ON CERTAIN HYPERABELIAN FUNCTIONS WHICH ARE EXPRESSIBLE BY THETA SERIES*

BY

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The moduli of periodicity of an integral u_i of the first kind on the Riemann surface

$$y^{\nu} = (x-a)^{\alpha}(x-b)^{\beta}(x-c)^{\gamma}(x-d)^{\delta}$$

are linearly expressible in terms of two such A_i, B_i when $\alpha+\beta+\gamma+\delta$ is a multiple of ν. If u_i be replaced by the new integral $u'_i = u_i/B_i$ the moduli of periodicity of u'_i will be of the form $m + n\omega_i$ in which $\omega_i = A_i/B_i$. Accordingly the table of periods for the p integrals u'_i will be expressible linearly in terms of the p parameters ω_i. The bilinear relations among the periods will enable us to reduce the linearly independent parameters ω_i to a number $p' < p$.† These remaining parameters will be connected by certain transcendental relations. Suppose, however, that after constructing the table of periods by aid of the Riemann surface, it is assumed that the p' linearly independent ω_i are also absolutely independent of one another. The table of periods as thus generalized will no longer be related to the given Riemann surface. It can nevertheless serve as a table of theta moduli, since the necessary bilinear relations and inequality conditions are satisfied. The table being now written in its homogeneous form as expressed in terms of A_i and B_i, let these undergo the transformation

$$(1)\qquad \begin{aligned} A'_i &= a_i A_i + b_i B_i, \\ B'_i &= c_i A_i + d_i B_i, \end{aligned}$$

and suppose that the result is equivalent to a linear transformation of the theta moduli the integer coefficients of which form the matrix

$$\left|\begin{array}{c|c} \alpha_{ik} & \beta_{ik} \\ \hline \gamma_{ik} & \delta_{ik} \end{array}\right|.$$

We thus obtain a group of transformations of the theta functions to which cor-

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† For more definite propositions concerning the relations among these ω_i, see a paper presented to the American Mathematical Society, September 7, 1905, by Mr. RICHARD MORRIS, a brief résumé of which is given in the Bulletin for November, 1905.

responds a group of transformations on the ω_i which is linear with respect to each variable. A uniform function of the ω_i which is unaltered for the given group of transformations is called, after Picard, a *hyperabelian* function. Hyperabelian functions which belong to a group generated in the manner above described, can evidently be expressed in terms of theta functions with zero arguments.

To make the matter more clear, let us take as a particular case the Riemann surface

$$y^n = (x-a)(x-b)(x-c)^{n-1}(x-d)^{n-1}.$$

To form a convenient picture of the surface and its cross-cuts, let a, c, b, d be placed at the vertices of a rectangle in the order just written, the sides ac and bd of which are branch-lines of the surface. Take any point P (inside the rectangle, for convenience) and denote by α_k a path starting at P in the kth sheet, winding once around a positively, and returning to P in the $(k+1)$th sheet. Let β_k be a similar path about b. Let γ_k be a path from P in the kth sheet, winding once around c, and returning in the $(k-1)$th sheet, and δ_k a similar path about d. The equation

$$a_k = \sum_{i=1}^{k} \alpha_i + \sum_{i=k}^{1} \beta_{i+1}$$

will be used to express symbolically that the path a_k is obtained by first describing the path α_1, then the path α_2, and so on in order. A canonical system of $2p$ cross-cuts a_k, b_k may then be constructed from the a_k just defined and cuts b_k defined by the relation

$$b_k = a_k + \delta_{k+1}.$$

If

$$u_k = \int \frac{(x-c)^{k-1}(x-d)^{k-1}}{y^k}\,dx \qquad (k=1,\,2,\,\cdots,\,n-1)$$

be taken as the p integrals of the first kind, the table of periods can readily be expressed in terms of the moduli of periodicity at a_1, b_1. If A_i, B_i denote the value of u_i when integrated along the paths a_1, b_1 respectively, and if we write $\rho = e^{2\pi i/5}$, the result for $n=5$ is:

$$\begin{array}{c|cccc|cccc|}
u_1' & -1, & -\rho^4, & -\rho^3, & -\rho^2 & \omega_1, & (1+\rho^4)\omega_1, & (-\rho-\rho^2)\omega_1, & -\rho\omega_1 \\
u_2' & -1, & -\rho^3, & -\rho, & -\rho^4 & \omega_2, & (1+\rho^3)\omega_2, & (-\rho^2-\rho^4)\omega_2, & -\rho^2\omega_2 \\
u_3' & -1, & -\rho^2, & -\rho^4, & -\rho & \omega_3, & (1+\rho^2)\omega_3, & (-\rho-\rho^3)\omega_3, & -\rho^3\omega_3 \\
u_4' & -1, & -\rho, & -\rho^2, & -\rho^3 & \omega_4, & (1+\rho)\omega_4, & (-\rho^3-\rho^4)\omega_4, & -\rho^4\omega_4
\end{array}.$$

Instead of applying the processes indicated above to this table, we will deduce a simpler case by means of a transformation. Introduce as new integrals $w_1 = \frac{1}{2}(u_1' - u_4')$, $w_2 = \frac{1}{2}(u_2' - u_3')$, $w_3 = \frac{1}{2}(u_1' + u_4')$, $w_4 = \frac{1}{2}(u_2' + u_3')$ and make the transformation

$$\left|\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & & & & \\ 0 & 0 & -1 & -1 & & 0 & & \\ 1 & 2 & 1 & 1 & & & & \\ 0 & 0 & -1 & 1 & & & & \\ \hline & & & & 2 & -1 & 0 & 0 \\ & 0 & & & 2 & 0 & -1 & -1 \\ & & & & 0 & 1 & 0 & 0 \\ & & & & 0 & 0 & -1 & 1 \end{array}\right|.$$

The above table then reduces to two hyperelliptic tables which, after multiplying the rows by B_1 and B_2 to make homogeneous and using the bilinear relations, are:

$$\text{(I)}\qquad \begin{array}{c} W_1 B_1 \\ W_2 B_2 \end{array}\left|\begin{array}{cc|cc} (\rho-\rho^4)B_1, & (\rho^3-\rho^2)B_2 & (1+\rho^4)A_1, & \rho^2 A_1 \\ (\rho^2-\rho^3)B_2, & (\rho-\rho^4)B_2 & (1+\rho^3)A_2, & \rho^4 A_2 \end{array}\right|,$$

$$\text{(II)}\qquad \begin{array}{c} W_3 B_1 \\ W_4 B_2 \end{array}\left|\begin{array}{cc|cc} (\rho+\rho^4)B_1, & (\rho^2+\rho^3)B_1 & (1-\rho^4)A_1, & (1+\rho-\rho^3-\rho^4)A_1 \\ (\rho^2+\rho^3)B_2, & (\rho+\rho^4)B_2 & (1-\rho^3)A_2, & (1+\rho^2-\rho-\rho^3)A_2 \end{array}\right|.$$

In table (I) change A_i, B_i to A_i', B_i' and substitute for these the expressions (1) Again, denote the elements in (I) by the usual notation $A_{ik} \mid B_{ik}$, and let these be subjected to the transformation

$$A'_{ik} = \sum_{l=1}^{2} (\alpha_{kl} A_{il} + \beta_{kl} B_{il}),$$

$$B'_{ik} = \sum_{l=1}^{2} (\gamma_{kl} A_{il} + \delta_{kl} B_{il})$$

of determinant 1, in which α_{ik}, β_{ik}, $\cdots$ are integers satisfying certain well known bilinear relations. Assume the two transformed tables thus obtained to be identical. On comparing like terms certain conditions are obtained from which the following results may be deduced.

For brevity write $\alpha_{1i} = \alpha_i$, $\beta_{1i} = \beta_i$, $\gamma_{1i} = \gamma_i$, $\delta_{1i} = \delta_i$. Then the theta transformation can be expressed in the form

$$\left|\begin{array}{cc|cc} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ \alpha_2 & \alpha_1+\alpha_2 & \beta_2 & \beta_1+\beta_2 \\ \hline \gamma_1 & \gamma_2 & \delta_1 & \delta_2 \\ \gamma_2 & \gamma_1+\gamma_2 & \delta_2 & \delta_1+\delta_2 \end{array}\right|$$

the coefficients being subject to the conditions

$$(2)\qquad \begin{aligned} \alpha_1\delta_1 + \alpha_2\delta_2 - \beta_1\gamma_1 - \beta_2\gamma_2 &= 1, \\ \alpha_1\delta_2 + \alpha_2(\delta_1 + \delta_2) - \gamma_1\beta_2 - \gamma_2(\beta_1 + \beta_2) &= 0. \end{aligned}$$

The coefficients in the corresponding transformation on the ω_i

$$(3)\qquad \omega_i' = \frac{a_i\omega_i + b_i}{c_i\omega_i + d_i} \qquad (i=1,2)$$

take the form

$$\begin{aligned} a_1 &= \delta_1 - \lambda\delta_2, & a_2 &= \delta_1 - \lambda'\delta_2, \\ b_1 &= (\rho - 1)(\gamma_1 - \lambda\gamma_2), & b_2 &= (\rho^2 - 1)(\gamma_1 - \lambda'\gamma_2), \\ c_1 &= \frac{1}{\rho - 1}(\beta_1 - \lambda\beta_2), & c_2 &= \frac{1}{\rho^2 - 1}(\beta_1 - \lambda'\beta_2), \\ d_1 &= \alpha_1 - \lambda\alpha_2, & d_2 &= \alpha_1 - \lambda'\alpha_2, \end{aligned}$$

in which $\lambda = \rho + \rho^4$ and $\lambda' = \rho^2 + \rho^3$. Also $a_i d_i - b_i c_i = 1$ on account of (2). This transformation may be reduced to one with real coefficients by means of the substitution $\omega_1 = i\rho^3 z_1$, $\omega_2 = i\rho z_2$.

Reducing table (I) to the normal form in the usual way and applying the formulæ of KRAZER and PRYM for the transformation of the theta functions we find

$$\Delta_A = (c_1\omega_1 + d_1)(c_2\omega_2 + d_2)(\pi i)^2,$$

and hence obtain a relation of the form

$$(4)\qquad \vartheta\left[\begin{smallmatrix} g' \\ h' \end{smallmatrix}\right](\omega_1', \omega_2') = C\sqrt{(c_1\omega_1 + d_1)(c_2\omega_2 + d_2)}\,\vartheta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](\omega_1, \omega_2),$$

in which C is a function of the coefficients of transformation only, and $\vartheta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](\omega_1, \omega_2)$ denotes the theta function with zero arguments and moduli

$$\begin{aligned} a_{11} &= \tfrac{1}{5}\pi i\left[(\rho^4 - \rho)(1 + \rho^4)\omega_1 + (\rho^3 - \rho^2)(1 + \rho^3)\omega_2\right], \\ a_{12} &= \tfrac{1}{5}\pi i\left[(\rho - \rho^3)\omega_1 + (\rho^2 - \rho)\omega_2\right], \\ a_{22} &= \tfrac{1}{5}\pi i\left[(\rho^4 - 1)\omega_1 + (\rho^3 - 1)\omega_2\right]. \end{aligned}$$

By means of formula (4) we may construct from quotients of theta series, functions of ω_1, ω_2 which are unaltered by the given group of transformations.

In a similar manner from table (II) we derive the theta transformations

$$\left|\begin{array}{cc|cc} \alpha_1 & \alpha_2 & \beta_1 & -\beta_2 \\ -\alpha_2 & \alpha_1 - 3\alpha_2 & -\beta_2 & -\beta_1 + 3\beta_2 \\ \hline \gamma_1 & \gamma_2 & \delta_1 & -\delta_2 \\ \gamma_2 & -\gamma_1 + 3\gamma_2 & \delta_2 & \delta_1 - 3\delta_2 \end{array}\right|,$$

the coefficients of which satisfy the conditions

$$\alpha_1\delta_1 - \alpha_2\delta_2 - \beta_1\gamma_1 + \beta_2\gamma_2 = 1,$$

$$\alpha_1\delta_2 + \alpha_2(\delta_1 - 3\delta_2) - \beta_2\gamma_1 + \gamma_2(-\beta_1 + 3\beta_2) = 0.$$

The coefficients in the transformation (3) are

$$a_1 = \delta_1 - \mu\delta_2, \qquad a_2 = \delta_1 - \mu'\delta_2,$$

$$b_1 = \frac{1}{\rho - \rho^3}(\gamma_1 - \mu\gamma_2), \qquad b_2 = \frac{1}{\rho^2 - \rho}(\gamma_1 - \mu'\gamma_2),$$

$$c_1 = (\rho - \rho^3)(\beta_1 - \mu\beta_2), \qquad c_2 = (\rho^2 - \rho)(\beta_1 - \mu'\beta_2),$$

$$d_1 = \alpha_1 - \mu\alpha_2, \qquad d_2 = \alpha_1 - \mu'\alpha_2,$$

in which $\mu = \rho + \rho^4 + 2$ and $\mu' = \rho^2 + \rho^3 + 2$.

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