VOLUMES AND AREAS*

BY

N. J. LENNES

Professor HILBERT's *Festschrift* (1899) contains a theory of areas of plane polygons independent of every axiom of continuity, and dependent on the remaining axioms of plane geometry, viz., those of incidence, order, parallelism, congruence. The main results obtained by HILBERT in this connection are the following:

(*a*) The measure of area of a triangle [(base × altitude)/2] is independent of the selection of the base.

(*b*) The measure of area of any polygon is independent of any particular decomposition. [The measure of area of a polygon being defined as the sum of the measures of area of a set of triangles into which the polygon may be decomposed].

(*c*) If two polygons have equal measures of area it does not follow that there exists a decomposition of them into finite sets of polygons which are congruent in pairs. If such decomposition does exist the polygons are said to be decompositionally congruent.

(*d*) If two polygons have equal measures of area then it is always possible to add polygons decompositionally congruent such that the resulting polygons shall be decompositionally congruent.†

The measures of volume are expressed in terms of elements which satisfy certain conditions imposed upon a segment calculus. Among other conditions they satisfy such conditions of order as are usually adapted for scalar quantities, viz.:

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† We notice here that HILBERT's theory requires a slight modification. HILBERT defines as follows: "Sind P und Q zwei inhaltsgleiche Polygone so mus es nach der Definition zwei flächengleiche [decompositionally congruent] Polygone P' und Q' geben, so dass das aus P und P' zusammengesetzte Polygon mit dem aus Q und Q' zusammengesetzten Polygon flächengleich ausfällt." *Festschrift*, p. 45.

In developing the theory it becomes necessary to adjoin polygons which are entirely predetermined, but this is not always possible. Following is an example of two polygons which cannot be thus adjoined: A regular sharp pointed star with a large number of sides and a regular convex polygon of a large number of sides.

If we regard the sum of a number of polygons not as one polygon formed by adjoining these polygons but as an aggregate of polygons which may be decomposed separately there is no such difficulty.

If a, b and c are any such scalar quantities then

(1) Of the three relations $a = b$, $a < b$, $b < a$ one and only one is valid.

(2) If $a = b$ and $b = c$ then $a = c$.

(3) If $a < b$ and $b < c$ then $a < c$.

The proposition listed as (d) makes it possible to define equal and unequal areas in terms of geometric congruence of a finite number of polygons in such manner that it shall correspond to the definition stated in terms of measure of area.

In space the situation is different. M. DEHN has proved * the following condition necessary in order that two polyhedrons shall be decompositionally congruent:

If two polyhedrons P' and P'' are decompositionally congruent then there exists a linear homogeneous function $f(\pi_1', \pi_2', \cdots, \pi_1'', \pi_2'', \cdots)$ with integral coefficients all different from zero such that

$$f(\pi_1', \pi_2', \cdots, \pi_1'', \pi_2'', \cdots) \equiv 0 \pmod{2R}$$

(where R denotes a right angle) in which $\pi_1', \pi_2', \cdots$ are the plane angles of the dihedral angles of P' and $\pi_1'', \pi_2'', \cdots$ are the plane angles of the dihedral angles of P''.

In the article cited, M. DEHN further shows that this condition is also necessary in order that it shall be possible to adjoin to P' and P'' polyhedrons decompositionally congruent so that the resulting polyhedrons shall be decompositionally congruent. Since it is not difficult to find two polyhedrons such that this condition is not satisfied no matter what may be their measures of volume, it follows that in respect to the volumes of such polyhedrons the theory must be essentially different from the theory of areas of polygons.

More recently,† S. O. SCHATUNOVSKY has discussed the measure of volumes of polyhedrons. He considers a scalar function μ (base $\times$ alt.) of a tetrahedron (μ is ultimately given the value $\frac{1}{3}$) which is called the measure of volume of the tetrahedron. The measure of volume of any polyhedron is defined as the sum of the measures of volume of the tetrahedrons into which the polyhedron may be decomposed. A proof is then given that this measure of volume of a polyhedron is independent of any particular decomposition of the polyhedron. Hence to every polyhedron P there corresponds a unique segment (number), denoted by $M(P)$, which is the measure of volume of the polyhedron. Evidently these measures of volume satisfy the conditions of order enumerated above.

It follows that if two polyhedrons P_1 and P_2 are decompositionally congruent then $M(P_1) = M(P_2)$. If P_1 and P_2 are decomposable into sets of polyhe-

*Mathematische Annalen, vol. 55 (1902), p. 465.

†Mathematische Annalen, vol. 57 (1903), p. 496.

drons P_1' and P_2' such that the elements of P_1' are congruent in pairs to part of the element of P_2' then $M(P_1) < M(P_2)$.

The converses of these propositions proved by SCHATUNOVSKY do not follow without the use of the archimedian axiom. From the hypothesis that $M(P_1) = kM(P_2)$, k being any segment (number) whatever different from zero, we can draw no conclusion whatever as to whether or not there exist P_1' and P_2' such that the elements of P_1' are congruent in pairs to a part or all of the elements of P_2' or vice versa.

The proof of this statement is identical with that suggested* by HILBERT † in proof of the proposition that two parallelograms of equal base and altitude cannot be proved decompositionally congruent without the use of the archimedian axiom. Hence it does not appear possible without the use of the archimedian axiom so to define volume of polyhedrons in terms of geometric congruence of a finite number of polyhedrons that it shall correspond to the measure of volume.

The object of this paper is to state a theorem provable only by means of the archimedian axiom and to base upon it a definition of volume in terms of congruence which shall correspond to the current definition in terms of measure of volume.

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* A detailed proof runs as follows:

Consider an equilateral triangle ABC whose sides are unity. It follows readily from the axioms at our disposal that any decomposition (into polygons) of this triangle will result in a set of polygons such that a segment connecting any two points in one of them will be equal to, or less than, unity.

On a non-archimedian line consider three points A, B, C', in the order ABC', such that $AB = 1$ while the point C' cannot be reached from B by laying off unit segments any finite number of times. At C' erect a perpendicular to the line AC' and on this perpendicular lay off a segment $C'D = \frac{1}{4}\sqrt{3}$ ($\frac{1}{2}$ the altitude of the equilateral triangle described above). Connect AD and BD.

The following lemma follows directly from the theory of proportion: Let M, N, M', N' be four points on the segment BC' such that $MN = M'N'$. Erect perpendiculars to the line BC' at these points. These perpendiculars intersect the sides AD and BD of the triangle ABD forming quadrilaterals. The quadrilateral two of whose sides lie on the perpendiculars at M' and N' is congruent to part of the quadrilateral two of whose sides lie on the perpendiculars at M and N, provided at least one of the points M and N lie between B and both M' and N'.

Any decomposition of the triangle ABD into polygons which shall be congruent in part with polygons of a decomposition of ABC must be such that a segment connecting any two points in one of them is equal to or less than unity. Hence any such polygon lies between a pair of lines perpendicular to BC' and not more than a unit distance apart. It follows from the lemma that any set of n such polygons may be obtained by suitably decomposing that part of the triangle ABD which lies between the perpendiculars to AC' at A and at a point E between B and C' such that $BE = n(AB)$. By this process we cannot reach a point P between B and C' such that $PC' \leqq n(AB)$ for any value whatever of n. Hence any decomposition of the triangle ABC and ABD into polygons of which any number n are congruent in pairs will always have a set of polygons resulting from the decomposition of ABD, not included in these n polygons, the sum of whose measures of area is greater than the measure of all of that part of the triangle ABD which lies between a perpendicular to the line BC' at P and the perpendicular to that line at C'. Consequently ABD cannot be exhausted even by an infinite limiting process, in spite of the fact that its measure of area is only one half that of ABC.

† *Festschrift*, p. 42; TOWNSEND's translation, p. 60.

THEOREM. *If $M(P_1)$ denotes the measure of volume of a polyhedron P_1 and $M(P_2)$ the measure of volume of a polyhedron P_2 and if $M(P_2) < M(P_1)$ then there exists a decomposition of P_1 and P_2 into sets of polyhedrons P_1' and P_2' such that the elements of P_2' are congruent in pairs with part* of the elements of P_1'.*

Proof: Denote by S_1 and S_2 the areas of the surfaces of P_1 and P_2 respectively, by l_1 and l_2 the total lengths of their edges and by v_1 and v_2 the numbers of their vertices. Consider a division of space into a set $[c]$ of equal cubes. † Denote by $[c_1]$ a subset of the set $[c]$ such that every cube of $[c_1]$ has at least one point in common with P_1 and by $[c_2]$ a subset of $[c]$ such that every cube of $[c_2]$ has at least one point in common with P_2. Denote further by $[c_1']$ all cubes each of which has at least one point in common with the boundary of P_1 and by $[c_2']$ all cubes each of which has a point in common with the boundary of P_2. Denote the diagonals of such cubes by k. Then all cubes of the set $[c]$ which have a point in common with a segment of length l lie within a rectangular parallelopiped of length $l + 2k$ its other dimensions being $2k$. Hence the total volume of such cubes is less than $4k^2(l + 2k)$. Therefore the total volume of all cubes of the set $[c]$ which have a point in common with an edge of the polyhedron P_1 is less than $4k^2 l_1 + 8k^3 v$. It is readily seen that if we add to this $2ks_1$ we shall have a sum greater than the measure of volume of the set $[c_1]$.

Denote by $f_1(k)$ the expressions $4k^2 l_1 + 8k^3 v_1 + 2ks_1$, and by $f_2(k)$ the similar expression $4k^2 l_2 + 8k^3 v_2 + 2ks_2$, which is greater than the measure of volume of the set $[c_2']$.

Let $M(P_1) - M(P_2) = \sigma$. Take k so that $f_1(k) + f_2(k) < \sigma$.‡ Then

$$f_1(k) + f_2(k) < M(P_1) - M(P_2)$$

or

$$M(P_2) + f_2(k) < M(P_1) - f_1(k)$$

and therefore

$$M[c_2] < \{M[c_1] - M[c_1']\}.$$

Hence for such values of k there is a larger number of cubes of the set $[c]$ which contain interior points of P_1 without containing any point of its boundary than there is in the complete set $[c_2]$. Hence there exists a decomposition of P_1 and P_2 into P_1' and P_2' such that the elements of P_2' are congruent in pairs to part of the elements of P_1' which was to be proven.

We arrange a relation between any two polyhedrons P_1, P_2 as to volume, so that in every case at least one of the relations

* By *part* is meant proper subset (not denoting the whole of P_1').

† This division is effected by means of parallel planes. The parallel line axiom is also used in forming the calculus of segments here used.

‡ It is at this point that the archimedian axiom is used.

$$\text{vol}(P_1) > \text{vol}(P_2), \qquad \text{vol}(P_1) = \text{vol}(P_2), \qquad \text{vol}(P_1) < \text{vol}(P_2)$$

holds by the following

DEFINITION. *The relation*

$$\text{vol}(P_1) > \text{vol}(P_2) \qquad \textit{or} \qquad \text{vol}(P_2) < \text{vol}(P_1)$$

implies that the two polyhedrons P_1, P_2 are decomposable into sets P_1', P_2' of polyhedrons such that part of the elements of P_1' are in a one-to-one way congruent to all of the elements of P_2'. The relation

$$\text{vol}(P_1) = \text{vol}(P_2)$$

implies that neither of the relations

$$\text{vol}(P_1) > \text{vol}(P_2), \qquad \text{vol}(P_2) < \text{vol}(P_2)$$

holds.

It follows from the above theorem that if $M(P_1) = M(P_2)$, then in the sense of the definition just given $\text{vol}(P_1) = \text{vol}(P_2)$ and conversely that if $\text{vol}(P_1) = \text{vol}(P_2)$ then $M(P_1) = M(P_2)$.

We have thus obtained in terms of congruence a definition of equal and unequal volumes such that if a and b represent the volume of any two polyhedrons then between a and b, there exists one and only one of the following relations

$$a = b, \qquad a < b, \qquad b < a.$$

Obviously these relations are transitive, as required by the statement on page 487.

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