpha\beta} \frac{\partial f}{\partial p_{\alpha\beta}} dp_{\alpha\beta},$$

and it therefore admits of an immediate interpretation. There are also the invariänts obtained in the preceding section which do not involve the magnitudes v_i $(i = 1, \cdots, n)$ and the remaining invariants are those solutions of the set of equations last given which involve these magnitudes.

We see that when certain conditions given in the preceding section hold, the quantities F_{A_i} $(i = 1, \cdots, n)$ are all zero.

There remains the system

$$\mathbf{S}\left(P_{ii} F_{P_{ii}} + \sum_{j=1}^n P_{ij} F_{P_{ij}}\right) - v_i F_{v_i} = \mu F,$$

$$\mathbf{S}\left(P_{jj} F_{P_{ji}} + \sum_{k=1}^n P_{kj} F_{P_{ki}}\right) - v_i F_{v_j} = 0,$$

$$\mathbf{S}\left(\sum_{\alpha\beta} P_{\alpha\beta} F_{P_{\alpha\beta}}\right) + \sum_{i=1}^n v_i F_{v_i} = \lambda F \quad (i, j, \alpha, \beta, = 1, \cdots, n).$$

The solution of these equations is a function of the algebraic invariants and covariants of the r quadratic forms

$$\sum_{ij} P_{ij} v_i v_j .$$

In addition, this function must be homogeneous in the variables v.

These forms

$$\sum_{ij} P_{ij} v_i v_j$$

are those which have been referred to earlier as the Characteristic Invariants of the expressions f.

Now suppose that u and the quantities v are all zero.

In this case F_Z and F_{p_i} $(i = 1, \cdots, n)$ are all zero unless all the r-row determinants of the matrix

$$\begin{Vmatrix} P_{1,11} & \cdots & P_{1,ij} & \cdots & P_{1,nn} \\ \vdots & & \vdots & & \vdots \\ P_{r,11} & \cdots & P_{r,ij} & \cdots & P_{r,nn} \end{Vmatrix}$$

vanish.

We assume that these conditions are not satisfied, and therefore F is a function of a_i, A_i, P_{ij}, c_{ij} $(i, j = 1, \cdots, n)$.

The equations of types (27) and (28) are satisfied by the magnitudes

$$L \equiv \sum_i A_i a_i + \sum_{jk} P_{jk} c_{jk} .$$

It may readily be shown that these are the only independent solutions of this set of equations unless it is possible to make

$$\sum_{i=1}^{r} S_i B_i + AK = 0$$

identically, where any n variables $y_1, y_2, \cdots, y_n$ are taken, and the magnitudes B_i are linear functions of these variables, K is a quadratic function of them, and

$$S_i \equiv \sum_{\alpha\beta} P_{i,\alpha\beta} y_\alpha y_\beta ,$$

$$A \equiv \sum_k a_k y_k .$$

This equation can be satisfied by making the linear functions all equal to A, multiplied by certain constant factors. This leads to solutions of our system of the type L.

If we assume that there is no other way of making the expression considered an identity, we see that the quantities L are the only invariants involving the variables $A_k (k = 1, \cdots, n)$.

It is readily seen that these expressions L are equivalent to the expressions df, subject to the conditions

$$u = 0, \qquad v_k = 0 \qquad (k = 1, 2, \cdots, n).$$

It is easy to show that the remaining solutions of the system are the algebraic invariants of the system of quadratic forms S_i and of the linear form A.

These forms S are again the characteristic invariants of the expressions f; they have however, in this case, the magnitudes y for independent variables.

Combining our results we have the following theorem:

All invariants, of the restricted type considered of r second order expressions f in one dependent and n independent variables are

(1) *Expressions of the type df.*

(2) *Algebraic invariants and covariants of the quadratic forms*

$$\sum_{ij} \frac{\partial f}{\partial p_{ij}} v_i v_j,$$

where the v's are the variables, and

$$v_k = dp_k - \sum_{i=1}^{n} p_{ki} dx_i,$$

provided

$$dz \neq \sum_{i=1}^{n} p_i dx_i,$$

and the v's are not zero.

(3) *Algebraic invariants of the quadratic forms*

$$\sum_{ij} \frac{\partial f}{\partial p_{ij}} y_i y_j,$$

and of the linear form

$$\sum_{i=1}^{n} y_i dx_i,$$

where the y's are the variables, provided that

$$dz = \sum_{i=1}^{n} p_i dx_i, \qquad dp_i = \sum_{k=1}^{n} p_{ik} dx_k.$$

In the above there are two sets of restrictions on the expressions f.

(1) The r-row determinants must not all be zero in the matrix

$$\left\| \begin{matrix} P_{1,11} & \cdots & P_{1,ij} & \cdots & P_{1,nn} \\ \vdots & & & & \\ P_{r,11} & \cdots & P_{r,ij} & \cdots & P_{r,nn} \end{matrix} \right\|$$

(2) It must not be possible to satisfy the identity

$$\sum_{i=1}^{r} S_i B_i + AK \equiv 0,$$

where S, B, A, K, are as previously defined, except by making the B's all constant multiples of A.

The case when the second of the above restrictions is removed requires further consideration. It is obvious that it must be removed if r is great enough, but the whole question will be left for future consideration. At present we content ourselves with a discussion of the case when n is two.

For one expression f, $SB + AK \equiv 0$, provided that S admits A as a factor, since we are neglecting the possibility of B being λA, where λ is a constant.

This gives $P_{11}a_2^2 - P_{12}a_1a_2 + P_{22}a_1^2 = 0$, as the condition for the existence of further integrals.

If this condition is satisfied, the equations of type (27) and (28) possess the two solutions

$$\alpha_1 = A_1 a_1 a_2 + P_{22}a_1c_{12} + P_{11}a_2c_{11},$$

$$\alpha_2 = A_2 a_1 a_2 + P_{11}a_2c_{12} + P_{22}a_1c_{11},$$

instead of the single one already given.

It may readily be shown that there are no new invariants, but if

$$I \equiv P_{11}a_2^2 - P_{12}a_1a_2 + P_{22}a_1^2 = 0,$$

then the two equations

$$\alpha_1 \equiv A_1 a_1 a_2 + P_{22}a_1c_{12} + P_{11}a_2c_{11} = 0,$$

$$\alpha_2 = A_2 a_1 a_2 + P_{11}a_2c_{12} + P_{22}a_1c_{22} = 0$$

are an invariant system.

We observe that I is an invariant.

The equations $I = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$, taken in conjunction with

$$dz = p_1dx_1 + p_2dx_2, \qquad dp_1 = p_{11}dx_1 + p_{12}dx_2, \qquad dp_2 = p_{12}dx_1 + p_{22}dx_2,$$

have an immediate and important interpretation in connection with the differential equation $f = 0$. They are precisely the equations for the characteristics of this differential equation.*

Now suppose that there are two expressions f. In this case there are four equations in seven variables of the types (27) and (28). They therefore possess in general three integrals. Two of these are already known, and are df_1, df_2

The remaining one may be expressed as the determinant

$$\begin{vmatrix} \lambda_{1,1}, & \lambda_{1,2}, & 0 \\ K_{1,1}, & K_{1,2}, & I_1 \\ K_{1,2}, & K_{2,2}, & I_2 \end{vmatrix} \equiv \Delta,$$

* GOURSAT, *Equations aux dérivées partielles du second ordre*, vol. 2, p. 174.

where

$$\lambda_1 = 2P_{11}a_2 - P_{12}a_1,$$
$$\lambda_2 = 2P_{22}a_1 - P_{12}a_2,$$
$$K_1 = a_1 a_2 A_2 + P_{11}a_2 c_{12} + P_{22}a_1 c_{22},$$
$$K_2 = a_1 a_2 A_1 + P_{22}a_1 c_{12} + P_{11}a_2 c_{11},$$
$$I = P_{22}a_1^2 - P_{12}a_1 a_2 + P_{11}a_2^2.$$

It may easily be shown that $\Delta/a_1 a_2$ satisfies the remaining system of equations, and therefore the complete system of integrals is df_1, df_2, $\Delta/a_1 a_2$, and the invariants of the binary forms

$$(P_{1,11},\ \tfrac{1}{2}P_{1,12},\ P_{1,22}\between *\ )^2, \qquad (P_{2,11},\ \tfrac{1}{2}P_{2,12},\ P_{2,22}\between *\ )^2, \qquad (a_1,\ a_2\between *\ ).$$

We omit for the present the interpretation of Δ, and proceed to consider the case in which there are three expressions f.

In this case the equations of types (27) and (28) have the five functionally independent integrals df_1, df_2, df_3 and

$$H \equiv \begin{vmatrix} P_{1,11} & A_{1,1} & P_{1,22} \\ P_{2,11} & A_{2,1} & P_{2,22} \\ P_{3,11} & A_{3,1} & P_{3,22} \end{vmatrix} - \begin{vmatrix} A_{1,2} & P_{1,12} & P_{1,22} \\ A_{2,2} & P_{2,12} & P_{2,22} \\ A_{3,2} & P_{3,12} & P_{3,22} \end{vmatrix},$$

$$K \equiv \begin{vmatrix} P_{1,11} & A_{1,2} & P_{1,22} \\ P_{2,11} & A_{2,2} & P_{2,22} \\ P_{3,11} & A_{3,2} & P_{3,22} \end{vmatrix} - \begin{vmatrix} P_{1,11} & P_{1,12} & A_{1,1} \\ P_{2,11} & P_{2,12} & A_{2,1} \\ P_{3,11} & P_{3,12} & A_{3,1} \end{vmatrix}.$$

We substitute these integrals in the remaining equations, and the complete system of independent integrals of the equations thus obtained is the system of invariants of the binary forms

$$(P_{1,11},\ \tfrac{1}{2}P_{1,12},\ P_{1,22}\between *\ )^2,$$
$$(P_{2,11},\ \tfrac{1}{2}P_{2,12},\ P_{2,22}\between *\ )^2,$$
$$(P_{3,11},\ \tfrac{1}{2}P_{3,12},\ P_{3,22}\between *\ )^2,$$
$$(K,\ H\between *\ ),$$
$$(a_1,\ a_2\between *\ ).$$

If we take the variables to be a_2, $-a_1$, we see that the solutions in question are the invariants and covariants of the binary forms

$$Ka_2 - Ha_1,$$
$$P_{1,11}a_2^2 - P_{1,12}a_1 a_2 + P_{1,22}a_1^2,$$
$$P_{2,11}a_2^2 - P_{2,12}a_1 a_2 + P_{2,22}a_1^2,$$
$$P_{3,11}a_2^2 - P_{3,12}a_1 a_2 + P_{3,22}a_1^2.$$

In addition to these there are the three solutions df_1, df_2, df_3, and these are all the functionally independent integrals of our type.

§ 4.

We shall consider to a small extent the more general type of expression f, that is to say the type in which there are more dependent variables than one.

Let there be m dependent variables, $z_1, \cdots, z_m$, and as before let there be n independent variables, $x_1, \cdots, x_n$. We use the same notation as on pp. 288, 289, and in addition $df/dx_\mu = A_\mu$.

In this case, the most general contact transformation possible may easily be shown to be a point transformation.

Let the expressions f_λ be of the first order, that is to say, let them be functions of the variables x_k, z_i, p_k^i $(i = 1, 2, \cdots, m;\ k = 1, 2, \cdots, n)$.

If F is any first order invariant we have

$$\frac{dF}{dt} = \left(\mu_0 \frac{d\Omega_0}{dt} + \mu_1 \frac{d\Omega_1}{dt}\right) F,$$

where

$$\frac{d\Omega_0}{dt} = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial x_k} + \sum_{i=1}^{m} \frac{\partial \zeta_i}{\partial z_i},$$

$$\frac{d\Omega_1}{dt} = \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{\partial \pi_k^i}{\partial p_k^i},$$

and μ_0, μ_1, have the meanings given on p. 288.

Expanding dF/dt and equating to zero the coefficients of ξ_k, ζ_i, $\partial\zeta_i/\partial x_\mu$, we see that the variables x, z, do not occur explicitly, and that the variables p, x, only occur through the variables A.

We assume F to be a function of the variables A, Z, P, and

$$a_k \equiv dx_k,\ v_i \equiv dz_i - \sum_{k=1}^{n} p_k^i dx_k \qquad (k = 1, \cdots, n,\ i = 1, \cdots, m).$$

The values of the various increments involved are the following:

$$\frac{d}{dt} a_k = \sum_{\lambda=1}^{m} \frac{\partial \xi_k}{\partial z_\lambda} v_\lambda + \sum_{\mu=1}^{n} \frac{d \xi_k}{dx_\mu} a_\mu,$$

$$\frac{d}{dt} v_i = \sum_{\lambda=1}^{m} \frac{\partial \theta_i}{\partial z_\lambda} v_\lambda,$$

$$-\frac{dA_k}{dt} = \sum_{r=1}^{n} \frac{d\xi_r}{dx_k} A_r + \sum_{i=1}^{m} \sum_{\mu=1}^{n} P_\mu^i \frac{d^2 \theta_i}{dx_\mu\, dx_k},$$

$$-\frac{dZ_\lambda}{dt} = \sum_{r=1}^{n} \frac{\partial \xi_r}{\partial z_\lambda} A_r + \sum_{s=1}^{m} \frac{\partial \theta_s}{\partial z_\lambda} Z_s + \sum_{i=1}^{m} \sum_{k=1}^{n} P_k^i \frac{d}{dx_k}\left(\frac{\partial \theta_i}{\partial z_\lambda}\right),$$

$$-\frac{dP^h_k}{dt} = \sum_{i=1}^{m} P^i_k \frac{\partial \theta_i}{\partial z_h} - \sum_{\sigma=1}^{n} P^h_\sigma \frac{d\xi_k}{dx_\sigma} \quad (h, \lambda = 1, 2, \cdots, m;\ k = 1, 2, \cdots, n).$$

Also

$$\mu_0 \frac{d\Omega_0}{dt} + \mu_1 \frac{d\Omega_1}{dt} = \mu \sum_{k=1}^{n} \frac{d\xi_k}{dx_k} + \nu \sum_{h=1}^{m} \frac{\partial \theta_h}{\partial z_h},$$

where μ and ν are constants depending on μ_0 and μ_1.

We substitute these values in the equation for F and equate the coefficients of the various derivatives of ξ and ζ to zero. We thus obtain the following system of equations for F:

$$(47) \qquad a_\sigma F_{a_k} - \mathbf{S} A_k F_{A_\sigma} + \mathbf{S} \sum_{i=1}^{m} P^i_\sigma F_{P^i_k} = 0 \qquad (\sigma \neq k),$$

$$(48) \qquad a_k F_{a_k} - \mathbf{S} A_k F_{A_k} + \mathbf{S} \sum_{i=1}^{m} P^i_k F_{P^i_k} = \mu F,$$

$$(49) \qquad v_\lambda F_{a_k} - \mathbf{S} A_k F_{Z_\lambda} = 0,$$

$$(50) \qquad v_\lambda F_{v_i} - \mathbf{S} Z_i F_{Z_\lambda} - \mathbf{S} \sum_{k=1}^{n} P^i_k F_{P^\lambda_k} = 0 \qquad (\lambda \neq i),$$

$$(51) \qquad v_i F_{v_i} - \mathbf{S} Z_i F_{Z_i} - \mathbf{S} \sum_{k=1}^{n} P^i_k F_{P^i_k} = \nu F,$$

$$(52) \qquad -\mathbf{S} P^i_\mu F_{A_k} - \mathbf{S} P^i_k F_{A_\mu} = 0,$$

$$(53) \qquad -\mathbf{S} P^i_k F_{Z_\lambda} = 0$$

$$(\sigma, k, \mu = 1, \cdots, n);\ (i, \lambda, = 1, \cdots, m).$$

The equations (49) give solutions of type

$$\Delta \equiv \sum_{k=1}^{n} a_k A_k + \sum_{i=1}^{m} v_i Z_i,$$

and the equations (53) then show that the Δ's cannot enter into F unless the number r of expressions f_λ is greater than mn.

From equations (52) we deduce that the variables A do not occur in F unless

$$r > \tfrac{1}{2} m(n+1).$$

In the case when m is unity, the equations are not all independent, and r need not satisfy this last condition.

Suppose that r is less than this number, then F is a function of the variables $v_1, \cdots v_m, P^1_1, \cdots P^m_n$, which satisfies the equations

$$\mathbf{S} \sum_{i=1}^{m} P^i_\sigma F_{P^i_k} = 0, \qquad (\sigma \neq k),$$

$$\mathbf{S}\sum_{i=1}^{m} P^i_k F_{P^i_k} = \mu F,$$

$$\mathbf{S}\sum_{k=1}^{n} P^i_k F_{P^\lambda_k} - v_\lambda F_{v_i} = 0,$$

$$\mathbf{S}\sum_{k=1}^{n} P^i_k F_{P^i_k} - v_i F_{v_i} = -\nu F.$$

The last two equations show that F must be an invariant or covariant of the linear forms

$$\sum_{i=1}^{m} P^i_{\lambda, k} v_i \qquad (k=1, 2, \cdots, n;\ \lambda=1, 2, \cdots, r).$$

The two first equations show that F must at the same time be an invariant of the linear forms

$$\sum_{k=1}^{n} P^i_{\lambda, k} y_i \qquad (i=1, 2, \cdots, m;\ \lambda=1, 2, \cdots, r).$$

BRYN MAWR COLLEGE.