DETERMINATION OF THE ABSTRACT GROUPS OF ORDER $p^2qr$;
$p, q, r$ BEING DISTINCT PRIMES*

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Since the publication† in 1899 of Professor Miller's "Report on recent progress in the theory of groups of finite order," Western‡ has published his determination of the groups of order $p^3q$, and Le Vasseur§ has discussed the order $p^2q^2$. This paper is devoted to the determination of all groups of the order $p^2qr$. It thus completes the discussion of the problem of groups whose orders are products of four primes. ||

With the exception of the group of order $2^2 \cdot 3 \cdot 5$, simply isomorphic with the icosahedron-group, all groups of order $p^2qr$ are solvable. The maximal self-conjugate subgroups will therefore serve as the basis of classification. The twelve possible arrangements of the factors of composition are

(1) $ppqr$, (2) $pprq$, (3) $pqpr$, (4) $pqrp$, (5) $prpq$, (6) $prqp$,
(7) $qppr$, (8) $qprp$, (9) $qrpp$, (10) $rqpp$, (11) $rppq$, (12) $rpqp$.

If for a given type of group precisely the arrangements $(i), (j), (k), \ldots$, of the factors of composition are possible, then we symbolize $^\dagger$ the group $(i, j, k, \ldots)$. Two groups having distinct symbols cannot be simply isomorphic.

The group $G$ always contains a maximal invariant subgroup ** of order $p^2q$, and may contain maximal subgroups †† of order $p^2r$ and $pqr$. We shall discuss

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in detail in this paper only two classes of groups: those possessing invariant subgroups of both the types $H_{p,q}$ and $H_{p,r}$, and those possessing maximal invariant subgroups of the type $H_{p,q}$ only. A detailed summary of the results obtained in the other classes is given at the end. We shall thus be concerned principally with the subgroups $H_{p,q}^\sigma (\sigma = q, r)$ all types of which are given in the following table, in which $\tau$ denotes the number of distinct types, while $(p)$ signifies (modulo $p$):

<table>
<thead>
<tr>
<th>$H_{p,q}^\sigma$</th>
<th>$S_{-1}^{1} S_{-3} S_{-1}^{-1} S_{3} S_{-3}^{-1} S_{2} S_{3} S_{-1}^{-1} S_{1} S_{2}$</th>
<th>Parameters</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = I$</td>
<td>$S$</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$II$</td>
<td>$S_{1}$</td>
<td>$S_{2}$</td>
<td>$S_{1}$</td>
</tr>
<tr>
<td>$III$</td>
<td>$S_{1}^c$</td>
<td>$S_{2}$</td>
<td>$S_{1}$</td>
</tr>
<tr>
<td>$IV$</td>
<td>$S_{1}^a$</td>
<td>$S_{2}$</td>
<td>$S_{1}$</td>
</tr>
<tr>
<td>$V$</td>
<td>$S_{1}^a$</td>
<td>$S_{2}^{a,b}$</td>
<td>$S_{1}$</td>
</tr>
<tr>
<td>$VI$</td>
<td>$S_{2}$</td>
<td>$S_{1}^{-1} S_{2}^{a+b}$</td>
<td>$S_{1}$</td>
</tr>
</tbody>
</table>

$\sigma = q, r$; $S_{1}^{a} = 1$, $S_{2}^{a} = 1$, $S_{3}^{a} = 1$.

§ 1. Determination of $\rho_{0,1}$.

By Sylow's theorem,† $N_{\sigma} = qr/\sigma, p, p^2, pqr/\sigma, p^2qr/\sigma$ or 1. If $N_{\sigma} = 1$ then $\rho_{0,1} = 1$, $\Omega$ being any operator of prime order in $G$. When $N_{\sigma} > 1$, the result of transforming the single conjugate set of $N_{\sigma}$ subgroups

$g_1, g_2, g_3, \cdots, g_{N_{\sigma}}$

by $\Omega$ is to permute them among themselves. Hence

$\Omega^{-1}(g_1, g_2, \cdots, g_{N_{\sigma}}) \Omega = (g_{1}, g_{2}, \cdots, g_{N_{\sigma}})$

It follows that $J_{g_{1},g_{2}} = 1$ and

(1) $N_{\sigma} - \rho_{0,1} \equiv 0 (\text{mod } \omega)$; $\rho_{0,1} \equiv 1$.

Next let $\omega = \sigma$. Then $N_{p} = (p^2 - 1)/(p - 1) = p + 1$, and

(2) $p + 1 - \rho_{0,p} \equiv 0 (\text{mod } \sigma)$.

Hence either $\rho_{0,p} = 0$ or else $\rho_{0,p} \equiv 2 (\omega = q, r)$. Now if the subgroup $I_{p}$ of $H_{p,q}^\sigma$ is cyclical the order of its group of isomorphisms is

$I = \phi (p^2) = \phi (p - 1)$.

*Throughout the paper $i$ denotes a non-integral mark of the $GF [p^2]$. Thus $c^i \equiv 1 (p)$ is an abbreviation for $c^i \equiv 1 (\text{mod } p, P)$, $P$ being any quadratic function irreducible modulo $p$.

† Sylow, Mathematische Annalen, vol. 5 (1872).
If $I_p$ is of type $[1, 1]$ its group of isomorphisms is simply isomorphic with the congruence group \{ $S_1$, $S_2$ ... \} of order $I = p(p - 1)^2(p + 1)$, where $S_1$ is
\[ y_1 \equiv a_{11}x_1 + a_{12}x_2 \pmod{p}, \quad y_2 \equiv a_{21}x_1 + a_{22}x_2 \pmod{p}, \]
or say
\[ S_1 = (a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2). \]

Since $\Omega$ corresponds to an isomorphism of $G$, \{ $\Omega$ \} corresponds to a subgroup of the group of isomorphisms of $G$ and $\omega$ divides $I$. Hence when $I_p$ is cyclical, or when $I_p = [1, 1]$ and $p \equiv 1(\sigma)$, $\rho_{p, \sigma} \equiv 2$. But when $p \equiv -1(\sigma)$ and $p$ is odd, $\rho_{p, \sigma} = 0$. Also since $\rho_{p, \sigma} \equiv 1$, $J_{q, \sigma}$ and $J_{r, \sigma}$ may be permutable. If
\[ S_2 = (b_{11}x_1 + b_{12}x_2, b_{21}x_1 + b_{22}x_2) \]
the necessary and sufficient conditions that $S_1 S_2 = S_2 S_1$ are
\[
\begin{align*}
\delta_{12} &= \begin{vmatrix} a_{12} & b_{12} \\
- a_{22} & b_{22} \end{vmatrix} \equiv 0, \quad \delta'_{12} = \begin{vmatrix} a_{21} & b_{21} \\
- a_{22} & b_{22} \end{vmatrix} \equiv 0, \\
\delta_{12} &= \begin{vmatrix} a_{12} & a_{21} \\
- b_{12} & b_{21} \end{vmatrix} \equiv 0.
\end{align*}
\]

§ 2. Class (9, 10), $p > q > r$.

We now consider the groups whose symbol is (9, 10), having the maximal subgroups $H_{pqr, i}$ and $H_{pqr, j}$ ($i, j = IV, V, VI$). Since $I_p$ is invariant in $G$ the existence of a subgroup of type IV excludes the possibility of a subgroup of type V or VI, and vice versa. There are thus five cases to consider.

[1] $i = j = IV$. Here $I_p = \{ P \}$ is cyclical and $P$ may be regarded as the generator of order $p^2$ in both $H$-subgroups. Since $\rho_{q, r} \equiv 1$, we may choose \{ $R$ \} permutable with $Q$ and, since $q > r$, $QR = RQ$, so that $G$ is defined by
\[ P^p = Q^p = R^r = 1, \quad Q^{-1}PQ = P^\alpha, \quad R^{-1}PR = P^\beta, \quad QR = RQ; \]
or for brevity $G = (\alpha : \beta : 1)$, where
\[ \alpha^q = 1, \quad \beta^r = 1(p^2), \quad p \equiv 1(qr), \quad r = 1. \]

[2] $i = j = V$. Let $H_{pqr, i} = \{ P_1, P_2, Q \}$, $H_{pqr, j} = \{ P_1, P_2, R \}$, wherein $QR = RQ$. We may write
\[ R^{-1}P_1R = P_1, \quad R^{-1}P_2R = P_2^\beta, \quad \alpha^q = 1(p), \quad \beta = \alpha^q. \]
\[ Q^{-1}P_1Q = P_1^\alpha P_2^\alpha, \quad Q^{-1}P_2Q = P_1^\alpha P_2^\alpha, \quad \alpha^q = 1(p), \quad \beta = \alpha^q. \]
and from the permutable isomorphisms of $I_p$
\[
J_Q = \left( \begin{array}{cc} P_1 & P_2 \\
\alpha P_1 & \alpha P_2 \end{array} \right), \quad J_R = \left( \begin{array}{cc} P_1 & P_2 \\
\alpha P_1 & \alpha P_2 \end{array} \right),
\]

* All congruences are taken modulo $n$ unless otherwise indicated.
\[ \delta_{12} = a_{12}(\alpha - \beta) \equiv 0, \quad \delta'_{12} = a_{21}(\alpha - \beta) \equiv 0. \]

Reserving for later treatment the ambiguous case \( h = 1 \), we deduce \( a_{12} = a_{21} = 0 \). Suppose next that
\[ \Delta^{-1} P'_i \Delta = P_{\Delta}^{b_{i1}, P_{\Delta}^{b_{i2}},} \quad (i = 1, 2). \]

Then
\[ (\Delta^{-1} P'_i \Delta) R = (\Delta^{-1} P_{\Delta}^{b_{i1}, P_{\Delta}^{b_{i2}}})^{-1} (\Delta^{-1} P'_i \Delta) R = P_{\Delta}^{b_{i1}, P_{\Delta}^{b_{i2}}}, \]
\[ b_{i1}(a_{11} - \gamma) \equiv 0, \quad b_{i2}(a_{22} - \gamma) \equiv 0, \quad \gamma \equiv 1, \]
\[ b_{i1}(a_{11} - \delta) \equiv 0, \quad b_{i2}(a_{22} - \delta) \equiv 0, \quad \delta \equiv \gamma^k. \]

Thus when \( h \neq 1, k \neq 1 \) we have one of the two equivalent results
\[ a_{11} \equiv \gamma, \quad a_{22} \equiv \delta \quad \text{or} \quad a_{11} \equiv \delta, \quad a_{22} \equiv \gamma. \]

In case \( h \neq 1, k = 1 \), the set \((5)\) becomes
\[ b_{i1}(a_{11} - \gamma) \equiv 0, \quad b_{i2}(a_{22} - \gamma) \equiv 0, \]
\[ b_{i2}(a_{11} - \gamma) \equiv 0, \quad b_{i2}(a_{22} - \gamma) \equiv 0, \]
and there are three possibilities to consider, viz.,
(i) \( a_{11} \equiv \gamma, \quad b_{11} \equiv 0, \quad b_{12} \equiv 0, \quad b_{21} \equiv 0, \quad b_{22} \equiv 0, \quad a_{22} \equiv \gamma; \)
(ii) \( a_{11} \equiv \gamma, \quad a_{22} \equiv \gamma, \quad b_{21} \equiv b_{22} \equiv 0, \quad b_{11} \equiv 0, \quad b_{12} \equiv 0; \)
(iii) \( a_{11} \equiv \gamma, \quad a_{22} \equiv \gamma. \)

Case (i) implies
\[ \Delta^{-1} P'_i \Delta = P_{\Delta}^{b_{11}, P_{\Delta}^{b_{12}}}, \quad \Delta^{-1} P'_i \Delta = P_{\Delta}^{b_{21}, P_{\Delta}^{b_{22}}}, \]
\[ \Delta^{-1} P'_i \Delta = \Delta^{-1} P'_i \Delta \quad \text{or} \quad P_{\Delta}^{b_{11}, P_{\Delta}^{b_{12}}} = P_{\Delta}^{b_{21}, P_{\Delta}^{b_{22}}}, \]
contrary to the independence of \( P'_i \) and \( P'_2 \). Likewise, case (ii) is excluded. Hence \( a_{11} \equiv a_{22} \equiv \gamma \).

In a similar manner, when \( h = 1, k \neq 1 \), we get \( a_{11} \equiv a_{22} \equiv \alpha \).

Next let \( h = 1, k = 1 \), so that
\[ \Delta^{-1} P'_i \Delta = P_{\Delta}^{a_{i1}}, \quad Q^{-1} P'_i Q = P_{\Delta}^{a_{i1}}. \quad (i = 1, 2). \]

One of the operations \( P'_1, P'_2 \) must be independent of \( P_1 \). As \( \gamma^i \equiv 1 \mod p \), we may assume that \( P'_1 \) and \( P'_2 \) are independent. These will generate \( I_{\Delta} \), so that
\[ Q^{-1} P'_1 Q = P_{\Delta}^{a_{11}, P_{\Delta}^{a_{12}}}, \quad \Delta^{-1} P'_i \Delta = P_{\Delta}^{b_{11}, P_{\Delta}^{b_{12}}}. \]
The abelian conditions from \( J_{\alpha} \) and \( J_{\alpha} \) are \([\text{Eq. (3)}]\)
\[ \delta_{12} = b_{12}(a_{11} - \delta) \equiv 0, \quad \delta'_{12} = a_{21}(b_{22} - \alpha) \equiv 0, \quad d_{12} = a_{21} b_{12} \equiv 0. \]

Thus three possibilities arise, viz.,
\[ a_{21} = 0, \quad b_{12} = 0, \quad a_{11} = \delta; \]
\[ a_{21} \neq 0, \quad b_{12} = 0, \quad b_{22} = \alpha; \]
\[ a_{21} = 0, \quad b_{12} = 0. \]

For (i), let \( P'_1 = P'_1 P'_2, \; P_2 = P'_1 P'_2, \) whence
\[ Q^{-1} P'_1 Q = P''_1 P''_2 = P''_1 P''_2, \]
\[ R^{-1} P'_2 R = P''_1 P''_2 = P''_1 P''_2, \]
\[ (\gamma - \delta) x = 0, \quad (\gamma - \delta) y = 0, \]
\[ w(b_{22} - \beta) = 0, \quad z(\alpha - \beta) + b_{12} w = 0. \]

Hence \( \gamma = \delta \) and \( k = 1; \) but as \( P'_1, \; P_2 \) are independent, \( w \neq 0, \; b_{22} \neq \beta, \)
\( \alpha \neq \beta \) and \( h \neq 1, \) contrary to hypothesis. Since (ii) is likewise excluded, we
have \( a_{21} = b_{12} = 0, \)
\[ Q^{-1} P_1 Q = P''_1, \quad R^{-1} P_2 R = P''_2, \]
\[ x(\alpha_1 - \gamma) = 0, \quad y(\delta - \gamma) = 0, \]
\[ z(\beta - \alpha) = 0, \quad w(b_{22} - \beta) = 0, \]
where \( x \neq 0, \; w \neq 0. \) Hence when \( \alpha = \beta, \; \delta = \gamma \) there results \( a_{11} = \gamma, \)
\( b_{22} = \alpha. \) We are thus led to a single set of defining relations:
\[ P''_1 = P''_2 = Q'' = R'' = 1, \quad P_1 P_2 = P_2 P_1, \quad Q^{-1} P_1 Q = P'_1, \]
\[ Q^{-1} P_2 Q = P'_2, \quad R^{-1} P_1 R = P'_1, \quad R^{-1} P_2 R = P'_2, \quad R Q = Q R, \]
\[ \alpha' = 1(p), \quad \gamma' = 1(p) \quad (h = 1, 2, \ldots, r - 1; \; k = 1, 2, \ldots, q - 1), \]
or, briefly, say \( G = (1: \gamma_0: 0: \gamma_0: \alpha_0: 0: \alpha_0: R): 1). \)

Proceeding to the determination of \( \tau \) we observe that there are, by hypothesis, two subgroups, \( \{ P_1 \}, \; \{ P_2 \}, \)
both permutable with \( Q \) and \( R. \) In any isomorphism of \( G \) with itself either \( \{ P_1 \} \sim \{ P_2 \}, \; \{ P_2 \} \sim \{ P_1 \} \) or else \( \{ P_1 \} \sim \{ P_1 \}, \; \{ P_2 \} \sim \{ P_2 \}. \)
Hence there are two choices of generators of order \( p. \) Every element of \( G \) is of the
form \( \Omega = R Q'' P''_1 P''_2. \) Hence \( \Omega' = R Q'' P''_1 P''_2, \) which transforms \( G \) in the same manner as \( Q_0 = Q'' \). Similarly \( R_0 = R'. \) Employing the new generators \( R_0, \; Q_0, \)
\( P_{1_0} = P_1, \; P_{2_0} = P_2, \) we get
\[ (1: \gamma_0: 0: \gamma_0: \alpha_0: 0: \alpha_0: 1) \sim (1: \gamma_0: 0: 0: \alpha_0: 0: \alpha_0: 1). \]

Hence any set of relations involving arbitrary primitive roots \( (\alpha', \; \gamma') \) can be
transformed into the original set. Next let \( P_{1_0} = P_2, \; P_{2_0} = P_1. \) Then
\[ (1: \gamma_0: 0: \gamma_0: \alpha_0: 0: \alpha_0: 1) \sim (1: \gamma_0: 0: 0: \alpha_0: 0: \alpha_0: 1). \]
if

\[ ky \equiv 1 \pmod{q}, \quad hx \equiv 1 \pmod{r}. \]

The group characterized by \([k, k]\) is thus isomorphic with \([x, y]\) when (6) is satisfied. Further \(r = 2\), \(\tau = \frac{1}{2}(q + 1)\), and when \(r\) is odd, \(\tau = \frac{1}{2}(qr + q + r + 1)\).

[3] \(i = VI, j = V\). When \(h = 1\) we have \(Q^{-1}P_i Q = P^e_j\) \((j = 1, 2)\).

Assuming that

\[
R^{-1}P'_1 R = P'_1 P'_2, \quad R^{-1}P'_2 R = P'_1 P'_2,
\]

we derive

\[
\begin{align*}
a_{11}x - z & \equiv 0, \quad x - (\nu^p + \iota - a_{11})z \equiv 0, \\
a_{22}y - w & \equiv 0, \quad y - (\nu^p + \iota - a_{22})w \equiv 0.
\end{align*}
\]

The elimination of \(x, y, z, w\) gives

\[
a_{jj}^2 - (\nu^p + \iota) a_{jj} + 1 \equiv 0 \quad (j = 1, 2),
\]

whence \(a_{jj} = \nu^p\) or \(\iota\). Hence \(a_{11}, a_{22}\) are galoisian imaginaries* and \(G\), for \(i = VI, j = V\), does not exist.

Before considering the ambiguous case \(h = 1\) a few general results must be established.

Let \(S\) and \(T\) be any set of generators of \(I_{p^*}\), so that \(G = \{S, T, Q, R\}\). We may write

\[
P'_1 = S^z T^v, \quad P'_2 = S^z T^w,
\]

Hence

\[
Q^{-1}S Q = S^{a_{11}} T^{a_{21}}, \quad Q^{-1}T Q = S^{a_{12}} T^{a_{22}}.
\]

Hence

\[
Q^{-1}P'_1 Q = P'_2 = S^z T^w = S^{a_{11} z + a_{12} s} T^{a_{21} z + a_{22} s},
\]

\[
Q^{-1}P'_2 Q = P'_1 r^p + t = S^{-z + (p + 1) z} T^{-y + (p + 1) w} = S^{a_{11} x + a_{12} y} T^{a_{21} z - a_{22} w},
\]

whence results the eliminant

\[
D = \begin{vmatrix} x & y & z & w \\ a_{11} & a_{12} & -1 & 0 \\ a_{21} & a_{22} & 0 & -1 \\ 1 & 0 & a_{11} - t & a_{12} \\ 0 & 1 & a_{21} & a_{22} - t \end{vmatrix} \equiv 0 \pmod{p},
\]

where \(t = \nu^p + \iota\). Its expansion gives

\[
D_{12}^2 - t(a_{11} + a_{22} - t)D_{12} + a_{22}^2 - a_{11}^2 + t(a_{11} - a_{22}) + 2a_{12}a_{21} + 1 \equiv 0.
\]

Now assume \(S = P_1\). Then, since \(p \equiv -1 \pmod{q}\), \(\rho_{Q, P} = 0\) and we may take \(Q^{-1}P_1 Q \equiv U\) as \(T\). Then

*Serret, Cours d'Algèbre Supérieure, cinq. ed. (1885), tome 2, sec. 3, chap. 3. See also Dickson, Linear Groups, pp. 14–19.
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\[
J_q = \left( \frac{P_{1}^{z_1} U_{z_2}}{P_{1}^{z_1+z_2} U_{x_1+a z_2}} \right), \quad J_q = 1,
\]

\[
D_{12} = \begin{vmatrix}
0 & a_{12} \\
1 & a_{22}
\end{vmatrix} \equiv (-a_{12})^{-1} \equiv 1 \pmod{p}.
\]

Now \(-a_{12}\) cannot be a primitive root of this congruence; for, if so \(p \equiv 1 \pmod{q}\), whereas \(p \equiv -1 \pmod{q}\) and \(q > r\). It follows that \(a_{12} \equiv -1 \pmod{p}\) and

\[
D \equiv (a_{22} - t)^2 \equiv 0, \quad a_{22} \equiv t \equiv \nu + \iota.
\]

This gives \(I_p, \{ P_1 U \}\) and

\[
Q^{-1} P_1 Q = U, \quad Q^{-1} U Q = P_1^{-1} U^{\iota + \iota},
\]

\[
R^{-1} P_1 R = P_1^z, \quad R^{-1} U R = P_1^z U^z,
\]

\[
\delta_{12} = \begin{vmatrix}
-1 & \xi \\
-\nu - \iota & \alpha - \eta
\end{vmatrix} \equiv 0, \quad \delta'_{12} = \begin{vmatrix}
1 & 0 \\
-\nu - \iota & \alpha - \eta
\end{vmatrix} \equiv 0,
\]

and thus, when \(h = 1, \eta \equiv \alpha, \xi \equiv 0 \pmod{p}\).

Inversely let \(P_2 = P_1^{\nu} U^{\nu}\). Then

\[
R^{-1} P_2 R = P_1^{\nu z} U^{\nu z} = P_1^{\nu z} U^{\nu z}
\]

and hence \(h = 1\). Thus when \(h = 1\) there exists a group

\[
G = \{ P_1, U, Q, R \} = (1:01: -1\nu + \iota: \alpha:0\alpha:1),
\]

where \(\alpha' \equiv 1 \pmod{p}\), \(p \equiv 1 \pmod{q}\), \(\tau = 1\). Also \(p \equiv -1 \pmod{q}\) and, in the \(GF[p^2]\), \(\nu' \equiv 1 \pmod{p}\).

[4] \(i = V, j = VI\). Since \(r\) is necessarily an odd prime, the argument of

[3] again gives for \(G\) a single type, \(G = (1: \gamma 0: 0\gamma : 01: -1\nu + \iota : 1)\), with \(\gamma^2 \equiv 1 \pmod{p}\). \(p \equiv 1 \pmod{q}\), \(\tau = 1\). Likewise \(p \equiv -1 \pmod{r}\); and \(\nu' \equiv 1 \pmod{p}\) in the \(GF[p^2]\).

[5] \(i = VI, j = VI\). Employing as in [3] the theory of the determinant \(D\) we are led to the same equations (7), viz.,

\[
Q^{-1} P_1 Q = U, \quad Q^{-1} U Q = P_1^{-1} U^{\iota + \iota}, \quad \nu' \equiv 1 \pmod{p}.
\]

Let us assume that

\[
R^{-1} P_1 R = P_2 = P_1^z U^z, \quad R^{-1} U R = P_1^z U^v.
\]

Then

\[
\delta_{12} = \begin{vmatrix}
-1 & z \\
-\nu' - \iota & x - \omega
\end{vmatrix} \equiv 0, \quad \delta'_{12} = \begin{vmatrix}
1 & y \\
-\nu' - \iota & x - \omega
\end{vmatrix} \equiv 0,
\]

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Thus

\[
d_{12} = \begin{vmatrix} -1 & 1 \\ z & y \end{vmatrix} \equiv 0, \quad D_{12} = \begin{vmatrix} x & z \\ y & w \end{vmatrix} \not\equiv 0.
\]

Thus

\[z \equiv -y, \quad w \equiv x + (\tau_1^p + \tau_1)y, \quad D_{12} \equiv x^2 + (\tau_1^p + \tau_1)xy + y^2.\]

Since

\[R^{-1}P_{1}R = P_{1}^{-1}P_{2}^{p+1}, \quad \tau_1^p \equiv 1(p),\]

so that

\[R^{-1}U^pR = P_{1}^{-1}U^{xy+(\tau_1^p + \tau_2^p)y^2} = P_{1}^{-1}U^{xy+(\tau_1^p + \tau_2^p)y^2}.\]

Since \(P_{1}\) and \(P_{2}\) are independent, \(y \not\equiv 0\); hence

\[(8) \quad 2x + (\tau_1^p + \tau_1)y - (\tau_1^p + \tau_2) \equiv 0,\]

\[(9) \quad y^2 - x^2 + (\tau_1^p + \tau_2)x - 1 \equiv 0.\]

From the latter we at once derive

\[D_{12} = x^2 + (\tau_1^p + \tau_1)xy + y^2 \equiv 1,\]

\[(10) \quad (\tau_2 - \tau_1^2)^2x^2 - (1 - \tau_1^2)(\tau_2 - \tau_1)\tau_2 + (1 - \tau_1^2)(\tau_2^2 - \tau_1^2) \equiv 0,\]

\[(11) \quad (\tau_1 - \tau_1^p)^2y^2 - (\tau_2 - \tau_1)\tau_2 \equiv 0.\]

There always exist integral solutions of \((10)\) and \((11)\), \(x = \epsilon_j, y = \sigma_j (j = 1, 2)\). Thus

\[R^{-1}P_{1}R = P_{1}^{\epsilon_1+(\tau_1^p + \tau_2^p)}U_{-\sigma_1}, \quad R^{-1}UR = P_{1}^{\sigma_1}U^{\epsilon_1}.\]

**Theorem.** The two general types of \(G\) characterized by the two distinct sets of solutions of \((10)\) and \((11)\), viz. \([\epsilon_1, \sigma_1]\) and \([\epsilon_2, \sigma_2]\) are simply isomorphic.

In proof, \(\sigma_2 \equiv -\sigma_1\), and congruence \((8)\) gives

\[2\epsilon_2 - (\tau_1^p + \tau_1)\sigma_1 - (\tau_1^p + \tau_2) \equiv 0, \quad \epsilon_2 \equiv \epsilon_1 + (\tau_1^p + \tau_1)\sigma_1.\]

Hence the two types of \(G\) are characterized by

\[R^{-1}P_{1}R = P_{1}^{\epsilon_1+(\tau_1^p + \tau_2^p)}U_{-\sigma_1}, \quad R^{-1}UR = P_{1}^{\sigma_1}U^{\epsilon_1},\]

and

\[R^{-1}P_{1}R = P_{1}^{\epsilon_1}U^{\sigma_1}, \quad R^{-1}UR = P_{1}^{-\sigma_1}U^{\epsilon_1+(\tau_1^p + \tau_2^p)}\sigma_1.\]

Let us select a new operation of order \(q\) from \([Q]\), e.g. \(Q' = Q^{-1}\). Then

\[Q'R = RQ', \quad Q'^{-1}UQ' = P_{1},\]

\[Q'^{-1}P_{1}Q' = U^{\eta}P_{1}^{\eta} = U^{-1}P_{1}^{\eta+1}, \quad r_j = \frac{\tau^{(q-j)p} - \tau^{q-j}}{\tau_1^{q-j} - \tau_1}.\]

The result of selecting \(Q'\) and \((\epsilon_2, \sigma_2)\) is thus to interchange \(P_{1}\) and \(U\) and to reproduce the relations given by \(Q\) and \((\epsilon_1, \sigma_1)\). Hence \([\epsilon_2, \sigma_2] \sim [\epsilon_1, \sigma_1]\).

The quantities \(\tau_1\) and \(\tau_2\) are marks of the \(GF[p^2]\) and in that field appertain
respectively to the exponents \( q \) and \( r \). Let \( \rho \) be any primitive root in the \( GF[p^2] \). It is easy to show that \( \tau = 1 \) and hence we may select
\[
\iota_1 \equiv \rho^{(p^2-1)/q}, \quad \iota_2 \equiv \rho^{(p^2-1)/r},
\]
thus
\[
G = (1:0:1:1, \ i_1^q + \iota : \epsilon + (\iota_1 + \iota_1) \sigma, -\sigma : \sigma \epsilon : 1),
\]
where
\[
\iota_1 \equiv \rho^{(p^2-1)/q}, \quad \iota_2 \equiv \rho^{(p^2-1)/r}, \quad \rho^{p^2-1} \equiv 1; \quad p \equiv -1(\text{mod} \ qr), \quad \tau = 1,
\]
\[
(\iota_1 - \iota_1)^2 \sigma^2 - (\iota_2 - \iota_2)^2 \equiv 0, \quad 2\epsilon + (\iota_1 + \iota_1) \sigma - (\iota_2 + \iota_2) \equiv 0.
\]

§ 8. The generating function \([k]\).

Consider the relation \( R^{-t}P_1R^t = P_1^{m}U^\nu \). From it
\[
uz+1 - (2x + t_1y)uz + (x^2 + t_1xy + y^2)uz-1 \equiv 0,
\]
\[
u_{z+1} - t_2uz + uz-1 \equiv 0 \quad (t_j = t_j + y; \ j = 1, 2),
\]
These recurring formulae give
\[
u_k \equiv [k]_2x - [k - 1]_2, \quad v_k \equiv [k]_2y,
\]
where
\[
[k]_j \equiv \frac{t_j^k - \iota_j^k}{\iota_j - \iota_j}.
\]

Following are some of the properties of the generating function \([k]_j\).

(12) \[
\frac{[k + 1]_j}{[k]_j} = \frac{1}{t_j + \iota_j + \iota_j + \cdots k \text{ terms}},
\]
(13) \[
[k]_j^2 - [k + 1]_j[k - 1]_j - 1 \equiv 0,
\]
(14) \[
[0]_j \equiv 0, \quad [1]_j \equiv 1, \quad [-k]_j \equiv -[k]_j,
\]
(15) \[
[k + 1]_j \equiv [2]_j[k]_j - [k - 1]_j,
\]
(16) \[
([k + 1]_j - [k - 1]_j - [2]_j) \iota_j^k \equiv (t_j^{k+1} - 1)(t_j^{k-1} - 1).
\]

§ 4. Class (10), \( p > q > r \).

We shall consider next groups possessing a single maximal self-conjugate subgroup \( H_{p^q} \) of non-abelian type (\( i = \text{III}, \text{IV}, \text{V}, \text{VI} \)). It is readily shown that class (10), with \( i = \text{III} \), must contain an invariant subgroup \( H_{p^q} \).

Class (10) remains to be considered.

\[
[i = \text{IV}. \quad \text{Here } H_{p^q, \text{IV}} = \{ P, Q \} \text{ and since } \{ P \} \text{ is self-conjugate in } G, \ R^{-1}PR = P^\beta. \quad \text{Since } \rho_{p^q} \equiv 1 \quad \text{[Eq. (1)]}, \ R^{-1}QR = Q^\gamma. \quad \text{Hence}
\]
\[
(QR)^{-1}P(QR) = P^\alpha = (QR)^{-1}P(RQ) = P^\beta \alpha \gamma, \quad \alpha \equiv 1(p^2), \quad \alpha \beta \gamma (\alpha \gamma - 1) \equiv 0 (\text{mod } p^2), \quad \gamma \equiv 1 (\text{mod } q).
\]

* Dickson, Linear Groups, p. 13.
Hence \( \{P_1, P_2, R\} \) is self-conjugate in \( \{P_1, P_2, Q, R\} = G \), contrary to hypothesis.

[2] \( \text{i} = V. \) Let \( H_{P_1, P_2, Q} = \{P_1', P_2, Q\} \). Assuming that

\[
R^{-1}P_1'R = P_1'^{\alpha_{11}}P_2'^{\alpha_{21}}, \quad R^{-1}P_2'R = P_1'^{\alpha_{12}}P_2'^{\alpha_{22}},
\]

we deduce

\[
a_{11}\alpha(\alpha^\gamma - 1) \equiv 0, \quad a_{21}(\beta^\gamma - \alpha) \equiv 0,
\]

\[
a_{22}\beta(\beta^\gamma - 1) \equiv 0, \quad a_{12}(\alpha^\gamma - \beta) \equiv 0,
\]

where \( \alpha^\gamma \equiv 1(p) \), \( \beta \equiv \alpha^h \). Now \( \gamma \equiv 1(\text{mod} q) \). Hence

\[
a_{11} \equiv 0, \quad a_{22} \equiv 0, \quad \alpha^h \equiv \alpha, \quad \alpha^\gamma \equiv \alpha^h \pmod{p},
\]

\[
\gamma \equiv h \pmod{q}, \quad \alpha^\gamma \equiv \alpha \pmod{p}, \quad \gamma^2 \equiv 1 \pmod{q}.
\]

But \( \gamma \) appertains to the exponent \( r \) modulo \( q \), and therefore \( r = 2 \) and \( \gamma \equiv -1 \pmod{q} \). Thus

\[
R^{-1}P_1'R = P_2'^{\alpha_{21}}, \quad R^{-1}P_2'R = P_1'^{\alpha_{12}}, \quad a_{12}a_{21} \equiv 1 \pmod{p}.
\]

Then \( P_1 = P_1'^{\alpha_{11}}, P_2, Q, R \), generate a group of order \( 2p^2q \), viz.,

\( G = \{1 : \pi_0 : 0 \alpha^\gamma : 01 : 10 : -1\}. \) Also \( p \equiv 1(q) \), \( r = 1 \).

[3] \( \text{i} = VI. \) It has been shown \([\S 1]\), that \( p \equiv \pm 1 \pmod{r} \).

(a) First let \( p = \pm 1(r) \). Then \( P_{R, P} \equiv 2 \) and two subgroups \( \{P_1\}, \{P_3\} \) may be selected which are permutable with \( R \). If

\[
Q^{-1}P_1Q = P_2, \quad Q^{-1}P_2Q = P_1^{-1}P_2^{\alpha^h+i},
\]

then

\[
R^{-1}P_1R = P_1^\alpha, \quad R^{-1}QR = Q^\gamma, \quad \gamma \equiv -1 \pmod{q}.
\]

Since \( I' \) is invariant in \( G \) we may assume that

\[
P_3 = P_1P_2^\alpha, \quad R^{-1}P_2R = P_1^\alpha P_2^\gamma,
\]

Hence

\[
(QR)^{-1}P_1(QR) = P_1^{\alpha}P_2^\gamma = (RQ^\gamma)^{-1}R_1(RQ^\gamma) = P_1^{-1}\beta(\gamma + 1)P_2^\gamma,
\]

\[
(QR)^{-1}P_2(QR) = P_1^{-1}\beta^{-1}[2]P_2^\gamma = (RQ^\gamma)^{-1}P_2(RQ^\gamma) = P_1^{-1}[\gamma - 1]\pi + \gamma P_2^\gamma[x + (\gamma + 1)y],
\]

\[
x \equiv -[\gamma - 1]\beta, \quad y \equiv [\gamma]\beta,
\]

\[
[\gamma]^2 = [\gamma - 1]^2 + [2][\gamma - 1] + 1,
\]

\[
\]

Now \( [\gamma] \equiv 0 \pmod{q} \). Since \( [-k] \equiv -[k] \) and

\[
[\gamma + 1] - [\gamma - 1] - [2] \equiv (\alpha^{\gamma+1} - 1)(\alpha^{\gamma-1} - 1) \equiv 0 \pmod{p},
\]

there results \( \gamma \equiv -1 \pmod{q} \), \( \gamma' \equiv (\gamma + 1)^{-1} \equiv +1 \pmod{q}, \) whence \( r = 2 \). If \( R^{-1}P_3R = P_3^\alpha \), then \( \alpha \equiv \pm 1 \pmod{p} \).
1906] OF ORDER $p^2qr$; $p$, $q$, $r$ BEING DISTINCT PRIMES 147

$w(y = 1) = 0, \quad xw + z(\beta = 1) = 0,$

$w(-\beta = 1) = 0, \quad [2] \beta w + z(\beta = 1) = 0.$

First let the upper sign hold. If $\beta = 1$, then $w = 0$ which is impossible, since $P_1$, $P_2$ are independent. Hence $\beta = -1, x = -[2], y = +[1] = +1$. Likewise if we use the lower sign, $\beta = +1, x = +[2], y = -[1] = -1$.

We thus obtain the two sets of defining relations:

$(1: 01: -1\iota^p + \iota: \mp 10: \iota^r + \iota^s, \pm 1: -1).$

To determine $\tau$, let $Q_0 = Q^*$, $R_0 = R$, $P_{10} = P_1$, $P_{20} = P_1^{[x-1]} P_2^{[z]}$; there results

$\{P_{10}, P_{20}, Q_0, R_0\} = (1: 01: -1\iota^p + \iota^x: \mp 10: \pm [x-1] \mp [2][x], \pm [x]: -1).$

But

$\pm [x-1] \mp [2][x] \mp [x+1] \mp [\iota^p + \iota^x] \mp [x-1],$

[Eq. (15)]. Hence

$\{P_{10}, P_{20}, Q_0, R_0\} = (1: 01: -1\iota^p + \iota^x: \mp 10: \pm (\iota^p + \iota^x), \pm 1: -1) \sim G.$

Thus the same defining relations are reproduced with $\iota$ replaced by $\iota^x$, and so $\tau = 1$.

It will now be proved that these two types are simply isomorphic. Select new operators as follows:

$q_1 = Q_1, r_1 = R, p_1 = P_1^a P_2^b, p_2 = P_1^{-a} P_2^{[z]+[2]b} = q_1^{-1} p_1 q_1.$

Then using the first set of defining relations we will have

$q_1^{-1} p_2 q_1 = p_1^{-1} p_2^{[z]+[2]b}, \quad r_1^{-1} p_1 r_1 = p_1, \quad r_1^{-1} p_2 r_1 = p_1^{[z]+[2]} p_2^{-1}, \quad r_1^{-1} q_1 r_1 = q_1^{-1}$

if

$2a + [2]b \equiv 0 \pmod{p}.$

Hence when a new operator $p_1 = P_1^a P_2^b$ is selected, where $a$ and $b$ are solutions of $2a + (\iota^p + \iota^s)b \equiv 0 \pmod{p}$, the first type is transformed into the second.

They are therefore isomorphic.

(b) When $p = -1(\imath), r$ odd, $\rho_{\imath, p} = 0$. As before, we deduce

$Q^{-1} P_1 Q = P_2, \quad Q^{-1} P_2 Q = P_1^{[z]+1}, \quad \iota^1 \equiv 1(p),$

$R^{-1} P_1 R = P_3, \quad R^{-1} P_3 R = P_1^{[z]+1}, \quad \iota^2 \equiv 1(p),$

Let $P_3 = P_1^x P_2^y$ and $R^{-1} P_2 R = P_4 = P_1^{[z]} P_2^{[x]}$. Then

$(17) \quad R^{-1} P_2 R = P_1^{(-[x]+1)+[2][z]} P_2^{[z]} P_2^{[y]} = P_1^{[y-1]} P_1^{[y]+[2][y]} P_2^{[y]} P_2^{[y]}.$

In addition to the latter, but not independent of them, we have the congruences derived from

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(18) \[ (QR)^{-1}P^*_q(QR) = (RQ^r)^{-1}P^*_{q'}(RQ^r). \]

The equations (17) and (18) give us the dialytic eliminant

\[ \Delta_{12} = \left\{ \ell_3^2 + \ell_2 \right\} \left\{ [\gamma]_1^2 - (\ell_3^2 + \ell_2) [\gamma]_1 + 1 \right\} (\ell_3^{r+1} - 1) (\ell_3^{r-1} - 1)^2 = 0. \]

Now \([\gamma]_1\) is an integer, and since \(r \neq 2\), and \(\gamma \neq -1\), it follows that \(\gamma \equiv 1 \pmod{q}\), contrary to hypothesis. Hence when \(p \equiv -1 \pmod{r}\) and \(r\) is odd, no corresponding group \(G\) exists.

The results of this section may be summarized in the following

**Theorem.** A group \(G_{pqr}(p > q > r)\) always contains a maximal self-conjugate subgroup \(H\) of order \(p^2q\). If \(H\) is the only maximal invariant subgroup of \(G\) and if \(r\) is odd, then \(N_q = 1\) and \(H\) is necessarily abelian. If \(r\) is even \((r = 2)\) and \(p \equiv 1 \pmod{q}\) there exists one type whose subgroup \(H_{p^2q}\) is non-abelian, and if \(r\) is even and \(p \equiv -1 \pmod{q}\) there exists a second type possessing a non-abelian \(H_{p^2}\). These two types of \(G\) contain respectively \(q\) and \(pq\) operators (and subgroups) of order \(2\), and in each type \(N_q = p^2\). Moreover, with exception of the two types just described, every group of order \(p^2qr(p > q > r)\), in which \(N_q \equiv 0 \pmod{q}\), possesses an abelian maximal self-conjugate subgroup \(H_{p^2q}\).

A general summary of all the existent types of \(G\) follows. Except for \(i\) and \(\rho\), every parameter occurring in the tables is an integer; while \(i\) and \(\rho\) are marks of the \(GF[p^2]\). See footnote on the second page of the paper.
Table 1. \( p > q > r \).

\( I_p \) non-cyclic; \( P_{10} = P_{20} = Q_n = R^r = 1 \), \( P_1P_2 = P_2P_1 \).
\( I_p \) cyclic; \( P_{10} = Q_n = R^r = 1 \).

Case (a), \( QR = RQ \).

<table>
<thead>
<tr>
<th>Class</th>
<th>( Q^{-1}P_2Q )</th>
<th>( Q^{-1}P_{1}Q )</th>
<th>( R^{-1}P_1R )</th>
<th>( R^{-1}P_{2}R )</th>
<th>Parameters</th>
<th>Arith. Rel.</th>
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<td>( P_1 )</td>
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<td>( \frac{1}{2}(q + 1) )</td>
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<td>( p = 1(r) )</td>
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</table>

\( \rho = \text{prim. root in} \ GF[p^\beta]; \iota_1, \iota_2 = \rho^{p^\beta - 1/q, r} \)
\( 2e + [2], \sigma - [2], \equiv 0 \rimag \)
\( (\iota_1 - \iota_2)^2 \sigma - (\iota_2 - \iota_2)^2 \equiv 0 \rimag \)

| \( P_1 \)       | \( P_2 \)       | \( P_2 \)       | \( P_1 \)       | \( P_1 \)       | \( \iota = 1(p) \) | \( p = 1(r) \) | 1     |

\( \gamma = 1(p) \rimag \)
Case (b). \( R^{-1} QR = Q^\gamma; \gamma = 1(q) \).

<table>
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<tr>
<th>Class.</th>
<th>( Q^{-1}P_iQ )</th>
<th>( Q^{-1}P_2Q )</th>
<th>( E^{-1}P_iR )</th>
<th>( E^{-1}P_2R )</th>
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<td>( P_3 )</td>
<td>( P_1^a )</td>
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<td>( h, k = 1, 2 \cdots r - 1 )</td>
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<td>( P_2 )</td>
<td>( P_1^{-1}P_2^r )</td>
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Table 2. \( q > p > r \).

\( I_{s^2} \) non-cyclical; \( P_i^s = Q^s = R^{\gamma} = 1 (i = 1, 2) \), \( P_1 P_2 = P_2 P_1 \), \( RP_1 = P_2 R \).

<table>
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<tr>
<th>Class.</th>
<th>( P_i^{-1}QP_1 )</th>
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</tbody>
</table>
OF ORDER \( p^aqr \); \( p, q, r \) BEING DISTINCT PRIMES

### Table 3. \( q > r > p \)

#### Case (a).

- \( I_p \) non-cyclical; \( P_i^p = Q^p = R^p = 1(i = 1, 2), \ P_1P_2 = P_2P_1, \ RQ = QR \)
- \( I_p \) cyclical; \( P_1^p = Q^p = R^p = 1, \ QR = RQ \)

<table>
<thead>
<tr>
<th>Class</th>
<th>( P_1^{-1}QP_1 )</th>
<th>( P_2^{-1}QP_2 )</th>
<th>( P_1^{-1}RP_1 )</th>
<th>( P_2^{-1}RP_2 )</th>
<th>Parameters</th>
<th>Arith. Rel.</th>
<th>( r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>[12345678]</td>
<td>( Q )</td>
<td>( R^a )</td>
<td></td>
<td></td>
<td>( \alpha_p = 1(r) )</td>
<td>( r = 1(p) )</td>
<td>1</td>
</tr>
<tr>
<td>[1237]</td>
<td>( Q )</td>
<td>( R^a )</td>
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<td>( \alpha_p = 1(r) )</td>
<td>( r = 1(p^2) )</td>
<td>1</td>
</tr>
<tr>
<td>[123456]</td>
<td>( Q^a )</td>
<td>( R^{ba} )</td>
<td></td>
<td></td>
<td>( \alpha_p = 1(q) )</td>
<td>( q = r = 1(p) )</td>
<td>( p - 1 )</td>
</tr>
<tr>
<td>[125]</td>
<td>( Q^a )</td>
<td>( R^{ba} )</td>
<td></td>
<td></td>
<td>( \alpha_p = 1(q) )</td>
<td>( q = 1(p^2) )</td>
<td>( p - 1 )</td>
</tr>
<tr>
<td>[234]</td>
<td>( Q^{ba} )</td>
<td>( R^a )</td>
<td></td>
<td></td>
<td>( \alpha_p = 1(q) )</td>
<td>( r = 1(p^2) )</td>
<td>( p - 1 )</td>
</tr>
<tr>
<td>[12]</td>
<td>( Q^a )</td>
<td>( R^{ba} )</td>
<td></td>
<td></td>
<td>( \alpha_p = 1(q) )</td>
<td>( q = r = 1(p^3) )</td>
<td>( p^3 - 1 )</td>
</tr>
<tr>
<td>[12345678]</td>
<td>( Q )</td>
<td>( Q )</td>
<td>( R )</td>
<td>( R^a )</td>
<td>( \alpha_p = 1(r) )</td>
<td>( r = 1(p) )</td>
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</tr>
<tr>
<td>[123]</td>
<td>( Q )</td>
<td>( Q^a )</td>
<td>( R )</td>
<td>( R^{ba} )</td>
<td>( \alpha_p = 1(q) )</td>
<td>( q = r = 1(p) )</td>
<td>( p - 1 )</td>
</tr>
<tr>
<td>[1235]</td>
<td>( Q )</td>
<td>( Q^a )</td>
<td>( R^a )</td>
<td>( R )</td>
<td>( \alpha_p = 1(r) )</td>
<td>( q = r = 1(p) )</td>
<td>1</td>
</tr>
</tbody>
</table>

#### Case (b). The simple group \( G_{151}, p = 2, q = 5, r = 3 \).

\( Q^a = 1, \ P^a = 1, \ (QP)^3 = 1, \ [R = QP] \).