ON HYPERCOMPLEX NUMBER SYSTEMS*

BY

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Generalization of the concept number system.

1. The usual theory relates to systems of numbers $\sum_{i=1}^{n} a_i e_i$ in which the coördinates a_i range independently over all real numbers or else over all ordinary complex numbers; for example, the real quaternion system, or the complex quaternion system. As an obvious generalization,† the coördinates may range independently over all the marks of any field F; for example, the rational quaternion system.

As a further generalization, the sets of coördinates $a_1, \cdots, a_n$ in the various numbers of a system may include only a part of the sets $b_1, \cdots, b_n$, each b_i ranging independently over F; for example, the integral quaternion system. The various coördinates $a_1, \cdots, a_n$ need not have the same range; for example, the numbers

$$(a + 2b\sqrt{2})e_1 + (c + 4d\sqrt{2})e_2 \quad (a, b, c, d \text{ arbitrary integers})$$

form a closed system under addition, subtraction, and multiplication, subject to the associative law, if we set

$$e_1^2 = 2\sqrt{2}e_1 - 2e_2, \qquad e_1e_2 = e_2e_1 = e_1, \qquad e_2^2 = e_2.$$

If we make these generalizations on the coördinates, but retain the usual conception of the units e_i, we obtain only subsystems of the usual number systems, the case of modular fields being an exception. It is otherwise if we generalize our conception of the units themselves, freeing them from the restriction ‡ of linear independence with respect to the set of all ordinary complex numbers, and assuming merely their linear independence with respect to the given field F.

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† DICKSON, *Definitions of a linear associative algebra by independent postulates*, Transactions, vol. 4 (1903), pp. 21–26.

‡ Retained implicitly by TABER, Transactions, vol. 5 (1904), p. 509.

By this generalization we may regard an algebraic field * as a number system; in particular, all ordinary complex numbers form a number system * with respect to the set of all real numbers, the units being 1, i.

The question of further generalization is considered in § 5.

Closed system of n-tuple elements with respect to a field F.

2. A set of n ordered marks $a_1, \cdots, a_n$ of F will be called an *n-tuple element* a. The symbol $a = (a_1, \cdots, a_n)$ employed is purely positional, without functional connotation. Its definition implies that $a = b$ if and only if $a_1 = b_1, \cdots, a_n = b_n$.

A system of n-tuple elements a in connection with n^3 fixed marks γ_{ijk} of F will be called a *closed system* if the following five postulates hold:

Postulate I. If a and b are any two elements of the system, then $s = (a_1 + b_1, \cdots, a_n + b_n)$ is an element of the system.

Definition. Addition of elements is defined by $a \oplus b = s$.

Postulate II. The element $0 = (0, \cdots, 0)$ occurs in the system.

Postulate III. If 0 occurs, then to any element a of the system corresponds an element a' of the system such that $a \oplus a' = 0$.

THEOREM. *The system is a commutative group under* $\oplus$.

Postulate IV. If a and b are any two elements of the system, then $p = (p_1, \cdots, p_n)$ is an element of the system, where

$$p_i = \sum_{j,k}^{1,\ldots,n} a_j b_k \gamma_{jki} \qquad (i = 1, \cdots, n).$$

Definition. Multiplication of elements is defined by $a \otimes b = p$.

Postulate V. The fixed marks γ satisfy the relations

$$\sum_{j=1}^{n} \gamma_{rtj}\gamma_{jki} = \sum_{j=1}^{n} \gamma_{tkj}\gamma_{rji} \qquad (r, t, k, i = 1, \cdots, n).$$

THEOREM. *Multiplication is associative and distributive.*

To make the system n *dimensional*, we add a sixth postulate:

Postulate VI. If $\tau_1, \cdots, \tau_n$ are marks of F such that $\tau_1 a_1 + \cdots + \tau_n a_n = 0$ for every element $(a_1, \cdots, a_n)$ of the system, then $\tau_1 = 0, \cdots, \tau_n = 0$.

THEOREM. *The system contains n elements* $\epsilon_i = (a_{i1}, \cdots, a_{in})$, $i = 1, \cdots, n$, *such that* $|a_{ij}| \neq 0$.

Indeed, by postulate VI there occurs an element $\neq 0$. Hence if the theorem is false, we may assume that the system contains ν elements $\epsilon_1, \cdots, \epsilon_\nu$, $1 \leqq \nu < n$, such that not every determinant of order ν in the matrix

* For these, PEIRCE's theorem that all but one of the units of the first group can be assumed nilpotent evidently fails. My conception of number systems is therefore wider than TABER's.

$$M_\nu \equiv \begin{bmatrix} a_{11} \cdots a_{1n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{\nu 1} \cdots a_{\nu n} \end{bmatrix}$$

vanishes, while all of order $\nu + 1$ in $M_{\nu+1}$ vanish, $(a_{\nu+11}, \cdots, a_{\nu+1n})$ being an arbitrary element of the system. To fix the notations, let

$$D = |a_{ij}| \neq 0 \qquad (i, j = 1, \cdots, \nu).$$

The expansion of $|a_{ij}| = 0$ $(i, j = 1, \cdots, \nu + 1)$ gives a relation

$$\sum_{i=1}^{\nu+1} d_i a_{\nu+1i} = 0, \qquad d_{\nu+1} = D \neq 0,$$

in contradiction with postulate VI.

Every n dimensional closed system a complex number system.

3. This identification may be made formally by establishing a one-to-one correspondence between a and $\sum a_i e_i$, where the e's have the multiplication table $e_j e_k = \sum \gamma_{jki} e_i$.

The following method seems preferable. We define the product* ρa of a mark ρ of F and an element $a = (a_1, \cdots, a_n)$ to be $(\rho a_1, \cdots, \rho a_n)$. In terms of the ϵ_i (end of § 2) any element a is expressible uniquely:

$$a \equiv \sum_{i=1}^{n} \rho_i \epsilon_i, \qquad a_j = \sum_{i=1}^{n} \rho_i a_{ij} \qquad (j = 1, \cdots, n),$$

the ρ's belonging to F. We introduce new units e_i obtained from the ϵ_i by applying the inverse of the transformation (a_{ij}):

$$\epsilon_i \equiv \sum_{j=1}^{n} a_{ij} e_j \qquad (i = 1, \cdots, n),$$

so that $e_1 = (1, 0, \cdots, 0)$; $\cdots$; $e_n = (0, 0, \cdots, 0, 1)$. Then

$$a \equiv \sum_{j=1}^{n} a_j e_j.$$

We note that the e_i are not in general elements of the system; likewise for $(\rho_1, \cdots, \rho_n)$.

On the independence of the postulates.

4. If F has a modulus q, postulates II and III are derivable from the others; in postulate III we may take $a_i' = (q - 1) a_i$. If F is the field of integers

* The associative law $\sigma(\rho a) = (\sigma\rho) a$, and the distributive law $\rho(a \oplus b) = \rho a \oplus \rho b$ then follow from these laws for marks and from I. We would give the same definition for $a\rho$ if needed.

modulo q, also postulate IV is redundant, since postulates I and VI insure the presence in the system of every element a, each a_i an integer (§ 2). In fact, there exist integers ρ_i such that

$$\sum_{i=1}^{n} \rho_i a_{ij} \equiv a_j \pmod{q} \qquad (j=1, \cdots, n)$$

If $n = 1$, $q = 2$ or 3, postulate I is likewise redundant. If $n = 1$, postulate V is redundant. Aside from these special cases, postulates I–VI are independent, as shown by the following systems:

(I) Elements $0, (\pm 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, \pm 1)$; each $\gamma_{ijk} = 0$.

(II) F non-modular; elements a, with each a_i an arbitrary positive integer; each $\gamma_{ijk} = 1$.

(III) Set (II) with element 0 added; each $\gamma_{ijk} = 0$.

(IV) $F \neq GF[q]$; elements a, with each a_i an arbitrary integral mark; each $\gamma_{ijk} = 0$ except γ_{111}, while $\gamma_{111} \neq$ integer.*

(V) Elements a with each a_i arbitrary in F; $n > 1$; $\gamma_{112} = \gamma_{221} = 1$, the remaining γ_{ijk} all zero. [Postulate V fails for $r = t = i = 1$, $k = 2$.]

(VI) Elements a with $a_2 = \cdots = a_n = 0$, a_1 arbitrary; $n > 1$; each $\gamma_{ijk} = 0$. An example for $n \geqq 1$ is the system with the single element 0; each $\gamma = 0$.

Further generalization of the concept number system.

5. The simplest example of a number system the coördinates of whose elements do not all belong to a field is furnished by a reducible system, the coördinates of the numbers of one subsystem belonging to a field of modulus p, those of a second subsystem to a field either without modulus or with modulus $\neq p$. To extend the definition (§ 2) to include this case, we take as elements $(a_1, \cdots, a_n)$, where $a_1, \cdots, a_{n_1}$ belong to a field F_1; $a_{n_1+1}, \cdots, a_{n_1+n_2}$ to F_2; etc. A similar change is to be made in postulate VI. Further, a γ_{jki} vanishes unless j, k, i belong to a single set of subscripts $1, \cdots, n_1$; etc.

Inversely, if in all the numbers $a_1 e_1 + \cdots + a_n e_n$ of a given system, $a_1, \cdots, a_{n_1}$ are marks of a field F_1 of modulus p; $a_{n_1+1}, \cdots, a_{n_1+n_2}$ are marks of a field $F_2, \cdots$, where each F_i $(i > 1)$ is either without modulus or with modulus $\neq p$, then the given system is reducible, the units of the subsystems being $e_1, \cdots, e_{n_1}$; $e_{n_1+1}, \cdots, e_{n_1+n_2}$; etc. Proof will be given here only for the special case † in which each of the fields F_i $(i > 1)$ has a modulus q_i. Let $1 \leqq j \leqq n_1$, $n_1 + 1 \leqq k \leqq n_1 + n_2$. Then

$$p(e_j e_k) = (pe_j) e_k = 0, \qquad q_2(e_j e_k) = e_j(e_k q_2) = 0.$$

* For $n = 2$ we may use the elements $A_1 + A_2 j$; $A_r = a_r + \alpha_r i$, $i^2 = j^2 = -1$, $ji = -ij$; viz. the real quaternion system expressed in two units $1, j$, but without γ's satisfying IV.

† Noted by Mr. WEDDERBURN, who gave a different proof.

Since p and q_2 are distinct primes, there exist integers P and Q such that $Pp + Qq_2 = 1$. Hence $e_j e_k = 0$.

It remains to examine the restriction that the coördinates of the numbers of a system (or of a subsystem of a reducible system) shall belong to a field. One would surely retain the assumption that the aggregate of these coördinates can be extended by adjunction to an aggregate A forming a commutative group under an operation called addition and such that a second operation called multiplication, obeying the associative and distributive laws, can be performed within A. We retain the commutative law for multiplication and assume that a product vanishes only when one factor vanishes, as otherwise in either case the aggregate of coördinates would be fully as complicated as a number system. Under these assumptions the aggregate forms part of a field. *

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* See KÖNIG, *Algebraischen Gröszen*, 1903, pp. 8–9.