ON A PROBLEM INCLUDING THAT OF SEVERAL BODIES AND ADMITTING OF AN ADDITIONAL INTEGRAL*

BY

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In the problem of three bodies BERTRAND† introduced certain quadratic functions of the coördinates of the bodies and of quantities proportional to the projections of the velocities on the axes of coördinates. BOUR‡ showed that BERTRAND's variables satisfy a certain system (S) of ordinary differential equations of the first order, and pointed out that the problem of three bodies may be considered as a particular solution of a more general problem whose equations are those of S and of which a certain integral D is known.

It is the object of the following note to write out the extension of these results to the case of any number of bodies.

Given a system of $n+1$ bodies consisting of a fixed body $(0, 0, 0; \mu)$ and n others $(x_i, y_i, z_i; m_i)$, mutually attracting one another by central forces varying directly as the masses and as any arbitrary function of the distance; to determine the motion of the n bodies about the fixed center we arrive at a system of $6n$ differential equations of the first order in the canonical form:

$$(1) \qquad \begin{cases} \dfrac{dx_i}{dt} = -\dfrac{\partial F}{\partial \xi_i}, & \dfrac{dy_i}{dt} = -\dfrac{\partial F}{\partial \eta_i}, \quad \dfrac{dz_i}{dt} = -\dfrac{\partial F}{\partial \zeta_i}, \\[2ex] \dfrac{d\xi_i}{dt} = \dfrac{\partial F}{\partial x_i}, & \dfrac{d\eta_i}{dt} = \dfrac{\partial F}{\partial y_i}, \quad \dfrac{d\zeta_i}{dt} = \dfrac{\partial F}{\partial z_i}, \end{cases} \qquad (i = 1, 2, \cdots, n),$$

where ξ_i, η_i, ζ_i are proportional to the projections of the velocities of the bodies on the axes of coördinates, and the function F is of the form

$$(2) \qquad F = U - \sum_{i=1}^{n} \frac{\xi_i^2 + \eta_i^2 + \zeta_i^2}{2m_i},$$

the force-function being designated by U.

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† BERTRAND, *Mémoire sur l'intégration des équations différentielles de la mécanique*, Journal de Liouville, ser. 1, vol. 17 (1852), pp. 393–436.

‡ BOUR, *Mémoire sur le problème des trois corps*, Journal de l'École Polytechnique, vol. 21 (1856), pp. 35–58.

Let new variables

$$(3)\qquad \begin{cases} q_{ij} = x_i x_j + y_i y_j + z_i z_j, & q_{ji} = q_{ij}, \\ r_{ij} = \xi_i \xi_j + \eta_i \eta_j + \zeta_i \zeta_j, & r_{ji} = r_{ij}, \\ s_{ij} = x_i \xi_j + y_i \eta_j + z_i \zeta_j, & s_{ji} \neq s_{ij} \end{cases} \qquad (i, j = 1, 2, 3, \cdots, n),$$

be introduced. These variables are of the same form as those employed by BERTRAND in the memoir cited. They are $n(2n+1)$ in number and are not all distinct. The relations among them may be set up by the aid of the following well known theorem of the theory of determinants: If k is a given number and i, j two numbers which may take all values from 1 to n, the determinant of n^2 elements whose general element is

$$(4)\qquad a_{ij} = \sum_{h=1}^{k} \alpha_{i,h} \beta_{j,h}$$

is equal to the product of the determinant of the α's by that of the β's for $k = n$, and is identically equal to zero for all values of k less than n.

From this theorem and the forms (3) it readily appears that the symmetrical determinant

$$(5)\qquad \Delta = \begin{vmatrix} q_{ij} & s_{ij} \\ s_{ij} & r_{ij} \end{vmatrix} \qquad (i, j = 1, 2, \cdots, n),$$

where q_{ij} represents the square of n^2 elements obtained by giving to i, j the values $1, 2, \cdots, n$, and all its minors down to and including all the $\frac{1}{2}\binom{2n}{4}\{\binom{2n}{4}+1\}$ which are determinants of the fourth order vanish, and that no one of the $\frac{1}{2}\binom{2n}{3}\{\binom{2n}{3}+1\}$ which are of the third order vanishes.

These $\frac{1}{2}\binom{2n}{4}\{\binom{2n}{4}+1\}$ conditions among $n(2n+1)$ quantities are far too numerous, but they can be reduced to proper bounds by means of the following theorem given in 1869 by KRONECKER*: If in the determinant of the nth order

$$M = |a_{ij}| \qquad (i, j = 1, 2, \cdots, n),$$

the minor of the mth order

$$|a_{11}, a_{22}, \cdots, a_{mm}| \qquad (m < n)$$

does not vanish, and the minors of the $(m+1)$th order

$$|a_{11}, a_{22}, \cdots, a_{mm}, a_{ik}| \quad (i, k = m+1, m+2, \cdots, n)$$

do vanish, then all the $(m+1)$th order subdeterminants of M vanish.

* KRONECKER, *Bemerkungen zur Determinanten-Theorie*, Crelle's Journal, vol. 72 (1870), pp. 152–175.

My colleague Dr. O. D. KELLOGG gave me a proof of the above theorem, believing the theorem to be new. Later I found that KRONECKER had published it as just cited.

Accordingly the vanishing of all the $\frac{1}{2}\binom{2n}{4}\{\binom{2n}{4}+1\}$ fourth order sub-determinants of the above symmetrical determinant Δ is a consequence of the vanishing of $(n-1)(2n-3)$ properly chosen independent fourth order sub-determinants, and this choice can be made in $\frac{1}{2}\binom{2n}{3}\{\binom{2n}{3}+1\}$ ways. Then by the aid of these independent relations $(n-1)(2n-3)$ of the variables (3) can be eliminated if they be employed in problem (1); there would remain $6n-3$ independent variables which would be sufficient since a loss of three from the original $6n$ independent variables can be accounted for by change in orientation. On making $n=2$ in Δ we have Bour's determinant D, the vanishing of which expresses the single relation among Bertrand's ten variables (3) in the problem of three bodies.

In the variables (3) the force-function U becomes

$$U=\sum_{i=1}^{n}\mu m_i f(\sqrt{q_{ii}})-\sum_{i=1}^{n}\sum_{j=1}^{n} m_i m_j f(\sqrt{q_{ii}+q_{jj}-2q_{ij}}); \tag{6}$$

accordingly the partial derivatives of F are of the form

$$\frac{\partial F}{\partial x_i}=\mu_i x_i+\sum_{j=1}^{n}\mu_{ij}x_j, \qquad \frac{\partial F}{\partial \xi_i}=-\frac{\xi_i}{m_i}, \tag{7}$$

where the quantities

$$\left\{\begin{aligned} \mu_i &= \mu m_i\frac{f'(\sqrt{q_{ii}})}{\sqrt{q_{ii}}}-\sum_{j=1}^{n}\mu_{ij}, \\ \mu_{ij} &= m_i m_j\frac{f'(\sqrt{q_{ii}+q_{jj}-2q_{ij}})}{\sqrt{q_{ii}+q_{jj}-2q_{ij}}}=\mu_{ji}, \end{aligned}\right. \tag{8}$$

are coefficients depending on the forces and expressed in terms of the q's alone.

Then in virtue of (1) the variables (3) satisfy the following system of ordinary differential equations:

$$\left\{\begin{aligned} \frac{dq_{ij}}{dt} &= \frac{s_{ij}}{m_j}+\frac{s_{ji}}{m_i}; \\ \frac{dr_{ij}}{dt} &= \mu_i s_{ij}+\mu_j s_{ji}+\mu_{ij}(s_{ii}+s_{jj})+\sum_{k=1}^{n}\mu_{jk}s_{ki}+\sum_{l=1}^{n}\mu_{il}s_{lj} \\ & \qquad\qquad (i,j=1,2,\cdots,n); \\ \frac{ds_{ij}}{dt} &= \mu_j q_{ij}+\mu_{ij}q_{ii}+\frac{r_{ij}}{m_i}+\sum_{k=1}^{n}\mu_{jk}q_{ik}; \end{aligned}\right. \tag{9}$$

these equations are the generalizations of Bour's equations in the problem of three bodies.

It may now be shown without difficulty that the determinant Δ equated to a constant gives an integral of the equations (9). This can be done perhaps most simply on remarking that Δ does not contain the μ's. Let ϕ be a function of all the q's, r's and s's; if it is an integral not containing the μ's its total derivative with regard to the time

$$(10) \qquad \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\frac{\partial\phi}{\partial q_{ij}}\frac{dq_{ij}}{dt}+\frac{\partial\phi}{\partial r_{ij}}\frac{dr_{ij}}{dt}+\frac{\partial\phi}{\partial s_{ij}}\frac{ds_{ij}}{dt}\right\}$$

should vanish independently of the μ's when the total derivatives are replaced by their values (9).

From the absolute term of the equation thus formed we have the equation *

$$(11) \qquad \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\left(\frac{s_{ij}}{m_j}+\frac{s_{ji}}{m_i}\right)\phi_{q_{ij}}+\frac{r_{ij}}{m_i}\phi_{s_{ij}}\right\}=0;$$

from the coefficients of the μ_i the following n equations:

$$(12) \qquad b_i \equiv 2w_i\phi_{v_i}+u_i\phi_{w_i}+\sum_{j=1}^{n}(s_{ij}\phi_{r_{ij}}+q_{ij}\phi_{s_{ji}})=0 \qquad (i=1,2,\cdots,n);$$

and finally from the terms in which the μ_{ij} appear the following $\frac{1}{2}n(n-1)$ equations:

$$(13) \qquad \begin{aligned} d_{ij} \equiv d_{ji} \equiv\ & 2s_{ji}\phi_{v_i}+2s_{ij}\phi_{v_j}+q_{ij}(\phi_{w_i}+\phi_{w_j})+(w_i+w_j)\phi_{r_{ij}} \\ & +u_i\phi_{s_{ij}}+u_j\phi_{s_{ji}}+\sum_{k=1}^{n}(s_{ik}\phi_{r_{jk}}+s_{jk}\phi_{r_{ik}} \\ & +q_{jk}\phi_{s_{ki}}+q_{ki}\phi_{s_{kj}})=0 \qquad (i,j=1,2,\cdots,n), \end{aligned}$$

where for brevity we have put

$$(14) \qquad \begin{aligned} q_{ii} &= u_i, \\ r_{ii} &= v_i, \\ s_{ii} &= w_i. \end{aligned}$$

Combining these $\frac{1}{2}n(n+1)+1$ equations (11), (12), (13) in all possible pairs, by Poisson's operation, we obtain the following complete system of $n(2n+1)$ linear partial differential equations of the first order:

* In a previous communication to Professor E. W. BROWN this equation was immediately broken up into the n equations $a_i=0$ which follow above. He pointed out to me that this led to confusion since the ξ's contain the masses. The correction was made, but curiously enough it left the resulting complete system unchanged.

$$
(15)\quad \begin{cases}
a_i \equiv 2w_i\phi_{u_i} + v_i\phi_{w_i} + \sum_{j=1}^{n}(s_{ji}\phi_{q_{ij}} + r_{ij}\phi_{s_{ij}}) = 0; \qquad b_i = 0; \\
c_i \equiv 2u_i\phi_{u_i} - 2v_i\phi_{v_i} + \sum_{j=1}^{n}(q_{ij}\phi_{q_{ij}} - r_{ij}\phi_{r_{ij}} + s_{ij}\phi_{s_{ij}} - s_{ji}\phi_{s_{ji}}) = 0; \qquad d_{ij} = 0; \\
e_{ij} \equiv 2q_{ij}\phi_{u_i} - 2r_{ij}\phi_{v_j} + s_{ji}(\phi_{w_i} - \phi_{w_j}) + u_j\phi_{q_{ij}} - v_i\phi_{r_{ij}} \\
\qquad + (w_j - w_i)\phi_{s_{ij}} + \sum_{k=1}^{n}(q_{jk}\phi_{q_{ik}} - r_{ki}\phi_{r_{jk}} + s_{jk}\phi_{s_{ik}} - s_{ki}\phi_{s_{kj}}) = 0; \\
f_{ij} \equiv 2s_{ij}\phi_{u_i} + 2s_{ji}\phi_{u_j} + r_{ij}(\phi_{w_i} + \phi_{w_j}) + (w_i + w_j)\phi_{q_{ij}} + v_j\phi_{s_{ij}} \\
\qquad + v_i\phi_{s_{ji}} + \sum_{k=1}^{n}(s_{kj}\phi_{q_{ik}} + s_{ki}\phi_{q_{jk}} + r_{jk}\phi_{s_{ik}} + r_{ki}\phi_{s_{jk}}) = 0; \\
d_{ji} = d_{ij}, \qquad e_{ji} \neq e_{ij}, \qquad f_{ji} = f_{ij} \quad (i, j = 1, 2, \cdots, n).
\end{cases}
$$

These equations (15) are the generalizations of those given by GRAVÉ * for the case $n = 2$.

On replacing ϕ by Δ in equations (15) it is at once seen that they are identically satisfied in virtue of the well-known theorem of the theory of determinants which states that the sum of the products of the elements of any line of a determinant by the algebraic complements of the minors of the corresponding elements of a parallel line is zero.

We have then in equations (9) a problem including the problem of several bodies and admitting of the integral $\Delta =$ constant.

In virtue of the existence of this solution the determinant of the system (15) of $n(2n + 1)$ equations in $n(2n + 1)$ variables vanishes.

It may be added that the equations (15) admit of another integral which is a quadratic function of the integrals of areas in the n-body problem.

The reader will have little difficulty in verifying that the $n(2n + 1)$ operators

$$
(16)\qquad a_i, \; b_i, \; c_i, \; d_{ij}, \; e_{ij}, \; f_{ij}
$$

constitute a continuous group of transformations in LIE's sense. It is hoped to study this group in detail in a subsequent note.

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*GRAVÉ, *Sur le problème des trois corps*, Nouvelles Annales de Mathématiques, ser. 3, vol. 15 (1896), pp. 537–547.