A FIFTH NECESSARY CONDITION

FOR A STRONG EXTREMUM OF THE INTEGRAL

$$\int_{x_0}^{x_1} F(x, y, y')\,dx^*$$

BY

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In a previous article† I called attention to the fact that *the conditions of Euler, Legendre, Jacobi and Weierstrass are not sufficient* for a strong extremum of the definite integral

$$J = \int_{x_0}^{x_1} F(x, y, y')\,dx.$$

Hence the question of further necessary conditions arises, and the object of the present paper is to derive *a fifth necessary condition.*

§ 1. *Preliminary form of condition* (V).

The terminology, and the assumptions concerning the function $F(x, y, y')$ and the admissible curves being the same as in § 3, *c*) of my *Lectures on the Calculus of Variations*, let

$$\mathfrak{C}_0: \qquad y = f_0(x), \qquad x_0 \leqq x \leqq x_1,$$

be an extremal of class C' which passes through the two given points $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$ and which lies in the interior of the region **R** to which the admissible curves are confined. We suppose that for the curve $\mathfrak{C}_0$ the conditions of LEGENDRE and JACOBI are fulfilled in the somewhat stronger form

(II′) $$F_{y'y'}\big(x, f_0(x), f_0'(x)\big) > 0 \text{ in } (x_0 x_1),$$

(III′) $$x_1 < x_0',$$

x_0' being the conjugate value to x_0.

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† *Some Instructive Examples in the Calculus of Variations*, Bulletin of the American Mathematical Society, vol. 9 (1902), p. 9. Compare also my *Lectures on the Calculus of Variations* (Chicago, 1904), p. 99.

We proceed at first exactly as in one of the proofs* for the necessity of WEIERSTRASS' condition:

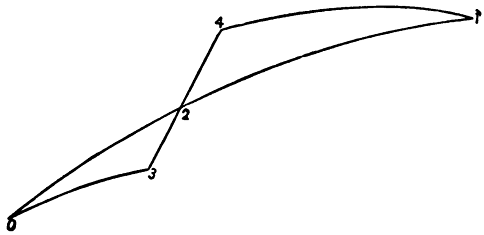
From (II′) and (III′) it follows that the set of extremals through the point P_0:

$$(1) \qquad y = \overset{0}{\phi}(x, \gamma)$$

furnishes an "improper" field† $\mathbf{S}$ about the arc $\mathfrak{E}_0$. Let now $P_2(x_2, y_2)$ be any point of $\mathfrak{E}_0$ to the right of P_0, i. e., $x_0 < x_2 \leqq x_1$, and take ρ so small that the neighborhood (ρ) of P_2 lies at the same time in the interior of the field $\mathbf{S}$ and of the region $\mathbf{R}$. Then if we choose $0 < h < \rho$, $|k| < \rho$ and denote by P_3 the point whose coördinates are

$$x_3 = x_2 - h, \qquad y_3 = y_2 - k,$$

there passes one and but one extremal P_0P_3 of the field through the point P_3. We draw the straight line P_3P_2 and vary the extremal $\mathfrak{E}_0$ by replacing the arc



P_0P_2 by the broken curve $P_0P_3P_2$. To this variation we may apply WEIERSTRASS' theorem and obtain, since the E-function vanishes along the extremal P_0P_3, for the total variation of J the expression

$$(2) \qquad \Delta J = \int_0^1 h\mathbf{E}\left(x, \bar{y};\, \overset{0}{p}(x, \bar{y}), \frac{k}{h}\right) dt,$$

where

$$x = x_2 - ht, \qquad \bar{y} = y_2 - kt,$$

and $\overset{0}{p}(x, \bar{y})$ denotes the slope at the point $(x, \bar{y})$ of the unique extremal of the field passing through the point $(x, \bar{y})$.

If we keep the line P_3P_2 (i. e., the ratio k/h) fixed and let the point P_3 approach the point P_2 along this line, we obtain WEIERSTRASS' condition. But if, on the contrary, we *revolve the line P_3P_2 about P_2 and let it approach the position parallel to the y-axis*, while the point P_3 moves on a line parallel to the x-axis (i. e., if we keep k fixed and let h approach zero), we obtain a new

* Compare E. R. HEDRICK, *On the sufficient conditions in the Calculus of Variations*, Bulletin of the American Mathematical Society, vol. 9 (1902), p. 14.

† Compare my *Lectures*, pp. 61 and 83, footnote 2; further KNESER, *Lehrbuch der Variationsrechnung*, § 27 and GOURSAT, *Cours d' Analyse*, vol. 2, p. 615, when the function F is analytic, and LUNN, in a paper which will be published in these Transactions, when F is not analytic.

necessary condition. For if the extremal $\mathfrak{E}_0$ furnishes a strong minimum for the integral J, the lower bound* of ΔJ for $h = +0$ must be positive or zero for all sufficiently small values of $|k|$.

The same reasoning can be applied to the set of extremals through the point P_1:

$$(3) \qquad y = \overset{1}{\phi}(x, \gamma)$$

and the variation $P_2 P_4 P_1$ (see figure). If we introduce the symbols

$$\epsilon_0 = -1, \qquad \epsilon_1 = +1,$$

the results of the two processes may be united in the one formula

$$(4) \qquad \underset{h=+0}{\mathbf{L}} \int_0^1 h\mathbf{E}\left(x, \bar{y};\, \overset{\lambda}{p}(x, \bar{y}), \frac{k}{h}\right) dt \geqq 0 \qquad (\lambda = 0, 1),$$

where

$$x = x_2 + \epsilon_\lambda ht, \qquad \bar{y} = y_2 + \epsilon_\lambda kt,$$

and $\overset{\lambda}{p}(x, \bar{y})$ denotes the slope at the point $(x, \bar{y})$ of the unique extremal passing through the points P_λ and $(x, \bar{y})$.

If we substitute in (4) for the $\mathbf{E}$-function its explicit expression

$$\mathbf{E}(x, y;\, p, \tilde{p}) = F(x, y, \tilde{p}) - F(x, y, p) - (\tilde{p} - p)F_{y'}(x, y, p),$$

the expression for ΔJ breaks up into four definite integrals two of which approach zero for $h = +0$, and we obtain the following *preliminary form of condition* (V):

$$(5) \qquad \underset{h=+0}{\mathbf{L}} \int_0^1 hF\left(x_2 + \epsilon_\lambda ht,\, y_2 + \epsilon_\lambda kt,\, \frac{k}{h}\right) dt$$
$$- k\int_0^1 F_{y'}\left(x_2, y_2 + \epsilon_\lambda kt, \overset{\lambda}{p}(x_2, y_2 + \epsilon_\lambda kt)\right) dt \geqq 0,$$

($\lambda = 0, 1$ when $x_0 < x_2 < x_1$; $\lambda = 0$ when $x_2 = x_1$; $\lambda = 1$ when $x_2 = x_0$).

These two inequalities must be satisfied for every value of k — positive or negative — of sufficiently small absolute value.

§ 2. *Final form of condition* (V).

From our assumptions concerning the function F and the properties† of the slope $\overset{\lambda}{p}(x, y)$ it follows by applying TAYLOR's formula that

$$\int_0^1 F_{y'}\left(x_2, y_2 + \epsilon_\lambda kt, \overset{\lambda}{p}(x_2, y_2 + \epsilon_\lambda kt)\right)dt = \overset{\lambda}{X}(x_2, y_2) + \frac{\epsilon_\lambda k}{2} \overset{\lambda}{X}_y(x_2, y_2) + k(k),$$

* "Untere Unbestimmtheitsgrenze" = "Unterer Limes," compare *Encyclopaedie*, II, A 1 (PRINGSHEIM), p. 14.

† Compare my *Lectures*, pp. 81, 82.

where (k) is an infinitesimal for $\mathrm{L}k = 0$, and

(6) $$\overset{\lambda}{X}(x, y) = F_{y'}(x, y, \overset{\lambda}{p}(x, y)).$$

Since the point P_2 lies on the extremal $\mathfrak{E}_0$,

(7) $$\overset{\lambda}{p}(x_2, y_2) = f'_0(x_2) \equiv y'_2,$$

and therefore

(8) $$\overset{\lambda}{X}(x_2, y_2) = F_{y'}(x_2, y_2, y'_2).$$

Again,

$$\overset{\lambda}{X}_y(x, y) = F_{y'y}(x, y, \overset{\lambda}{p}(x, y)) + F_{y'y'}(x, y, \overset{\lambda}{p}(x, y))\,\overset{\lambda}{p}_y(x, y)$$

and*

$$\overset{\lambda}{p}_y(x, y) = \frac{\overset{\lambda}{\phi}_{x\gamma}(x, \gamma)}{\overset{\lambda}{\phi}_\gamma(x, \gamma)},$$

γ being replaced by the inverse function: $\gamma = \overset{\lambda}{\psi}(x, y)$ obtained by solving the equation: $y = \overset{\lambda}{\phi}(x, y)$ with respect to γ.

If we substitute in these formulas x_2, y_2 for x, y, we must give γ the particular value γ_0 which corresponds to the extremal $\mathfrak{E}_0$ since P_2 lies on $\mathfrak{E}_0$. Hence if we write for brevity

(9) $$F_{y'y}(x, f_0(x), f'_0(x)) = Q(x),$$
$$F_{y'y'}(x, f_0(x), f'_0(x)) = R(x),$$

we obtain

$$\overset{\lambda}{X}_y(x_2, y_2) = Q(x_2) + R(x_2)\frac{\overset{\lambda}{\phi}_{x\gamma}(x_2, \gamma_0)}{\overset{\lambda}{\phi}_\gamma(x_2, \gamma_0)}.$$

This expression may still be thrown into a different form by introducing the general solution

$$y = f(x, \alpha, \beta)$$

of EULER's differential equation. For if we denote by α_0, β_0, the special values of the constants of integration α, β, which furnish the extremal $\mathfrak{E}_0$ and put

(10) $$\Delta(x, x_\lambda) = f_\alpha(x, \alpha_0, \beta_0) f_\beta(x_\lambda, \alpha_0, \beta_0) - f_\beta(x, \alpha_0, \beta_0) f_\alpha(x_\lambda, \alpha_0, \beta_0),$$

then†

$$\overset{\lambda}{\phi}_\gamma(x, \gamma_0) = C_\lambda \Delta(x, x_\lambda),$$

where C_λ is a constant different from zero. Hence

(11) $$\overset{\lambda}{X}_y(x_2, y_2) = Q(x_2) + R(x_2)\frac{\Delta_x(x, x_\lambda)}{\Delta(x, x_\lambda)}.$$

* Compare my *Lectures*, pp. 81, 82.

† Compare, for instance, my *Lectures*, p. 62.

Substituting the values of $\overset{\lambda}{X}(x_2, y_2)$ and $\overset{\lambda}{X}_y(x_2, y_2)$ in (5), our condition may be written in the following form:

$$\text{(V)}\quad \underset{h=+0}{\mathbf{L}} \int_0^1 hF\left(x_2 + \epsilon_\lambda ht, y_2 + \epsilon_\lambda kt, \frac{k}{h}\right) dt - kF_{y'}(x_2, y_2, y_2') - \frac{\epsilon_\lambda k^2}{2}\left(Q(x_2) + R(x_2)\frac{\Delta_x(x, x_\lambda)}{\Delta(x, x_\lambda)}\right) + k^2(k) \geqq 0,$$

in which it is immediately applicable to examples.

We now divide (V) by k and then let k approach zero. If we put

$$(12)\qquad U_\lambda(k, x_2) = \frac{1}{k} \underset{h=+0}{\mathbf{L}} \int_0^1 hF\left(x_2 + \epsilon_\lambda ht, y_2 + \epsilon_\lambda kt, \frac{k}{h}\right) dt$$

and

$$(13)\qquad \underset{k=+0}{\mathbf{L}} U_\lambda(k, x_2) = \overset{+}{G}_\lambda(x_2), \qquad \underset{k=-0}{\mathbf{L}} U_\lambda(k, x_2) = \overset{-}{G}_\lambda(x_2),$$

where $+\infty$ and $-\infty$ must be included among the possible values of $U_\lambda(k, x_2)$, and $\overset{\pm}{G}_\lambda(x_2)$, we reach the following

THEOREM: *In order that the extremal* $\mathfrak{E}_0$, *for which conditions* (II′) *and* (III′) *are supposed to be satisfied, may furnish a strong minimum for the integral*

$$J = \int_{x_0}^{x_1} F(x, y, y')\,dx,$$

it is necessary that

$$(\mathrm{V}_a)\qquad \begin{aligned} \overset{+}{G}_\lambda(x_2) - F_{y'}(x_2, y_2, y_2') &\geqq 0 \\ \overset{-}{G}_\lambda(x_2) - F_{y'}(x_2, y_2, y_2') &\leqq 0 \end{aligned}$$

for $\lambda = 0$ *and for* $\lambda = 1$, *when* $x_0 < x_2 < x_1$; *for* $\lambda = 0$, *when* $x_2 = x_1$; *for* $\lambda = 1$, *when* $x_2 = x_0$.

If the four (or two, when $x_2 = x_0$ or x_1) inequalities (V_a) are satisfied *with the inequality sign*, then also (V) holds for all sufficiently small values of $|k|$.

But it may happen that one or several of these inequalities hold *with the equality sign.* In this case we cannot go back to (V), unless a further condition be added. We obtain it by dividing (V) by k^2 before passing to the limit. If we put

$$(14)\qquad \underset{k=\pm 0}{\mathbf{L}} \frac{U_\lambda(k, x_2) - F_{y'}(x_2, y_2, y_2')}{k} = \overset{\pm}{H}_\lambda(x_2),$$

including again the values $+\infty$ and $-\infty$ among the possible values of $\overset{\pm}{H}_\lambda(x_2)$, we obtain the following

COROLLARY: *If one of the conditions* (V_a) *is satisfied in the form of an equality*

$$(15)\qquad \overset{\pm}{G}_\lambda(x_2) - F_y'(x_2, y_2, y_2') = 0,$$

then, with the same meaning of $\pm$ *and* λ, *the following additional condition must hold* *

$$(\mathrm{V}_b) \qquad \overset{\pm}{H}_\lambda(x_2) - \frac{\epsilon_\lambda}{2} Q(x_2) - \frac{\epsilon_\lambda}{2} R(x_2) \frac{\Delta_x(x_2, x_\lambda)}{\Delta(x_2, x_\lambda)} \geqq 0,$$

where the functions $Q(x)$, $R(x)$, $\Delta(x, x_\lambda)$ *are defined by* (9) *and* (10).

If condition (V_b) is satisfied with the equality sign, another condition must be added, derived from the terms of the third order in the expansion of (V), etc., etc.

§ 3. *The special case when* $F(x, y, p)$ *admits expansions into power series in the vicinity of* $p = \pm\infty$.

The lower bounds U, G, H can be easily computed when $F(x, y, p)$ admits expansions into power series of the form

$$(16) \qquad F(x, y, p) = p^{n_1} \sum_{i,j,l} A^{(1)}_{i,j,l}(x - x_2)^i (y - y_2)^j \left(\frac{1}{p}\right)^l$$

convergent for $|x - x_2| < d_1$, $|y - y_2| < d_1$, $p > R_1$, and

$$(17) \qquad F(x, y, p) = |p^{n_2}| \sum_{i,j,l} A^{(2)}_{i,j,l}(x - x_2)^i (y - y_2)^j \left(\frac{1}{p}\right)^l$$

convergent for $|x - x_2| < d_2$, $|y - y_2| < d_2$, $p < -R_2$, d_1, d_2, R_1, R_2 being positive and the indices i, j, l running from 0 to $+\infty$; n_1 and n_2 are real, but need not be integers.

Under these assumptions we find easily

$$\int_0^1 hF\left(x_2 + \epsilon ht, y_2 + \epsilon kt, \frac{k}{h}\right) dt = \frac{|k|^n}{h^{n-1}} \sum_{i,j,l} \frac{\epsilon^{i+j}}{i+j+1} A_{ijl} k^{i+j} \left(\frac{h}{k}\right)^{l+i},$$

where n and A_{ijl} have to be replaced by n_1 and $A^{(1)}_{ijl}$ or by n_2 and $A^{(2)}_{ijl}$ according as k is positive or negative. Hence if we put

$$(18) \qquad B_{\mu\nu} = \sum_{i=0}^{\infty} A_{i,\mu-i,\nu-i},$$

with the understanding that every A_{ijl} with a negative index is zero, and suppose that

$$B_{\mu\nu} = 0 \qquad \text{for} \qquad \begin{cases} \mu = 0, 1, 2, \cdots, \\ \nu = 0, 1, 2, \cdots, s-1, \end{cases}$$

* If we consider the set of extremals through an arbitrary point $P(\xi, \eta)$ of $\mathfrak{E}_0$ different from P_2 we obtain a condition analogous to (V_b), in which x_λ is replaced by ξ and ϵ_λ by the sign of $(\xi - x_2)$. If we reduce the left-hand side to a fraction with the denominator $\Delta(x_2, \xi)$, numerator and denominator considered as functions of ξ are solutions of JACOBI's differential equation. Applying STURM's oscillation-theorem we obtain the result that the inequality in question is satisfied for every $\xi \neq x_2$ between x_0 and x_1, whenever it is satisfied for $\xi = x_0$ and $\xi = x_1$, so that no new condition is reached by this apparent generalization.

but that not all the coefficients $B_{\mu s}$ are equal to zero, we get for the above integral the value

$$\frac{|k|^{n-s}\delta^s}{h^{n-s-1}}\left\{\sum_{\mu=0}^{\infty}\frac{\epsilon^\mu}{\mu+1}B_{\mu s}k^\mu+(h)\right\},$$

where (h) is an infinitesimal and $\delta = k/|k|$.

We have now to distinguish three cases:

Case I. $n - s - 1 > 0$.

Suppose $B_{\mu s} = 0$ for $\mu = 0, 1, \cdots, r-1$, but $B_{rs} \neq 0$, then *condition* (V) *reduces to the inequality*

(19) $$\epsilon_\lambda^r \delta^{r+s} B_{rs} > 0.$$

If $x_0 < x_2 < x_1$, (19) must hold for $\epsilon_\lambda = +1$ and for $\epsilon_\lambda = -1$, hence r *must be even.* If $x_2 = x_0$, (19) must hold for $\epsilon_\lambda = +1$, if $x_2 = x_1$, for $\epsilon_\lambda = -1$.

Case II. $n - s - 1 = 0$.

In this case the expansion of the left-hand side of (V) according to powers of k begins with the terms

$$k\,[\delta^{s+1}B_{0s} - F_{y'}(x_2, y_2, y_2')] + \frac{\epsilon_\lambda k^2}{2}[\delta^{s+1}B_{1s} - \overset{\lambda}{X}_y(x_2, y_2)] + \cdots.$$

Hence it follows that *condition* (V) *reduces to the inequality*

(20) $$\delta\,[\delta^{s+1}B_{0s} - F_{y'}(x_2, y_2, y_2')] \geqq 0,$$

and *if*

(21) $$\delta^{s+1}B_{0s} - F_{y'}(x_2, y_2, y_2') = 0,$$

the condition must be added

(22) $$\epsilon_\lambda\left(\delta^{s+1}B_{1s} - Q(x_2) - R(x_2)\frac{\Delta_x(x_2, x_\lambda)}{\Delta(x_2, x_\lambda)}\right) \geqq 0.$$

Case III. $n - s - 1 < 0$.

In this case $U_\lambda(k, x_2) = 0$ for $\lambda = 0, 1$ and *condition* (V) *reduces to*

(23) $$-\delta F_{y'}(x_2, y_2, y_2') \geqq 0$$

and *if*

(24) $$F_{y'}(x_2, y_2, y_2') = 0$$

the condition must be added

(25) $$-\epsilon_\lambda\left(Q(x_2) + R(x_2)\frac{\Delta_x(x_2, x_\lambda)}{\Delta(x_2, x_\lambda)}\right) \geqq 0.$$

If $x_0 < x_2 < x_1$, conditions (22) and (24) must be satisfied for $\lambda = 0$ *and* for $\lambda = 1$; if $x_2 = x_0$, for $\lambda = 1$; if $x_2 = x_1$, for $\lambda = 0$.

In applying these conditions we must, in general, treat separately the two cases where k is positive ($n = n_1$, $A_{ijl} = A_{ijl}^{(1)}$, $\delta = +1$) and where k is negative ($n = n_2$, $A_{ijl} = A_{ijl}^{(2)}$, $\delta = -1$).

But when n_1 is an integer and *the same expansion holds for positive and negative values of* p — which happens, for instance, whenever $F(x, y, p)$ is a *rational function of* p — we have

$$n_1 = n_2, \text{ say } = n,$$

$A^{(2)}_{ijl} = (-1)^{n_1} A^{(1)}_{ijl}$, and therefore

$$r_1 = r_2, \text{ say } = r; \qquad s_1 = s_2, \text{ say } = s.$$

Hence it follows that the conditions for positive and negative values of k combine in the following manner:

Case I. $n - s - 1 > 0$,

(26) $$n + r + s \text{ must be } \textit{even}, \qquad \epsilon_\lambda^r B^{(1)}_{rs} > 0.$$

Case II. $n - s - 1 = 0$,

(27) $$B^{(1)}_{0s} - F_{y'}(x_2, y_2, y_2') = 0,$$

(28) $$\epsilon_\lambda \left(B^{(1)}_{1s} - Q(x_2) - R(x_2) \frac{\Delta_x(x_2, x_\lambda)}{\Delta(x_2, x_\lambda)} \right) \geqq 0.$$

Case III. $n + s - 1 < 0$.

Same conditions as in case II with $B^{(1)}_{0s} = 0$, $B^{(1)}_{1s} = 0$.

§ 4. *Examples.*

In this last section we propose to give examples for the different cases discussed in § 3. They will be so selected that not only conditions (II′) and (III′) are satisfied, but also WEIERSTRASS' condition in the somewhat stronger form

(IV′) $$\mathbf{E}\left(x, f_0(x); f_0'(x), \tilde{p}\right) > 0$$

for $x_0 \leqq x \leqq x_1$, and for every finite $\tilde{p} \neq f_0'(x)$, in order to show at the same time that our condition is *independent of* WEIERSTRASS' *condition.*

Case I. To this case belongs the example which I have given in the article referred to in the introduction:

$$F = ay'^2 - 4byy'^3 + 2bxy'^4,$$

$$a > 0, \qquad b > 0, \qquad (x_0, y_0) = 0, \qquad (x_1, y_1) = (1, 0).$$

The extremals are straight lines; in particular $\mathfrak{E}_0$ is the segment of the x-axis between $x = 0$ and $x = 1$. Conditions (II′), (III′) and (IV′) are satisfied Further we find easily for the integral

$$S = \int_0^1 hF\left(x_2 + \epsilon_\lambda ht, y_2 + \epsilon_\lambda kt, \frac{k}{h}\right) dt$$

the value

$$S = \frac{k^2}{h^3}(2bk^2 x_2 - bk^2 \epsilon_\lambda h + ah^2),$$

since $y_2 = 0$.

If $0 < x_2$ we obtain $U_\lambda(k, x_2) = +\infty$, for $\lambda = 0, 1$, and condition (V) is satisfied.

If $x_2 = 0$, in which case (V) must be satisfied for $\epsilon_\lambda = +1$, the term $-bk^4/h^2$ determines the sign and therefore $U_1(k, 0) = -\infty$, and (V) is not satisfied. Hence there exists *no strong minimum* if, as we suppose, the interval of integration extends to the point $x = 0$.

Also the following example, due to CARATHEODORY,* belongs to this case, viz.,

$$F = y'^2 - y^2 y'^4.$$

Here the extremals are, in general, not straight lines; but the particular line

$$\mathfrak{E}_0: \qquad y = 0 \equiv f_0(x)$$

is an extremal. Since

$$F_{y'y'}\big(x, f_0(x), f_0'(x)\big) = 2$$

condition (II′) is fulfilled, and we can always take x_0 and x_1 so close together that also (III′) is fulfilled. Again

$$\mathbf{E}\big(x, f_0(x), f_0'(x), \tilde{p}\big) = \tilde{p}^2;$$

hence also (IV′) is fulfilled.

Nevertheless there exists *no strong minimum*. For

$$S = \frac{k^2}{h^3}\left(-\frac{k^4}{3} + h^2 k^2\right),$$

and therefore $U_\lambda(k, x_2) = -\infty$ for $\lambda = 0, 1$ and for every x_2.

Case II. The examples for the remaining two cases will be of the form

$$F = \frac{L + My' + Ny'^2}{(1 + y'^2)^n},$$

where L, M, N are functions of x and y. We choose them so that all the non-singular extremals are straight lines, i. e., so that

$$F_y - F_{y'x} - y'F_{y'y} \equiv 0.$$

This leads to the following partial differential equations

$$\frac{\partial L}{\partial y} - \frac{\partial M}{\partial x} = 0, \qquad n\frac{\partial L}{\partial x} - \frac{\partial N}{\partial x} = 0,$$

$$(29) \qquad (2n+1)\frac{\partial L}{\partial y} + (2n-1)\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} = 0,$$

$$n\frac{\partial M}{\partial y} + (n-1)\frac{\partial N}{\partial x} = 0, \qquad (2n-1)\frac{\partial N}{\partial y} = 0.$$

* Archiv der Mathematik und Physik, ser. 3, vol. 10 (1906).

For $n \neq 0, 1, \frac{1}{2}$, the most general solution of these differential equations is

$$(30) \qquad L = ax + b, \qquad M = -(n-1)ay + c, \qquad N = nax + d,$$

$a, b, \cdots$ being arbitrary constants.

For $n = \frac{1}{2}$, the most general solution is

$$(31) \qquad \begin{aligned} L &= 2ax^2 + ay^2 + 2dx + by + c, \\ M &= 2axy + bx + dy + e, \\ N &= ax^2 + 2ay^2 + dx + 2by + f. \end{aligned}$$

The extremals being straight lines, condition (III′) is always satisfied.

We take the two given points on the x-axis, so that

$$\mathfrak{C}_0: \qquad y = 0 \equiv f_0(x).$$

Then

$$(32) \qquad \begin{aligned} F_{y'}\big(x, f_0(x), f_0'(x)\big) &= M(x, 0), \qquad Q(x) = M_y(x, 0), \\ R(x) &= 2\big(N(x, 0) - nL(x, 0)\big). \end{aligned}$$

In order to obtain an example for case II, we choose $n = \frac{1}{2}$, and give the arbitrary constants in (31) the values $a = 0, b = 1, c = 1, d = 0, e = 0, f = 1$, so that

$$F = \frac{(y+1) + xy' + (2y+1)y'^2}{\sqrt{1 + y'^2}},$$

the square root being positive. Then

$$R(x) = +1,$$

so that (II′) is satisfied. Again, if we put $\tilde{p} = \tan\tilde{\theta}$, where $-\pi/2 < \tilde{\theta} < +\pi/2$, we get

$$\mathbf{E}\big(x, f_0(x); f_0'(x), \tilde{p}\big) = \frac{1 - \cos\tilde{\theta}}{\cos\tilde{\theta}}(1 - x\sin\tilde{\theta}).$$

Hence condition (IV′) will be satisfied if

$$(33) \qquad -1 \leqq x_0 < x_1 \leqq 1.$$

For the integral S we find easily

$$S = \frac{k^2(\epsilon_\lambda k + 1) + hk\left(x_2 + \epsilon_\lambda \frac{h}{2}\right) + h^2\left(\epsilon_\lambda \frac{k}{2} + 1\right)}{\sqrt{h^2 + k^2}};$$

Further

$$F_{y'}[x, f_0(x), f_0'(x)] = x, \qquad Q(x) = 0, \qquad \Delta(x, x_\lambda) = x - x_\lambda.$$

Hence the inequality (V) takes the form

$$(\delta - x_2)k + \epsilon_\lambda k^2\left(\delta - \frac{1}{2(x_2 - x_\lambda)}\right) + k^2(k) \geqq 0,$$

where again $\delta = k/|k|$.

The discussion of this inequality leads to the following result:

When $-1 < x_0 < x_1 < +1$ *condition* (V) *is always fulfilled. But when* $x_1 = +1$, *or* $x_0 = -1$, *condition* (V) *introduces a new restriction of the interval beyond the restriction* (33) *already introduced by* WEIERSTRASS' *condition* (IV′), *viz., when* $x_1 = +1$, *we must have* $x_0 \geqq \frac{1}{2}$, *and when* $x_0 = -1$, *we must have* $x_1 \leqq \frac{1}{2}$.

Case III. In order to obtain an example for this case, we take $n = 2$, and give the constants in (30) the values $a = -1$, $b = 0$, $c = 0$, $d = 1$, so that

$$F = \frac{-x + yy' + (1 - 2x)y'^2}{(1 + y'^2)^2}.$$

Here we find

$$R(x) = 2,$$

so that condition (II′) is satisfied. Further

$$\mathbf{E}\left(x, f_0(x); f_0'(x), \tilde{p}\right) = \frac{\tilde{p}^2(1 + x\tilde{p}^2)}{(1 + \tilde{p}^2)^2};$$

hence also (IV′) *is satisfied provided that*

$$0 \leqq x_0.$$

We see at once that

$$\mathop{\mathbf{L}}_{h=+0} S = 0.$$

And since

$$F_{y'}[x, f_0(x), f_0'(x)] \equiv 0, \qquad Q(x) = 1, \qquad \Delta(x, x_\lambda) = x - x_\lambda,$$

condition (V) reduces to

$$-\epsilon_\lambda\left(1 + \frac{2}{x_2 - x_\lambda}\right) \geqq 0.$$

The discussion of this inequality easily leads to the following result: *In the present example condition* (V) *is equivalent to a restriction of the length of the interval* $(x_0 x_1)$, *viz:*

$$x_1 - x_0 \leqq 2.$$

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