ON THE ANALYTIC EXTENSION OF FUNCTIONS DEFINED BY DOUBLE POWER SERIES*

BY

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Problem and Theorem.

1. The object of the present paper is to study the following general problem: Given the double power series

$$\sum_{m=0}^{m=\infty}\sum_{n=0}^{n=\infty} a(m, n)x^m y^n, \tag{1}$$

defining a function $f(x, y)$ of the two independent complex variables x, y within the regions $|x| < R_1$, $|y| < R_2 (R_1 > 0,\ R_2 > 0)$, to determine the value of this function at any point in its domain of existence.†

It is our special purpose to demonstrate the following theorem:

THEOREM. *If in the series* (1) *the function* $a(u, v)$ *considered as a function of the two independent, complex variables* u, $v (u = \alpha + i\beta,\ v = \gamma + i\delta)$ *is single valued and analytic for those values of* u, v *for which* $\alpha \geqq 0$, $\gamma \geqq 0$ *and if there exists a constant* c *such that for the same values of* u, v

$$|a(u, v)| < c, \tag{2}$$

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† The importance of this problem in the theory of functions of two or more complex variables has long been recognized, but the results obtained in this field appear to be very meager. For a general statement respecting both the problem and the literature upon it see HADAMARD, *La série de Taylor et son prolongement analytique* (Paris, 1900), chap. IX, § 5. This problem is, in fact, the natural generalization of the problem of the analytic extension of a function defined by a single (instead of a double) power series (1), this latter problem being one which has been solved within recent years in a large variety of cases. Especially we may mention in this connection the results embodied in the following memoirs: *Sur les séries divergentes et les fonctions définies par un développement de Taylor*, by LE ROY in the Annales de la Faculté des Sciences de Toulouse, ser. 2, vol. 2 (1900); *Quelques applications d'une formule sommatoire générale*, by LINDELÖF in the Acta Societatis Scientiarum Fennicae, vol. 31 (1902); *Sur la fonction définie par une série de Maclaurin*, by FORD in the Journal de Mathématiques, ser. 5, vol. 9, pp. 223-232 (1903). Up to the present time the authors cited appear not to have generalized their theorems for the case of two or more variables. We propose, therefore, in the present paper to furnish such a generalization for the third of the memoirs just cited. We shall confine ourselves, for the sake of brevity, to the case of *two* variables, although the generalization for n variables can be deduced in a similar manner, i. e., by the use of CAUCHY'S generalized integral theorem for functions of n variables.

then the function $f(x, y)$ defined by (1) possesses a branch which is single valued and analytic throughout the x, y planes, exception being made of those point pairs x, y in which either x or y lies upon the rectilinear cut $0 \cdots +\infty$ in its plane. Moreover, this branch is defined by the formula

$$f(x, y) = a(0, 0) - \tfrac{1}{2}\int_{-\infty}^{\infty} a(\tfrac{1}{2} + i\beta, 0)\frac{(-x)^{\frac{1}{2}+i\beta}}{\cosh \pi\beta}\, d\beta$$

$$(3) \qquad - \tfrac{1}{2}\int_{-\infty}^{\infty} a(0, \tfrac{1}{2} + i\delta)\frac{(-y)^{\frac{1}{2}+i\delta}}{\cosh \pi\delta}\, d\delta$$

$$+ \tfrac{1}{4}\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} a(\tfrac{1}{2} + i\beta, \tfrac{1}{2} + i\delta)\frac{(-x)^{\frac{1}{2}+i\beta}(-y)^{\frac{1}{2}+i\delta}}{\cosh \pi\beta \cosh \pi\delta}\, d\delta,$$

*where in terms of the principal branch of the logarithm function**

$$(-x)^{\frac{1}{2}+i\beta} = e^{(\frac{1}{2}+i\beta)\log(-x)}, \qquad (-y)^{\frac{1}{2}+i\delta} = e^{(\frac{1}{2}+i\delta)\log(-y)}$$

and where, if desired, we may invert the order of integration in the improper double integral.†

Preliminary Results.

2. In order to prove this theorem we begin by observing several elementary relations to which frequent reference will be made.

First: If u and x are two complex quantities having the respective forms:

$$u = \alpha + i\beta, \qquad x = \rho e^{i\phi}, \qquad -2\pi < \phi \leqq 0,$$

then, always in terms of the principal branch * of the logarithm function,

$$\text{(I)} \qquad (-x)^u = e^{u\log(-x)} = (-x)^\alpha e^{-\beta(\phi+\pi)} e^{i\beta\log\rho},$$

and further

$$\text{(II)} \qquad \sin \pi u = \Omega(\alpha, \beta)\sinh \pi\beta,$$

where $\Omega(\alpha, \beta)$ is a function of α and β, such that, for all (real) values of α and β,

$$\text{(III)} \qquad |\Omega(\alpha, \beta)| \geqq 1.‡$$

* So that, for

$$x = \rho e^{i\phi}, \qquad -2\pi < \phi \leqq 0,$$

we have

$$\log(-x) = \log\rho + i(\phi + \pi).$$

† This theorem holds true under much less restrictive conditions, as appears from an examination of the analysis employed in the subsequent proof, but, for the sake of simplicity we shall confine ourselves as above. It will be observed, however, that this case covers a large class of series (1) in which $R_1 > 1$, $R_2 > 1$.

It is also desirable to remark that $f(x, y)$ as defined by (3) coincides in value with (1) whenever $|x| < R_1$, $|y| < R_2$ and neither x nor y is real and positive. For point pairs x, y in which either x or y is real and positive formula (3) in general loses significance, regardless of what $|x|$, $|y|$ may be.

‡ The proof of this and of the other relations is given below.

Again: If $\Phi(u)$ and $U(u)$ are two functions of the complex variable u, each single valued and analytic in a region A of the u plane and such that $U(u)$ vanishes in A only at the points $u = \lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ which are zeros of the first order, then, for any closed contour C_n within A and surrounding the points $u = \lambda_1, \lambda_2, \cdots, \lambda_n$ we have

$$\text{(IV)} \qquad \frac{1}{2\pi i}\int_{C_n} \frac{\Phi(u)}{U(u)}\,du = \sum_1^n \frac{\Phi(\lambda_n)}{U'(\lambda_n)},$$

the indicated integration being extended over C_n in the positive direction.

Relation (IV) admits of the following generalization:

Given two complex planes P_1 and P_2 to which belong respectively the two independent complex variables u, v and given two functions $\Phi(u, v)$, $U(u, v)$ of these variables, each single valued and analytic for all values of u, v within the regions A_1, A_2 of P_1, P_2 respectively. Furthermore, suppose that the function $U(u, v)$ is of the form $U(u, v) = E(u)F(v)$, in which $E(u)$ does not vanish within A_1 except at the points $u = \lambda_1, \lambda_2, \cdots, \lambda_m$ which are zeros of the first order, and let us suppose likewise that $F(v)$ does not vanish within A_2 except at the points $v = \mu_1, \mu_2, \mu_3, \cdots, \mu_n$ which are zeros of the first order. Then, if we designate by C_m, C_n closed contours lying within A_1, A_2, respectively, and surrounding the points $u = \lambda_1, \lambda_2, \cdots, \lambda_m$; $v = \mu_1, \mu_2, \cdots, \mu_n$, we have

$$\text{(V)} \qquad \frac{1}{(2\pi i)^2}\int_{C_m} du \int_{C_n} \frac{\Phi(u, v)}{U(u, v)}\,dv = \sum_{g=1}^{g=m}\sum_{h=1}^{h=n} \left[\frac{\Phi(u, v)}{\dfrac{\partial^2 U}{\partial u\,\partial v}}\right]_{\substack{u=\lambda_g\\ v=\mu_h}}$$

the integrations being extended over C_m and C_n in the positive direction.

Formula (I) is readily seen, while the relation (II, III) appears as follows:

$$\sin \pi u = \sin(\pi\alpha + i\pi\beta) = \sin \pi\alpha \cosh \pi\beta + i \cos \pi\alpha \sinh \pi\beta = \Omega(\alpha, \beta)\sinh \pi\beta,$$

where

$$\Omega(\alpha, \beta) = \sin \pi\alpha \coth \pi\beta + i \cos \pi\alpha.$$

But, for the function $\Omega(\alpha, \beta)$ thus defined, we have

$$|\Omega(\alpha, \beta)| = \sqrt{\sin^2 \pi\alpha \coth^2 \pi\beta + \cos^2 \pi\alpha}$$

and, since for β real $|\coth \pi\beta| \geqq 1$, this last relation gives also

$$|\Omega(\alpha, \beta)| \geqq 1.$$

Relation (IV) is a direct consequence of CAUCHY's integral theorem when we observe that the residue of the function $\Phi(u)/U(u)$ at the pole $u = \lambda_n$ is $\Phi(\lambda_n)/U'(\lambda_n)$.

Similarly, relation (V) results from a well-known theorem* in the theory of

* Cf. OSGOOD, *Encyklopädie*, II B1, § 42.

functions of two or more complex variables, it being observed in the present instance that the residue of the function $\Phi(u, v)/U(u, v)$ at the pole $u = \lambda_g$, $v = \mu_h$ is

$$\left[(u-\lambda_g)(v-\mu_h)\frac{\Phi(u, v)}{U(u, v)}\right]_{\substack{u=\lambda_g\\ v=\mu_h}} = \left[\frac{\Phi(u, v)}{\dfrac{\partial^2 U}{\partial u \partial v}}\right]_{\substack{u=\lambda_g\\ v=\mu_h}}$$

It is desirable to prove also the following lemma respecting the properties of double improper integrals:

Lemma. Given the function $f(\beta, t)$ of the two real variables β, t continuous throughout the region $\beta \geqq g$, $t \geqq a$ and satisfying the following conditions:

(*a*) $$|f(\beta, t)| \leqq \phi(\beta) \quad [\beta \geqq g,\ t \geqq a,\ \phi(\beta) \text{ continuous},]$$

(*b*) $$\int_g^\infty \phi(\beta)\, d\beta \text{ converges},$$

(*c*) $$t^\kappa |f(\beta, t)| \leqq \phi(\beta) \qquad (\beta \geqq g,\ t \geqq \tau \geqq a,\ \kappa > 1),$$

then both the double integrals

(4) $$\int_a^\infty dt \int_g^\infty f(\beta, t)\, d\beta, \qquad \int_g^\infty d\beta \int_a^\infty f(\beta, t)\, dt$$

converge and their values are equal.

Proof. In the first of the integrals (4) the single improper integral

(5) $$\int_g^\infty f(\beta, t)\, d\beta,$$

when considered for any special value of $t \geqq a$, converges absolutely, since, for such a value of t we may write as a result of (*a*) and (*b*),

$$|f(\beta, t)| \leqq \phi(\beta) \qquad (\beta \geqq g),$$

where $\phi(\beta)$ is continuous and such that the integral

$$\int_g^\infty \phi(\beta)\, d\beta$$

exists. Moreover, we shall now show that the integral (5) defines a *continuous* function $F_1(t)$ of t when $t \geqq a$ such that the limiting integral

(6) $$\int_a^\infty F_1(t)\, dt$$

exists, thus demonstrating that the first of the double integrals (4) has a meaning.

Now, the improper integral (5) will define a function of t, continuous when

$t \geqq a$, provided that for all such values of t the same integral converges *uniformly*,* i. e., an arbitrarily small positive quantity ϵ having been chosen, it shall be possible to find a positive quantity G (independent of t) such that whenever $t \geqq a$ we may write

$$(7) \qquad \left| \int_G^s f(\beta, t)\, d\beta \right| < \epsilon \qquad (s > G).$$

That such a relation exists appears as follows: Having chosen ϵ, let us take G, as we may do by virtue of (b), so large that

$$(8) \qquad \int_G^s \phi(\beta)\, d\beta < \epsilon \qquad (s > G).$$

Then availing ourselves of (a) we may write for $t \geqq a$

$$(9) \qquad \left| \int_G^s f(\beta, t)\, d\beta \right| \leqq \int_G^s |f(\beta, t)|\, d\beta \leqq \int_G^s \phi(\beta) d\beta < \epsilon \qquad (s > G).$$

But if the function

$$(10) \qquad F_1(t) = \int_g^\infty f(\beta, t)\, d\beta$$

is continuous when $t \geqq a$, then for any special value $r > a$ the integral

$$\int_a^r F_1(t)\, dt$$

will exist.† Moreover if we consider only those values of t for which $t \geqq \tau$ we may write, as a result of (c),

$$|F_1(t)| \leqq \int_g^\infty |f(\beta, t)|\, d\beta \leqq \frac{1}{t^\kappa} \int_g^\infty \phi(\beta)\, d\beta,$$

i. e., when $t \geqq \tau$ we shall have

$$|F_1(t)| \leqq \frac{c}{t^\kappa} \qquad c = \int_g^\infty \phi(\beta)\, d\beta \qquad (\kappa > 1).$$

Whence, the integral (6) converges absolutely.

Having shown that the first of the integrals (4) has a meaning, we are to show that the same is true of the second of these integrals and that the two limits thus defined are equal. Moreover, these further statements will now follow, as we know from a well known theorem,‡ under the following conditions:§

* Cf. Osgood, Annals of Mathematics, ser. 2, vol. 3 (1902), p. 132 (Theorem A).

† Cf. Picard, *Traité d' analyse*, vol. 1, chap. 1.

‡ Cf. Osgood, l. c., p. 135.

§ It will be recalled that the function $f(\beta, t)$ is continuous when $t \geqq a$, $\beta \geqq g$, thus fulfilling the preliminary conditions demanded by the theorem in question.

1° The integrals

$$\int_g^\infty f(\beta, t)\,d\beta, \qquad \int_a^\infty f(\beta, t)\,dt$$

shall respectively converge uniformly throughout every fixed interval $a \leqq t \leqq b$, $g \leqq \beta \leqq h$.

2° The integral

$$\int_a^t dt \int_g^\infty f(\beta, t)\,d\beta$$

shall converge uniformly throughout the unlimited interval $t \geqq a$.

In other words, in order to complete the proof it suffices to establish the following three relations:

An arbitrarily small positive quantity ϵ having been chosen, it is possible to determine

(A) a positive quantity G_1 (independent of t) such that, b being any fixed quantity $> a$, we may write

$$\left| \int_{G_1}^s f(\beta, t)\,d\beta \right| < \epsilon \qquad (a \leqq t \leqq b,\ s > G_1).$$

(B) a positive quantity G_2 (independent of β) such that, h being any fixed quantity $> g$, we may write

$$\left| \int_{G_2}^r f(\beta, t)\,dt \right| < \epsilon, \qquad (g \leqq \beta \leqq h,\ r > G_2).$$

(C) a positive quantity G_3 (independent of t) such that we may write

$$\left| \int_a^t dt \int_{G_3}^s f(\beta, t)\,d\beta \right| < \epsilon \qquad (t \geqq a,\ s > G_3).$$

We proceed to establish these relations in the order here indicated.

The fact that relation (A) exists follows *a fortiori* from the existence already shown of relation (7). In fact, if relation (7) holds for the unlimited interval $t \geqq a$ it must hold for the portion $a \leqq t \leqq b$ of this interval, so that for the establishment of (A) it suffices to take $G_1 = G$ where G is defined as in (8).

As to relation (B), let us choose G_2 in the following manner: Representing by M a quantity as great as the greatest value taken by $\phi(\beta)$ when $g \leqq \beta \leqq h$,* let us choose $G_2 > \tau$ and such that for the given value of $\kappa\,(\kappa > 1)$ we shall have

$$\int_{G_2}^r \frac{dt}{t^\kappa} < \frac{\epsilon}{M} \qquad (r > G_2).$$

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*Since $\phi(\beta)$ is continuous by hypothesis when $\beta \geqq g$, we may evidently take for M the upper limit of $\phi(\beta)$ for the interval $g \leqq \beta \leqq h$. For an arithmetic proof of the existence of such an upper limit, see STOLZ, *Allgemeine Arithmetik*, vol. I, chap. 9, § 1.

Then, by virtue of (c) we may write for all values of β such that $g \leqq \beta \leqq h$,

$$\left|\int_{G_2}^{r} f(\beta, t)\,dt\right| \leqq \int_{G_2}^{r} |f(\beta, t)|\,dt \leqq \phi(\beta)\int_{G_2}^{r}\frac{dt}{t^{\kappa}} < M\int_{G_2}^{r}\frac{dt}{t^{\kappa}} < \epsilon \qquad (r > G_2),$$

and this suffices for the proof of the relation in question.

In order to establish relation (C) we first observe that the quantity τ may, without further restrictions in our hypotheses, be considered positive, so that the quantity $\tau^{1-\kappa}(\kappa > 1)$ may in any case be considered a positive constant. This premised, let us now choose G_3, as we may do by virtue of (b), so that

$$\int_{G_3}^{s}\phi(\beta)\,d\beta < \frac{\epsilon}{2c} \qquad \left(c = \tau - a + 2\frac{\tau^{1-\kappa}}{\kappa - 1},\ s > G_3\right),$$

With this choice of G_3 let us first consider only those values of t such that $a \leqq t \leqq \tau$. Utilizing (a) we may then write

$$\left|\int_a^t dt\int_{G_3}^{s} f(\beta, t)\,d\beta\right| \leqq \int_a^t dt\int_{G_3}^{s}|f(\beta, t)|\,d\beta \leqq \int_a^{\tau} dt\int_{G_3}^{s}\phi(\beta)\,d\beta$$

$$= (\tau - a)\int_{G_3}^{s}\phi(\beta)\,d\beta < c\int_{G_3}^{s}\phi(\beta)\,d\beta < \frac{\epsilon}{2},$$

i. e., for our choice of G_3 we have the following relation

$$\left|\int_a^t dt\int_{G_3}^{s} f(\beta, t)\,d\beta\right| < \frac{\epsilon}{2} < \epsilon \qquad (a \leqq t \leqq \tau,\ s > G_3). \tag{11}$$

Secondly, for those values of $t > \tau$ we now have

$$\left|\int_a^t dt\int_{G_3}^{s} f(\beta, t)\,d\beta\right| \leqq \left|\int_a^{\tau} dt\int_{G_3}^{s} f(\beta, t)\,d\beta\right| + \left|\int_{\tau}^{t} dt\int_{G_3}^{s} f(\beta, t)\,d\beta\right|, \tag{12}$$

in which the first term of the second member is less than $\epsilon/2$ [see (11)], while the same is true also of the second term, since by virtue of (c) we may write

$$\left|\int_{\tau}^{t} dt\int_{G_3}^{s} f(\beta, t)\,d\beta\right| \leqq \int_{\tau}^{t} dt\int_{G_3}^{s}|f(\beta, t)|\,d\beta \leqq \int_{\tau}^{t}\frac{dt}{t^{\kappa}}\int_{G_3}^{s}\phi(\beta)\,d\beta$$

$$= \frac{1}{\kappa - 1}(\tau^{1-\kappa} - t^{1-\kappa})\int_{G_3}^{s}\phi(\beta)\,d\beta < \frac{2\tau^{1-\kappa}}{\kappa - 1}\int_{G_3}^{s}\phi(\beta)\,d\beta < c\int_{G_3}^{s}\phi(\beta)\,d\beta < \frac{\epsilon}{2},$$

i. e., for our choice of G_3 relation (12) yields the following:

$$\left|\int_a^t dt\int_{G_3}^{s} f(\beta, t)\,d\beta\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad (t > \tau,\ s > G_3). \tag{13}$$

In summary, then, relations (11) and (13) give (C) as desired.

Proof of the Theorem.

3. The preceding relations having been noted, we proceed with the proof of the theorem stated in § 1.

For this purpose let us write the series (1) in the form

$$(14)\quad a(0,0)+\sum_1^\infty a(m,0)x^m+\sum_1^\infty a(0,n)y^n+\sum_1^\infty\sum_1^\infty a(m,n)x^m y^n.$$

Then, as a result of the hypothesis concerning the function $a(u,v)$ we may at once apply a theorem established elsewhere* to each single series here appearing. Thus the second and third terms of expression (14) may be replaced respectively by the second and third terms in the right hand member of (3), so that for the proof of the theorem of § 1 we shall here need to consider merely the function defined by the double series

$$(15)\qquad \sum_1^\infty\sum_1^\infty a(m,n)x^m y^n,$$

showing that this function is equal to the last term of (3).

For this purpose we use relation (V) of § 2, choosing in the present instance, in the notation of that relation,

$$A_1 = \textit{the positive half of the } u \textit{ plane},$$

$$A_2 = \textit{the positive half of the } v \textit{ plane},$$

$$\Phi(u,v)=\pi^2 a(u,v)(-x)^u(-y)^v,$$

$$U(u,v)=\sin\pi u\sin\pi v,$$

and taking as the contours C_m, C_n the rectangles formed by the straight lines

$$(C_m)\qquad u=\tfrac{1}{2}+i\beta,\qquad u=\alpha\pm ig,\qquad u=\tfrac{1}{2}+2p+i\beta,$$

$$(C_n)\qquad v=\tfrac{1}{2}+i\delta,\qquad v=\gamma\pm ih,\qquad v=\tfrac{1}{2}+2q+i\delta,$$

in which g and h are any two positive quantities, while p and q are any two positive integers.

As a result of the relation in question we arrive at the following equation

$$\sum_1^{2p}\sum_1^{2q} a(m,n)x^m y^n = -\frac{1}{4\pi^2}\int_{C_m} du\int_{C_n}\frac{\Phi(u,v)}{U(u,v)}dv$$

$$= -\tfrac{1}{4}\int_{C_m} du\int_{C_n} a(u,v)\frac{(-x)^u(-y)^v}{\sin\pi u\sin\pi v}dv;$$

or, placing

$$(16)\qquad \Psi(u,v)=-\frac{a(u,v)(-x)^u(-y)^v}{4\sin\pi u\sin\pi v},$$

*Cf. FORD, Journal de Mathématiques, ser. 5, vol. 9 (1903), p. 223.

we have

$$\sum_{1}^{2p}\sum_{1}^{2q} a(m, n)x^m y^n = \int_{C_m} du \int_{C_n} \Psi(u, v)\, dv. \tag{17}$$

We proceed to study in detail the second member of equation (17), availing ourselves of the hypotheses and of equation (16). By breaking up the integral

$$\int_{C_n} \Psi(u, v)\, dv$$

into the four component integrals which come respectively from the integration over the four sides $v = \gamma - ih$, $v = \frac{1}{2} + 2q + i\delta$, $v = \gamma + ih$, $v = \frac{1}{2} + i\delta$ of the rectangle C_n, the second member of equation (17) takes the form

$$\int_{C_m}\Big(\int_{\frac{1}{2}}^{\frac{1}{2}+2q} \Psi(u, \gamma - ih)\, d\gamma + i\int_{-h}^{h} \Psi(u, \tfrac{1}{2} + 2q + i\delta)\, d\delta$$
$$+ \int_{\frac{1}{2}+2q}^{\frac{1}{2}} \Psi(u, \gamma + ih)\, d\gamma + \int_{h}^{-h} \Psi(u, \tfrac{1}{2} + i\delta)\, d\delta\Big)\, du.$$

Similarly, by breaking up the integrations with respect to u in each of the four expressions just obtained the second member of (17) becomes expressible finally as the sum of sixteen double definite (proper) integrals, i. e., takes the form

$$\begin{aligned}
&\int_{\frac{1}{2}}^{\frac{1}{2}+2p} d\alpha \int_{\frac{1}{2}}^{\frac{1}{2}+2q} \Psi(\alpha - ig, \gamma - ih)\, d\gamma + i\int_{-g}^{g} d\beta \int_{\frac{1}{2}}^{\frac{1}{2}+2q} \Psi(\tfrac{1}{2} + 2p + i\beta, \gamma + ih)\, d\gamma \\
&+ \int_{\frac{1}{2}+2p}^{\frac{1}{2}} d\alpha \int_{\frac{1}{2}}^{\frac{1}{2}+2q} \Psi(\alpha + ig, \gamma - ih)\, d\gamma + i\int_{g}^{-g} d\beta \int_{\frac{1}{2}}^{\frac{1}{2}+2q} \Psi(\tfrac{1}{2} + i\beta, \gamma - ih)\, d\gamma \\
&+ i\int_{\frac{1}{2}}^{\frac{1}{2}+2p} d\alpha \int_{-h}^{h} \Psi(\alpha - ig, \tfrac{1}{2} + 2q + i\delta)\, d\delta - \int_{-g}^{g} d\beta \int_{-h}^{h} \Psi(\tfrac{1}{2} + 2p + i\beta, \tfrac{1}{2} + 2q + i\delta)\, d\delta \\
&+ i\int_{\frac{1}{2}+2p}^{\frac{1}{2}} d\alpha \int_{-h}^{h} \Psi(\alpha + ig, \tfrac{1}{2} + 2q + i\delta)\, d\delta - \int_{g}^{-g} d\beta \int_{-h}^{h} \Psi(\tfrac{1}{2} + i\beta, \tfrac{1}{2} + 2q + i\delta)\, d\delta \\
&+ \int_{\frac{1}{2}}^{\frac{1}{2}+2p} d\alpha \int_{\frac{1}{2}+2q}^{\frac{1}{2}} \Psi(\alpha - ig, \gamma + ih)\, d\gamma + i\int_{-g}^{g} d\beta \int_{\frac{1}{2}+2q}^{\frac{1}{2}} \Psi(\tfrac{1}{2} + 2p + i\beta, \gamma + ih)\, d\gamma \\
&+ \int_{\frac{1}{2}+2p}^{\frac{1}{2}} d\alpha \int_{\frac{1}{2}+2q}^{\frac{1}{2}} \Psi(\alpha + ig, \gamma + ih)\, d\gamma + i\int_{g}^{-g} d\beta \int_{\frac{1}{2}+2q}^{\frac{1}{2}} \Psi(\tfrac{1}{2} + i\beta, \gamma + ih)\, d\gamma \\
&+ i\int_{\frac{1}{2}}^{\frac{1}{2}+2p} d\alpha \int_{h}^{-h} \Psi(\alpha + ig, \tfrac{1}{2} + i\delta)\, d\delta - \int_{-g}^{g} d\beta \int_{h}^{-h} \Psi(\tfrac{1}{2} + 2p + i\beta, \tfrac{1}{2} + i\delta)\, d\delta \\
&+ i\int_{\frac{1}{2}}^{\frac{1}{2}+2p} d\alpha \int_{h}^{-h} \Psi(\alpha - ig, \tfrac{1}{2} + i\delta)\, d\delta - \int_{g}^{-g} d\beta \int_{h}^{-h} \Psi(\tfrac{1}{2} + i\beta, \tfrac{1}{2} + i\delta)\, d\delta.
\end{aligned} \tag{18}$$

We shall now examine each of these double integrals in detail.

From the definition of $\Psi(u, v)$ given by (16), together with relation (I) of §1, we have for all (real) values of $\alpha, \beta, \gamma, \delta (\alpha > 0, \gamma > 0)$.

$$(19)\quad \Psi(\alpha + i\beta, \gamma + i\delta) = (-x)^\alpha (-y)^\gamma a(\alpha + i\beta, \gamma + i\delta) \frac{e^{i\beta[\log\rho + i(\phi+\pi)]} e^{i\delta[\log\sigma + i(\psi+\pi)]}}{-4\sin\pi(\alpha + i\beta)\sin\pi(\gamma + i\delta)},$$

where $\phi = \arg x$, $\psi = \arg y$, $\rho = \operatorname{mod} x$, $\sigma = \operatorname{mod} y$. Whence, by use of (II), the first term of (18) becomes

$$I_1 = \frac{e^{g(\phi+\pi)} e^{-ig\log\rho} e^{h(\psi+\pi)} e^{-ih\log\sigma}}{-4\sinh\pi g \sinh\pi h} \int_{\frac{1}{2}}^{\frac{1}{2}+2p} \frac{(-x)^\alpha d\alpha}{\Omega(\alpha, -g)} \int_{\frac{1}{2}}^{\frac{1}{2}+2q} \frac{a(\alpha - ig, \gamma - ih)(-y)^\gamma}{\Omega(\gamma, -h)} d\gamma.$$

Again, using (2) and (III), we now obtain

$$|I_1| < c\rho^{\frac{1}{2}+2p}\sigma^{\frac{1}{2}+2q} \frac{e^{g(\phi+\pi)} e^{h(\psi+\pi)}}{4\sinh\pi g \sinh\pi h} \int_{\frac{1}{2}}^{\frac{1}{2}+2p} d\alpha \int_{\frac{1}{2}}^{\frac{1}{2}+2q} d\gamma.$$

Whence, if neither x nor y is real and positive, we have but to recall that our values of $\phi = \arg x$ and $\psi = \arg y$ are always taken such that for the given point x, y we have $\phi < 0$, $\psi < 0$ in order to write for the point x, y, in question

$$\lim_{g=\infty,\, h=\infty} I_1 = 0.$$

Similarly, if we bear in mind throughout the restrictions

$$(20)\qquad -2\pi < \phi < 0, \qquad -2\pi < \psi < 0,$$

it appears by means of (II), (III) and (2) that the third, ninth and eleventh terms of expression (18) approach the limit zero for the given values of x, y when $g = \infty$, $h = \infty$.

We pass on, then, to consider the second term of (18). By use of (19) and the relation

$$(21)\quad \sin\pi(\tfrac{1}{2} + 2n + i\beta) = \sin\left(\frac{\pi}{2} + 2n\pi + i\pi\beta\right) = \cos i\pi\beta = \cosh\pi\beta \qquad (n = 0, 1, 2, 3, \cdots),$$

this term becomes

$$I_2 = \frac{e^{h(\psi+\pi)} e^{-ih\log\rho}(-x)^{\frac{1}{2}+2p}}{-4i\sinh\pi h} \int_{-g}^{g} \frac{e^{i\beta[\log\rho + i(\phi+\pi)]}}{\cosh\pi\beta} d\beta \int_{\frac{1}{2}}^{\frac{1}{2}+2q} a(\tfrac{1}{2} + 2p + i\beta, \gamma - ih) \frac{(-y)^\gamma}{\Omega(\gamma, -h)} d\gamma.$$

Whence, again using (III) and (2), we may write

$$|I_2| < c\rho^{\frac{1}{2}+2p}\sigma^{\frac{1}{2}+2q} \frac{e^{h(\psi+\pi)}}{4\sinh\pi h} \int_{-g}^{g} \frac{e^{-(\phi+\pi)\beta}}{\cosh\pi\beta} d\beta \int_{\frac{1}{2}}^{\frac{1}{2}+2q} d\gamma,$$

so that for our point x, y we have as before

$$\lim_{g=\infty,\, h=\infty} I_2 = 0.*$$

Similarly, the same fact appears for the fourth, fifth, seventh, tenth, twelfth, thirteenth and fifteenth terms of (18).

If then we confine ourselves to values x, y such as indicated, we may replace expression (18) by the sum of those of its terms which we have not as yet considered in detail — viz.: the sixth, eighth, fourteenth and sixteenth — it being understood, however, that we now take as limits of integration in these terms $g=\infty$, $h=\infty$. In other words, expression (18) may be written in the following form involving improper double integrals:

$$\begin{aligned}(22)\quad & -\int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} \Psi(\tfrac{1}{2}+2p+i\beta,\, \tfrac{1}{2}+2q+i\delta)d\delta + \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} \Psi(\tfrac{1}{2}+i\beta,\, \tfrac{1}{2}+2q+i\delta)d\delta \\ & + \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} \Psi(\tfrac{1}{2}+2p+i\beta,\, \tfrac{1}{2}+i\delta)d\delta - \int_{-\infty}^{\infty} d\beta \int_{-\infty}^{\infty} \Psi(\tfrac{1}{2}+i\beta,\, \tfrac{1}{2}+i\delta)d\delta.\end{aligned}$$

As before we shall now examine each of these double integrals in detail.

In the first place, in order to be assured that the first term of (22) has a meaning we should show † that each of the terms of the following expression:

$$\begin{aligned}(23)\quad & -\int_{0}^{\infty} d\beta \int_{0}^{\infty} \Psi(\tfrac{1}{2}+2p+i\beta,\, \tfrac{1}{2}+2q+i\delta)d\delta - \int_{-\infty}^{0} d\beta \int_{0}^{\infty} \Psi(\tfrac{1}{2}+2p+i\beta,\, \tfrac{1}{2}+2q+i\delta)d\delta \\ & -\int_{0}^{\infty} d\beta \int_{-\infty}^{0} \Psi(\tfrac{1}{2}+2p+i\beta,\, \tfrac{1}{2}+2q+i\delta)d\delta - \int_{-\infty}^{0} d\beta \int_{-\infty}^{0} \Psi(\tfrac{1}{2}+2p+i\beta,\, \tfrac{1}{2}+2q+i\delta)d\delta,\end{aligned}$$

has a meaning

Confining our attention at first to the first term of (23) we shall now show that in case the usual conditions (20) are satisfied by x, y this term has a meaning, thus showing that the same term defines a function of x, y throughout the entire x, y planes, with the exception of points on the positive real axes of x and y. Moreover, we shall establish this by means of the lemma of § 2, so that it will also result from our investigations that the order of integration may be inverted in this integral if desired.

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* It will be recalled that owing to relations (20) the expression

$$\lim_{g=\infty} \int_{-g}^{g} \frac{e^{-(\phi+\pi)\beta}}{\cosh \pi\beta}\, d\beta = \int_{-\infty}^{0} \frac{e^{-(\phi+\pi)\beta}}{\cosh \pi\beta}\, d\beta + \int_{0}^{\infty} \frac{e^{-(\phi+\pi)\beta}}{\cosh \pi\beta}\, d\beta$$

has a meaning.

† Cf. STOLZ, *Differential- und Integralrechnung*, vol. I, chap. 10, § 9.

According to the lemma of § 2 we may say that the first term of (23) will have a meaning under the following conditions: *

$$(a) \quad |\Psi(\tfrac{1}{2}+2p+i\beta, \tfrac{1}{2}+2q+i\delta)| \leqq L(\beta) \quad [\beta \geqq 0,\ \delta \geqq 0,\ L(\beta) \text{ continuous}],$$

$$(b) \qquad \int_0^\infty L(\beta)\,d\beta \text{ converges},$$

$$(c) \qquad \delta^\kappa |\Psi(\tfrac{1}{2}+2p+i\beta, \tfrac{1}{2}+2q+i\delta)| \leqq L(\beta) \quad (\beta \geqq 0,\ \delta \geqq \delta_1 \geqq 0,\ \kappa > 1).$$

Now from (19) and (21) we have

$$(24) \quad \begin{aligned} \Psi(\tfrac{1}{2}+2p+i\beta, \tfrac{1}{2}+2q+i\delta) &= -a(\tfrac{1}{2}+2p \\ &\quad + i\beta, \tfrac{1}{2}+2q+i\delta)\frac{(-x)^{\frac{1}{2}+2p}(-y)^{\frac{1}{2}+2q}e^{i\beta[\log\rho+i(\phi+\pi)]}e^{i\delta[\log\sigma+i(\psi+\pi)]}}{4\cosh\pi\beta\cosh\pi\delta}. \end{aligned}$$

Whence, using (2), we obtain for all values (real) of β and δ and for any set of values for p and q taken from the sequences $p=0, 1, 2, 3, \cdots$; $q=0, 1, 2, 3, \cdots$ the relation

$$(25) \quad |\Psi(\tfrac{1}{2}+2p+i\beta, \tfrac{1}{2}+2q+i\delta)| < \frac{c}{4}\rho^{\frac{1}{2}+2p}\sigma^{\frac{1}{2}+2q}\frac{e^{-(\phi+\pi)\beta}e^{-(\psi+\pi)\delta}}{\cosh\pi\beta\cosh\pi\delta}.$$

But, if M_ψ represents a positive quantity as large as the largest value taken by $e^{-(\psi+\pi)\delta}/\cosh\pi\delta\,(\psi > -2\pi)$ when $\delta \geqq 0$ † we obtain from (25) the relation

$$|\Psi(\tfrac{1}{2}+2p+i\beta, \tfrac{1}{2}+2q+i\delta)| < L(\beta) = M_\psi\frac{c\rho^{\frac{1}{2}+2p}\sigma^{\frac{1}{2}+2q}}{4}\frac{e^{-(\phi+\pi)\beta}}{\cosh\pi\beta} \qquad (\beta \geqq 0,\ \delta \geqq 0)$$

and thus condition (a) becomes satisfied. Moreover, by virtue of the relation $\phi > -2\pi$ condition (b) becomes satisfied also. As to condition (c), if we represent by δ_1 a positive quantity so large that for $\delta \geqq \delta_1$ we have

$$\delta^2\frac{e^{-(\psi+\pi)\delta}}{\cosh\pi\delta} < M_\psi \qquad (M_\psi \text{ defined as above})$$

we obtain from (25) the relation

$$\delta^2|\Psi(\tfrac{1}{2}+2p+i\beta, \tfrac{1}{2}+2q+i\delta)| < M_\psi\frac{c\rho^{\frac{1}{2}+2p}\sigma^{\frac{1}{2}+2q}}{4}\frac{e^{-(\phi+\pi)\beta}}{\cosh\pi\beta} = L(\beta) \qquad (\beta \geqq 0,\ \delta \geqq \delta_1),$$

so that this condition is also satisfied by virtue of the relation $\psi > -2\pi$.

Similarly, by availing ourselves of the lemma of § 2 and of the relations (20)

* We observe that the function $\Psi(\frac{1}{2}+2p+i\beta, \frac{1}{2}+2q+i\delta)$ is continuous when $\beta \geqq 0$, $\delta \geqq 0$ [see (19)]. Also, it is to be understood throughout the discussion immediately following that fixed values have been assigned to x, y, p and q; hence also to ρ, σ, ϕ and ψ.

† It will be recalled here that $\cosh\pi\delta = \frac{1}{2}(e^{\pi\delta}+e^{-\pi\delta})$ so that for the given value of ψ the number M_ψ certainly exists.

and (25) we see that conditions (a), (b) and (c) are satisfied by each of the remaining three terms of (23).

The first term of (22) therefore has a meaning for the point x, y in question. Moreover, the same can now be said of the remaining three terms of (22), since the reasoning which we have already given holds for any values of p and q taken from the sequences $p = 0, 1, 2, \cdots$; $q = 0, 1, 2, \cdots$, and the second, third and fourth terms of (22) correspond respectively to the sets $(p = 0, q = q)$, $(p = p, q = 0)$, $(p = 0, q = 0)$.

The existence of the four terms of expression (22) having been shown, it also follows directly from relation (25) that if we make the further restrictions $|x| = \rho < 1$, $|y| = \sigma < 1$, then the first term of (22) will approach the limit zero when either

$$(26)\qquad \begin{cases} 1)\ p \textit{ increases indefinitely, } q \textit{ remaining constant;} \\ 2)\ q \textit{ increases indefinitely, } p \textit{ remaining constant;} \\ 3)\ p \textit{ and } q \textit{ increase indefinitely in any manner.} \end{cases}$$

In fact, writing as before this term in the form (23), we see from (25) that the first term of (23) is less in absolute value than

$$(27)\qquad \frac{c}{4}\rho^{\frac{1}{2}+2p}\sigma^{\frac{1}{2}+2q}\int_0^\infty \frac{e^{-(\phi+\pi)\beta}}{\cosh \pi\beta}\,d\beta \int_0^\infty \frac{e^{-(\psi+\pi)\delta}}{\cosh \pi\delta}\,d\delta,$$

in which each improper integral has a meaning by virtue of relations (20). Moreover, since $\rho < 1$, $\sigma < 1$, the expression (27) is seen to approach zero when either one of the conditions (26) is satisfied, whence the same is true of the term in question, i. e., the first term of (23). Likewise, the same properties are seen to be possessed by the second, third and fourth terms of (23) and hence by the first term of (22).

Moreover, since the second term of (22) is the special case arising from its first term when p has the constant value zero, it follows from what we have just seen that under the same conditions for x, y this second term approaches the limit zero when $q = \infty$. Likewise, the third term of (22) which is obtained from its first term by placing therein $q = constant = 0$ approaches the limit zero when $= \infty$.

In summary then we may write, whenever $|x| < 1$, $|y| < 1$ (positive real values excluded),

$$\sum_1^\infty \sum_1^\infty a(m, n)x^m y^n = -\int_{-\infty}^\infty d\beta \int_{-\infty}^\infty \Psi(\tfrac{1}{2} + i\beta, \tfrac{1}{2} + i\delta)\,d\delta = I(x, y),$$

where

$$(28)\qquad I(x, y) = \frac{1}{4}\int_{-\infty}^\infty d\beta \int_{-\infty}^\infty a(\tfrac{1}{2} + i\beta, \tfrac{1}{2} + i\delta)\frac{(-x)^{\frac{1}{2}+i\beta}(-y)^{\frac{1}{2}+i\delta}}{\cosh \pi\beta \cosh \pi\delta}\,d\delta.$$

Moreover, we have now shown that this expression (28) has a meaning for all

values of x, y, provided merely that we choose $\phi = \arg x$, $\psi = \arg y$ so that relations (20) exist, and that we may, if desired, invert the order of integration.

We proceed to show that the function $I(x, y)$ thus defined not only has a meaning throughout any region of the x, y planes in which relations (20) exist, but that in these larger regions it is a *single valued analytic* function of the independent variables x, y thus completing the proof of the theorem in question.

That the function $I(x, y)$ is singled valued within such a region follows from the fact that the functions [see (I)]

$$(-x)^{\frac{1}{2}+i\beta} = \sqrt{-x}e^{-\beta(\phi+\pi)}e^{i\beta\log\rho}, \qquad (-y)^{\frac{1}{2}+i\delta} = \sqrt{-y}e^{-\delta(\psi+\pi)}e^{i\delta\log\sigma} \qquad (\beta,\ \delta \text{ real})$$

are single valued in the region.

In order to prove the analytic nature of $I(x, y)$ we begin by writing

$$I(x, y) = -\int_0^\infty d\beta \int_0^\infty \Psi(\tfrac{1}{2}+i\beta, \tfrac{1}{2}+i\delta)\,d\delta - \int_{-\infty}^0 d\beta \int_0^\infty \Psi(\tfrac{1}{2}+i\beta, \tfrac{1}{2}+i\delta)\,d\delta$$

$$(29) \qquad -\int_0^\infty d\beta \int_{-\infty}^0 \Psi(\tfrac{1}{2}+i\beta, \tfrac{1}{2}+i\delta)\,d\delta - \int_{-\infty}^0 d\beta \int_{-\infty}^0 \Psi(\tfrac{1}{2}+i\beta, \tfrac{1}{2}+i\delta)\,d\delta.$$

Turning to the first term of this expression let us write

$$(30) \qquad \int_0^\infty d\beta \int_0^\infty \Psi(\tfrac{1}{2}+i\beta, \tfrac{1}{2}+i\delta)\,d\delta = \int_0^\infty F_1(\beta, x, y)\,d\beta,$$

where

$$(31) \qquad F_1(\beta, x, y) = \int_0^\infty \Psi(\tfrac{1}{2}+i\beta, \tfrac{1}{2}+i\delta)\,d\delta.$$

Now, in order that expression (30) shall represent a function of x, y analytic in the region mentioned above it suffices to show

1) that for any special value of $\beta \geqq 0$ the integral (31) converges *uniformly* for all values of x, y lying within two arbitrary regions T_1, T_2 of the x, y planes respectively, which regions do not cut (or touch) the positive portions of the real axes of x, y.

2) that for the same values of x, y the integral

$$(32) \qquad \int_0^\infty F_1(\beta, x, y)\,d\beta.$$

also converges *uniformly*.

In fact, under these circumstances the function $F_1(\beta, x, y)$, $(\beta \geqq 0)$ will be analytic throughout T_1, T_2 and likewise the same will be true of the function (32) or (30)*.

* Cf. Osgood, *Encyklopädie*, vol. 2, B 1, § 43.

Now, for any special point x, y in T_1, T_2 we have by placing $p = q = 0$ in (25)

$$(33) \qquad |\Psi(\tfrac{1}{2} + i\beta, \tfrac{1}{2} + i\delta)| < \frac{c}{4} \rho^{\frac{1}{2}} \sigma^{\frac{1}{2}} \frac{e^{-(\phi+\pi)\beta}}{\cosh \pi\beta} \frac{e^{-(\psi+\pi)\delta}}{\cosh \pi\delta} \qquad (\beta, \delta \text{ real}).$$

Moreover, relations (20) are satisfied for every such point x, y throughout T_1, T_2 inclusive of boundary values.

Whence, if we represent by ρ_0, σ_0 respectively, the greatest values taken by ρ, σ, upon the boundaries of T_1, T_2 and by $\bar{\phi}$ and $\bar{\psi}$, respectively, the least values taken by ϕ and ψ upon the same boundaries, we may write [see (33)] for all points x, y in T_1, T_2

$$(34) \quad |\Psi(\tfrac{1}{2} + i\beta, \tfrac{1}{2} + i\delta)| < \frac{c}{4} \sqrt{\rho_0 \sigma_0} \frac{e^{-(\bar{\phi}+\pi)\beta}}{\cosh \pi\beta} \frac{e^{-(\bar{\psi}+\pi)\delta}}{\cosh \pi\delta}, \quad \begin{pmatrix} \beta \geqq 0, & -2\pi < \bar{\phi} < 0 \\ \delta \geqq 0, & -2\pi < \bar{\psi} < 0 \end{pmatrix}.$$

Whence we see that having chosen an arbitrarily small positive quantity ϵ, we may determine a quantity G_1 (dependent upon ϵ and β), so large that for any special value of $\beta \geqq 0$ we may write for all points x, y in T_1, T_2.

$$\int_{G_1}^{s} |\Psi(\tfrac{1}{2} + i\beta, \tfrac{1}{2} + i\delta)| d\delta < \frac{c}{4} \sqrt{\rho_0 \sigma_0} \frac{e^{-(\bar{\phi}+\pi)\beta}}{\cosh \pi\beta} \int_{G_1}^{s} \frac{e^{-(\bar{\psi}+\pi)\delta}}{\cosh \pi\delta} d\delta < \epsilon \qquad (s > G_1),$$

i. e., condition 1) becomes satisfied by the first term of (29).

Moreover, from (34) we now have for the same points x, y

$$|F_1(\beta, x, y)| \leqq \int_0^\infty |\Psi(\tfrac{1}{2} + i\beta, \tfrac{1}{2} + i\delta)| d\delta < \frac{c}{4} \sqrt{\rho_0 \sigma_0} \frac{e^{-(\bar{\phi}+\pi)\beta}}{\cosh \pi\beta} \int_0^\infty \frac{e^{-(\bar{\psi}+\pi)\delta}}{\cosh \pi\delta} d\delta.$$

Whence, for any positive quantity G_2, we may write under the same conditions for x, y

$$(35) \quad \left| \int_{G_2}^{r} F_1(\beta, x, y) d\beta \right| < \frac{c}{4} \sqrt{\rho_0 \sigma_0} \int_{G_2}^{r} \frac{e^{-(\bar{\phi}+\pi)\beta}}{\cosh \pi\beta} d\beta \int_0^\infty \frac{e^{-(\bar{\psi}+\pi)\delta}}{\cosh \pi\delta} d\delta \quad (r > G_2).$$

But, since relations (20) hold in particular when $\phi = \bar{\phi}$, while the improper integral here appearing has a meaning by virtue of the relation $\bar{\psi} < -2\pi$, we may take G_2 so large that the second member of (35) will become (and remain) less than any preassigned positive quantity η (independent of x and y); i. e., condition 2° becomes satisfied by the first term of (29).

Thus, the first term of expression (29) has the analytic properties desired and, by availing ourselves again of relations (33) and (20), the same result is seen to be true of the three remaining terms of (29) and therefore of the expression $I(x, y)$ defined in (28).

ANN ARBOR,
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